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HARMONIC UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY USING SALAGEAN INTEGRAL OPERATOR

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Abstract. In this paper we define and investigate a new class of harmonic functions defined by using Salagean integral operator with varying arguments. We obtain coefficient inequalities, extreme points and distortion bounds.

Keywords: harmonic, analytic, varying arguments, sense preserving, Salagean integral operator.

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1. Introduction

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f = h + \overline{g}, \tag{1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$, $z \in D$ (see [3]).

Denote by S_H the class of functions f of the form (1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f_z^{'}(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \;\; g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{2}$$

In 1984 Clunie and Shell-Small [3] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

Salagean integral operator I^n is defined as follows (see [8])

(i) $I^0 f(z) = f(z)$;

(ii) $I^1 f(z) = If(z) = \int\limits_0^z f(t)t^{-1}dt$;

....

(iii) $I^n f(z) = I(I^{n-1} f(z))$ $(n \in \mathbb{N} = \{1, 2, 3, ...\})$.

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In [4], Cotirla defined Salagean integral operator for harmonic univalent functions $f(z)$ such that $h(z)$ and $g(z)$ are given by (2) as follows

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, \tag{3}$$

where

$$I^n h(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k \text{ and } I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

With the help of the modified Salagean integral operator we let $E_H(m, n; \gamma, \rho)$ be the family of harmonic functions $f = h + \overline{g}$, which satisfy the following condition [6]

$$Re\left\{\left(1+\rho e^{i\alpha}\right)\frac{I^n f(z)}{I^m f(z)} - \rho e^{i\alpha}\right\} \geq \gamma \tag{4}$$

$$\left(\alpha \in \mathbb{R}, 0 \leq \gamma < 1, \rho \geq 0, m \in \mathbb{N}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m > n, \text{ and } z \in U\right),$$

where $I^n f$ is defined by (3), we note that
(i) Taking $\alpha = 0$, $E_H\left(n+1, n; 2\beta-1, 1\right) = H\left(n, \beta\right) (0 \leq \beta < 1)$ (see Cotirla [4]).
(ii) Taking $m = n + q$, $E_H\left(n+q, n; \gamma, \rho\right) = H_{\rho,q}\left(n, \gamma\right) (q \in \mathbb{N})$ (see Guney and Sakar [5]).

Also we note that, by the special choices of α, γ, ρ, m and n, we obtain the following special cases

$$\text{(i) Taking } \alpha = 0, \text{ then } E_H(m, n, 2\beta - 1, 1) = H\left(m, n; \beta\right) = \left\{f \in S_H : \right.$$

$$\left. Re\left\{\frac{I^n f(z)}{I^m f(z)}\right\} > \beta \left(0 \leq \beta < 1; m \in \mathbb{N}; n \in \mathbb{N}_0; m > n; z \in U\right)\right\};$$

$$\text{(ii) } E_H(n+1, n; \gamma, \rho) = E_H\left(n; \gamma, \rho\right) = \left\{f \in S_H : Re\left\{\left(1+\rho e^{i\alpha}\right)\frac{I^n f(z)}{I^{n+1} f(z)} - \right.\right.$$

$$\left.\left. -\rho e^{i\alpha}\right\} \geq \gamma \left(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; n \in \mathbb{N}_0; z \in U\right)\right\};$$

$$\text{(iii) } E_H(1, 0; \gamma, \rho) = E_H\left(\gamma, \rho\right) = \left\{f \in S_H : Re\left\{\left(1+\rho e^{i\alpha}\right)\frac{f(z)}{If(z)} - \rho e^{i\alpha}\right\} \geq \gamma \right.$$

$$\left. \left(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; z \in U\right)\right\}.$$

Also we define the subclass $V_{\overline{H}}(m, n; \gamma, \rho)$ consists of harmonic functions $f_n = h + \overline{g}_n$ in $E_H(m, n; \gamma, \rho)$ such that h and g_n are the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g_n(z) = \sum_{k=1}^{\infty} b_k z^k \tag{5}$$

and there exists a real number ϕ such that, mod 2π,

$$\arg(a_k) + (k-1)\phi \equiv \pi, \; k \geq 2 \text{ and } \arg(b_k) + (k+1)\phi \equiv (n-1)\pi, \; k \geq 1. \tag{6}$$

Also we note that, by the special choices of α, γ, m and n, we obtain:
(i)Taking $\alpha = 0$, $V_{\overline{H}}\left(n+1, n; 2\beta-1, 1\right) = V_{\overline{H}}\left(n, \beta\right)$;
(ii) Taking $\alpha = 0$, $V_{\overline{H}}(m, n, 2\beta - 1, 1) = V_{\overline{H}}\left(m, n; \beta\right)$;
(iii) $V_{\overline{H}}(n+1, n; \gamma, \rho) = V_{\overline{H}}\left(n; \gamma, \rho\right)$;
(iv) $V_{\overline{H}}(1, 0; \gamma, \rho) = V_{\overline{H}}\left(\gamma, \rho\right)$.

