TWMS J. Pure Appl. Math., V.4, N.1, 2013, pp.44-51

AN EXISTENCE OF THE TIME OPTIMAL CONTROL FOR THE OBJECT DESCRIBED BY THE HEAT CONDUCTIVE EQUATION WITH NON-CLASSICAL BOUNDARY CONDITION

K.K. HASANOV[1], L.K. HASANOVA[1]

Abstract. Many practical problems in mathematical formulation are reduced to the problems which require to transfer the system from one state to given another one. These problems are usually called time optimal problems and are investigated in the mathematical theory of optimal processes. For the systems with the distributed parameters, these investigations meet serious difficulties in differ from the systems with the concentrated parameters. Time optimal problems arise in various technological processes, which are described by the equations of heat conductivity, diffusions, filtrations. For example, time optimal problem appears in the control of diffusion process of neutrons in the nuclear reactor. Until now, mainly the problems with classical boundary conditions have been considered [2, 4, 5, 8, 9, 10].

Keywords: time optimal control, spectral problem, relay, necessary condition, generalized solution.

AMS Subject Classification: 49J20, 49Jxx.

1. Introduction

The systems of automatic control of the concentrated parameter objects have been relatively well studied. Important results on the existence of the time optimal control and the number of switching of concentrated parameters optimal control have been obtained [1, 3, 8].

The similar problems are actual for the distributed parameter systems as well [2, 4, 5, 9, 10].

In the present paper, we consider time optimal control problem for the linear heat conductive equation with non-classical boundary condition. The theorems on the existence of the time optimal control and relay property of the control are proved.

2. Problem statement

Let the controlled system be described by the function $y(x,t)$ that in the domain $D = \{0 \le x \le 1,\ 0 \le t \le T\}$ satisfies the equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + p(x)\,u(t)\,, \tag{1}$$

on the boundary of D the initial

$$y(x,0) = y_0(x) \tag{2}$$

and the boundary conditions

$$y(0,t) = 0, \quad y_x(0,t) = y_x(1,t)\,, \tag{3}$$

where $p(x) \neq 0,\ \ y_0(x)$ are given functions, $u(t)$ is a control.

[1]Baku State University, Baku, Azerbaijan,
e-mail: kazimhasanov@mail.ru

Manuscript received December 2011.

As a set of admissible controls we consider the functions

$$u(t) \in U_\partial = \{u(t) \in L_2(0,T) : |u(t)| \leq 1, \quad a.e \ (0,T)\}$$

almost everywhere on $(0,T)$, and the solution of the problem corresponding to the given control is taken almost everywhere [7]. Note that for $p(x) \in L_2(0,1)$, $y_0(x) \in W_2^1(0,1)$, $y_0(0) = 0$ and for the admissible control $u(t)$ there exists the almost everywhere unique solution $y(x,t)$ of the problem (1)-(3) that by the means of the function

$$G(x,s,t) = X_0(x)\,Y_0(s) + \sum_{k=1}^{\infty} \left\{ X_{2k}(x)\,Y_{2k}(s) + \left[X_{2k-1}(x) - 2\sqrt{\lambda_k} t X_{2k}(x) \right] Y_{2k-1}(s) \right\} e^{-\lambda_k t}$$

may be represented in the form

$$y(x,t) = \int_0^1 G(x,s,t)\,y_0(s)\,ds + \int_0^t \int_0^1 G(x,s,t-\tau)\,p(s)\,u(\tau)\,ds d\tau, \tag{4}$$

where λ_k are the eigenvalues, $\{X_k(x)\}$ is a system of eigen and adjoint functions of the spectral problem

$$X''(x) + \lambda X(x) = 0,$$

$$X(0) = 0, \ \ X'(0) = X'(1) \ ,$$

$\{Y_k(x)\}$ - eigen and adjoint functions of the adjoint problem [6].

For the given function $\varphi(x)$ from $L_2(0,1)$ it is required to find a control $u^*(t)$ such that corresponding solution $y^*(x,t)$ of the problem (1)-(3) satisfy the condition

$$y^*(x,\tau_0) = \varphi(x), \quad \tau_0 \in (0,T), \tag{5}$$

where τ_0 is the lower bound of the values $\tau \in (0,T)$ for which the condition

$$y(x,\tau) = \varphi(x) \tag{6}$$

is fulfilled for some $\tau \in (0,T)$; $y(x,t)$ is a solution of problem (1)-(3) corresponding to some admissible control $u(t)$.

3. Existence of the optimal control

Here we prove a theorem on the existence of the time optimal control.