2. Main results

Unless otherwise mentioned, we assume in the reminder of this paper that, $\alpha \in \mathbb{R}$, $0 \leq \gamma < 1$, $\rho \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$. We begin with a sufficient coefficient condition for functions in the class $E_H(m,n;\gamma,\rho)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be such that h and g are given by (2). Furthermore,*

$$\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}\left|a_k\right| + \tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}\left|b_k\right|\right] \leq 2, \tag{7}$$

where $a_1 = 1$. Then $f \in E_H(m,n;\gamma,\rho)$.

Proof. We need to show that if (7) holds then the condition (4) is satisfied, then we want to prove that

$$Re\left\{\frac{\left(1+\rho e^{i\alpha}\right)I^n f(z) - \rho e^{i\alpha} I^m f(z)}{I^m f(z)}\right\} = Re\frac{A(z)}{B(z)} \geq \gamma. \tag{8}$$

Using the fact that $Re\{w\} > \gamma$ if and only if $|1-\gamma+w| > |1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \geq 0, \tag{9}$$

where $A(z) = \left(1+\rho e^{i\alpha}\right)I^n f(z) - \rho e^{i\alpha} I^m f(z)$ and $B(z) = I^m f(z)$. Substituting for $A(z)$ and $B(z)$ in the left side of (9) we obtain

$$\begin{aligned}
&\left|\left(1+\rho e^{i\alpha}\right)I^n f(z) - \rho e^{i\alpha} I^m f(z) + (1-\gamma)\, I^m f(z)\right| - \\
&-\left|\left(1+\rho e^{i\alpha}\right)I^n f(z) - \rho e^{i\alpha} I^m f(z) - (1+\gamma) I^m f(z)\right| = \\
=\ &\Bigg|(2-\gamma)\, z + \sum_{k=2}^{\infty}\left[\left((1+\rho e^{i\alpha})k^{-n} + \left(1-\gamma-\rho e^{i\alpha}\right)k^{-m}\right)\right] a_k z^k + \\
&+(-1)^n\sum_{k=1}^{\infty}\left[(1+\rho e^{i\alpha})k^{-n} - (-1)^{m-n}\left(\rho e^{i\alpha}+\gamma-1\right)k^{-m}\right] \times \\
&\times\overline{b_k z^k}\Bigg| - \Bigg|\gamma z - \sum_{k=2}^{\infty}\left[(1+\rho e^{i\alpha})k^{-n} - \left(1+\gamma+\rho e^{i\alpha}\right)k^{-m}\right] a_k z^k - \\
&-(-1)^n\sum_{k=1}^{\infty}\left[(1+\rho e^{i\alpha})k^{-n} - (-1)^{m-n}\left(1+\gamma+\rho e^{i\alpha}\right)k^{-m}\right]\overline{b_k z^k}\Bigg| \geq \\
\geq\ & 2(1-\gamma)|z| - 2\sum_{k=2}^{\infty}\left[(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}\right]|a_k|\,|z|^k - \\
&-2\sum_{k=1}^{\infty}\left[(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}\right]|b_k|\,|z|^k \geq \\
&\geq\ 2(1-\gamma)|z|\left\{1 - \sum_{k=2}^{\infty}\frac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}|a_k|\,|z|^{k-1} - \right. \\
&\left. -\sum_{k=1}^{\infty}\frac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\,|z|^{k-1}\right\}.
\end{aligned}$$

By using (7), then the last expression is non-negative, then (9) is satisfied.

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \tfrac{1-\gamma}{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}} x_k z^k + \sum_{k=1}^{\infty} \tfrac{1-\gamma}{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}} \overline{y_k z^k}, \tag{10}$$

where $\sum\limits_{k=2}^{\infty} |x_k| + \sum\limits_{k=1}^{\infty} |y_k| = 1$, shows that the coefficient bound given by (7) is sharp. □

In the following theorem, it is shown that the condition (7) is also necessary for function $f_n = h + g_n$, where h and g_n are of the form (5).

Theorem 2.2. *Let $f_n = h + g_n$, where h and g_n are given by (5). Then $f_n \in V_{\overline{H}}(m, n; \gamma, \rho)$, if and only if the coefficient condition (7) holds.*

Proof. Since $V_{\overline{H}}(m, n; \gamma, \rho) \subseteq E_H(m, n; \gamma, \rho)$, we only need to prove the "only if" part of the theorem. For functions $f_n = h + g_n$, where h and g_n are given by (5), the inequality (4) with $f = f_n$ is equivalent to

$$Re \left\{ \frac{(1+\rho e^{i\alpha})[z + \sum\limits_{k=2}^{\infty} k^{-n} a_k z^k + (-1)^n \sum\limits_{k=1}^{\infty} k^{-n} \overline{b}_k \overline{z}^k]}{z + \sum\limits_{k=2}^{\infty} k^{-m} a_k z^k + (-1)^m \sum\limits_{k=1}^{\infty} k^{-m} \overline{b}_k \overline{z}^k} \right\} -$$

$$-Re \left\{ \frac{(\gamma+\rho e^{i\alpha})[z + \sum\limits_{k=2}^{\infty} k^{-m} a_k z^k + (-1)^m \sum\limits_{k=1}^{\infty} k^{-m} \overline{b}_k \overline{z}^k]}{z + \sum\limits_{k=2}^{\infty} k^{-m} a_k z^k + (-1)^m \sum\limits_{k=1}^{\infty} k^{-m} \overline{b}_k \overline{z}^k} \right\} > 0.$$

The above condition holds for all values of $\alpha \in \mathbb{R}$ and $z \in U$. Upon choosing ϕ according (6) and substituting $\alpha = 0$ and $z = re^{i\phi} (0 < r < 1)$, we must have

$$\frac{E}{1 - \left[\sum\limits_{k=2}^{\infty} k^{-m} |a_k| - (-1)^{m+n-1} \sum\limits_{k=1}^{\infty} k^{-m} |b_k| \right] r^{k-1}} > 0, \tag{11}$$

where

$$\begin{aligned} E \;=\; & (1-\gamma) - \left(\sum_{k=2}^{\infty} \left[(1+\rho)k^{-n} - (\gamma+\rho)\, k^{-m} \right] |a_k| \right) r^{k-1} - \\ & - \left(\sum_{k=1}^{\infty} \left[(1+\rho)k^{-n} - (-1)^{m-n} (\gamma+\rho)\, k^{-m} \right] |b_k| \right) r^{k-1}. \end{aligned}$$

If the inequality (7) does not hold, then E is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (11) is negative. But this is a contradiction, the proof of Theorem 2 is completed. □

We now obtain the distortion bounds for functions in $V_{\overline{H}}(m, n; \gamma, \rho)$.

Theorem 2.3. *Let $f_n = h + g_n$, where h and g_n are given by (5) and $f_n \in V_{\overline{H}}(m, n; \gamma, \rho)$. Then for $|z| = r < 1$, we have*

$$|f_n(z)| \leq (1 + |b_1|)\, r + \left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}} - \tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}} |b_1| \right] r^2 \tag{12}$$

and

$$|f_n(z)| \geq (1 + |b_1|)\, r - \left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}} - \tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}} |b_1| \right] r^2. \tag{13}$$

Proof. We prove the first inequality.
Let $f_n \in V_{\overline{H}}(m,n;\gamma,\rho)$, we have

$$\begin{aligned}
|f_n(z)| &\leq (1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^k \leq (1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^2 \leq \\
&\leq (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\sum_{k=2}^{\infty}\tfrac{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}{1-\gamma}(|a_k|+|b_k|)\,r^2 \leq \\
&\leq (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\times \\
&\quad\times\sum_{k=2}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\right]r^2 \leq \\
&\leq (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\left[1-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}|b_1|\right]r^2 \leq \\
&\leq (1+|b_1|)\,r+\left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-2^{-m}(\gamma+\rho)}-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-2^{-m}(\gamma+\rho)}|b_1|\right]r^2.
\end{aligned}$$

The proof of the second inequality is similar, thus it is left. □

The bounds given in Theorem 3 for functions $f_n = h + \overline{g}_n$ such that h and g_n are given by (6) also hold for functions $f = h + \overline{g}$ such that h and g are given by (2) if the coefficient condition (7) is satisfied.
Using the same technique used earlier by Aghalary [1] we introduce the extreme points of the class $V_{\overline{H}}(m,n;\gamma,\rho)$.

Theorem 2.4. *The closed convex hull of the class $V_{\overline{H}}(m,n;\gamma,\rho)$ (denoted by $clcoV_{\overline{H}}(m,n;\gamma,\rho)$) is*

$$\left\{f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\overline{\sum_{k=1}^{\infty}b_kz^k}\in E_H(m,n;\gamma,\rho):\right.$$
$$\left.\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\right]\leq 2\right\},$$

where a_1=1. Set $\lambda_k=\frac{1-\gamma}{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}$ and $\mu_k=\frac{1-\gamma}{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}$.
For b_1 fixed, $|b_1|\leq\frac{1-\gamma}{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}$, the extreme points of the class $V_{\overline{H}}(m,n;\gamma,\rho)$ are

$$\left\{z+\lambda_kxz^k+\overline{b_1}z\right\}\cup\left\{\overline{z+\mu_kxz^k+b_1z}\right\}, \tag{14}$$

where $k\geq 2$ and $|x|=1-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}$.

Proof. Any function $f\in V_{\overline{H}}(m,n;\gamma,\rho)$ may be expressed as

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|\,e^{i\beta_k}z^k+\overline{b_1z}+\overline{\sum_{k=2}^{\infty}|b_k|\,e^{i\delta_k}z^k},$$

where the coefficients satisfy the inequality (7). Set

$$h_1(z)=z,\;\; g_1(z)=b_1z,\;\; h_k(z)=z+\lambda_ke^{i\beta_k}z^k,\;\; g_k(z)=b_1z+\mu_ke^{i\delta_k}z^k, k=2,3,\ldots\,.$$

Writing $X_k=\frac{|a_k|}{\lambda_k}$, $Y_k=\frac{|b_k|}{\mu_k}$, $k=2,3,\ldots$ and $X_1=1-\sum\limits_{k=2}^{\infty}X_k$, $Y_1=1-\sum\limits_{k=2}^{\infty}Y_k$, we have

$$f(z)=\sum_{k=1}^{\infty}\left(X_kh_k(z)+\overline{Y_kg_k(z)}\right).$$

In particular, setting $f_1(z) = z + \overline{b_1 z}$ and $f_k(z) = z + \lambda_k x z^k + \overline{b_1 z} + \overline{\mu_k y z^k}$,

$$\left(k \geq 2, |x| + |y| = 1 - \frac{(1+\rho) - (-1)^{m-n}(\gamma+\rho)}{1-\gamma}|b_1|\right),$$

we see that extreme points of the class $V_{\overline{H}}(m, n; \gamma, \rho)$ are contained in $\{f_k(z)\}$. To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$\begin{aligned} f_1(z) &= \frac{1}{2}\left\{f_1(z) + \lambda\left(1 - \frac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_1|\right)z^2\right\} + \\ &+ \frac{1}{2}\left\{f_1(z) - \lambda\left(1 - \frac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_1|\right)z^2\right\}, \end{aligned}$$

a convex linear combination of functions in the class $V_{\overline{H}}(m, n; \gamma, \rho)$. Next we will show if both $|x| \neq 0$ and $|y| \neq 0$, then f_k is not an extreme point.

Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1 + \epsilon$ and $B = 1 - \left|\frac{\epsilon x}{y}\right|$, we then see that both

$$t_1(z) = z + \lambda_k x A z^k + \overline{b_1 z + \mu_k y B z^k}$$

and

$$t_2(z) = z + \lambda_k x (2 - A) z^k + \overline{b_1 z + \mu_k y (2 - B) z^k},$$

are in the class $V_{\overline{H}}(m, n; \gamma, \rho)$ and note that

$$f_k(z) = \frac{1}{2}(t_1(z) + t_2(z)).$$

The extremal coefficient bounds shows that functions of the form (14) are the extreme points for the class $V_{\overline{H}}(m, n; \gamma, \rho)$, then the proof of Theorem 4 is completed. □

Now we will examine the closure properties of the class $V_{\overline{H}}(m, n; \gamma, \rho)$ under the generalized Bernardi-Libera-Livingston integral operator (see [2, 7]) $L_c(f)$ which is defined by

$$L_c(f(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt \quad (c > -1). \tag{15}$$

Theorem 2.5. *Let $f_n = h + g_n \in V_{\overline{H}}(m, n; \gamma, \rho)$, where h and g_n are given by (5). Then $L_c(f_n(z))$ belongs to the class $V_{\overline{H}}(m, n; \gamma, \rho)$.*

Proof. From the representation of $L_c(f_n(z))$, it follows that

$$\begin{aligned} L_c(f_n(z)) &= \frac{c+1}{z^c}\int_0^z t^{c-1}(h(t) + \overline{g}_n(t))\, dt = \\ &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left\{t + \sum_{k=2}^{\infty} a_k t^k + \overline{\sum_{k=1}^{\infty} b_k t^k}\right\} dt = \\ &= z + \sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k}, \end{aligned}$$

where $A_k = \frac{c+1}{c+k} a_k$, $B_k = \frac{c+1}{c+k} b_k$. Therefore, we have,

$$\sum_{k=1}^{\infty}\left[\frac{(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}}{1-\gamma}\frac{c+1}{c+k}|a_k| + \frac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}\frac{c+1}{c+k}|b_k|\right] \leq$$

$$\leq \sum_{k=1}^{\infty} \left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma} |a_k| + \tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma} |b_k| \right] \leq 2,$$

and the proof of Theorem 5 is completed. □

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