Theorem 3.1. *If there exists a control $u(t)$ such that corresponding solution $y(x,t)$ of the problem (1)-(3) satisfies to the condition (6) for some $\tau \in (0,T)$, then there exists a time optimal control $u^*(t)$ as well, i.e corresponding solution $y^*(x,t)$ of the problem (1)-(3) satisfies to the condition $\tau_0 = \inf\{\tau\}$.*

Proof. If the functions $y_0(x)$ and $\varphi(x)$ are equivalent on [0,1], we'll assume that the control time equals zero, i.e. $\tau_0 = 0$. Now suppose that the functions $y_0(x)$ and $\varphi(x)$ are not equivalent. Let $\{u_n(t)\}$ be a minimizing sequence of the admissible controls for which the sequence $\{y_n(x,t)\}$ of appropriate solutions satisfy the conditions

$$y_1(x,\tau_1) = \varphi(x) \ , \ \ y_2(x,\tau_2) = \varphi(x) \ , \ \ y_3(x,\tau_3) = \varphi(x) \ , \ ... \tag{7}$$

and

$$\tau_1 < \tau_2 < \tau_3 < ..., \ \ \lim_{n\to\infty} \tau_n = \tau_0. \tag{8}$$

Since the set of admissible controls U_∂ is bounded in $L_2(0,T)$, from the sequence $\{u_n(t)\}$ we can choose such a subsequence (denote it also by $\{u_n(t)\}$), that

$$u_n(t) \to u^*(t) \ \ weakly \ \ in \ \ L_2(0,\tau_0), \ u^*(t) \in U_\partial. \tag{9}$$

Then from the equality

$$y_n(x,t) = \int\limits_0^1 G(x,s,t)\, y_0(s)\, ds + \int\limits_0^t \int\limits_0^1 G(x,s,t-\tau)\, p(s)\, u_n(\tau)\, ds d\tau,$$

passing to limit as $n \to \infty$, we get

$$y^*(x,t) = \int\limits_0^1 G(x,s,t)\, y_0(s)\, ds + \int\limits_0^t \int\limits_0^1 G(x,s,t-\tau)\, p(s)\, u^*(\tau)\, ds d\tau\,. \tag{10}$$

Consequently, $y^*(x,t)$ is a solution of the problem (1)-(3) by the control $u^*(t)$.

Show that

$$y_n(x,\tau_n) \to y^*(x,\tau_0) \ \ weakly \ in \ L_2(0,1)\,. \tag{11}$$

Since

$$y_n(x,\tau_n) - y^*(x,\tau_0) = [y_n(x,\tau_n) - y_n(x,\tau_0)] + [y_n(x,\tau_0) - y^*(x,\tau_0)] \tag{12}$$

taking into account

$$\lim_{n\to\infty} y_n(x,\tau_0) = y^*(x,\tau_0)\,, \tag{13}$$

we must show that the relation

$$y_n(x,\tau_n) - y_n(x,\tau_0) \to 0 \ weakly \ \ in \ \ L_2(0,1)\,. \tag{14}$$

is valid.

From thus one may get (11).

From the identity

$$y_n(x,\tau_n) - y_n(x,\tau_0) = \int\limits_{\tau_0}^{\tau_n} \frac{\partial y_n(x,t)}{\partial t} dt$$

for any function $\eta(x) \in L_2(0,1)$ we have

$$\int\limits_0^1 [y_n(x,\tau_n) - y_n(x,\tau_0)]\, \eta(x)\, dx = \int\limits_0^1 \int\limits_{\tau_0}^{\tau_n} \frac{\partial y_n(x,t)}{\partial t} \eta(x)\, dx dt.$$

Hence applying the Cauchy-Bunyakovsky inequality, we get

$$\left| \int\limits_0^1 [y_n(x,\tau_n) - y_n(x,\tau_0)]\, \eta(x)\, dx \right| \le \left(\int\limits_0^1 \left(\int\limits_{\tau_0}^{\tau_n} \frac{\partial y_n(x,t)}{\partial t} dt \right)^2 dx \right)^{\frac{1}{2}} \left(\int\limits_0^1 \eta^2(x)\, dx \right)^{\frac{1}{2}} \le$$

$$\le (\tau_n - \tau_0)^{\frac{1}{2}} \left(\iint\limits_D \left(\frac{\partial y_n(x,t)}{\partial y} \right)^2 dx dt \right)^{\frac{1}{2}} \left(\int\limits_0^1 \eta^2(x) dx \right)^{\frac{1}{2}}.$$

Taking into account the boundedness of the sequence $\left\{ \frac{\partial y_n(x,t)}{\partial t} \right\}$ in $L_2(D)$, we get the affirmation of (14).

Considering into account the equalities (13) and (14), from (12) we get the affirmation of (11). Then it follows from (8) that

$$y^*(x,\tau_0) = \varphi(x).$$

The theorem is proved. □

4. A theorem on relay property

There we'll prove theorems on relay property of the high speed optimal-control.

Theorem 4.1. *Let the condition of theorem 1 be fulfilled and $u^*(t)$ be a time optimal control for the problem* (1)-(3), (5). *Then*

$$|u^*(t)| = 1 \text{ a. e. on } (0,\tau_0). \tag{15}$$

Proof. The existence of the time optimal control $u^*(t)$ follows from theorem 1. Lets prove that for almost all $t \in (0,\tau_0)$ condition (15) is fulfilled. Denote by $y^*(x,t)$ the almost everywhere solution of the problem (1)-(3) by the control $u^*(t)$. Suppose that relation (15) is not fulfilled. Then for rather small number $\varepsilon > 0$ there exsts the set $e(\varepsilon)$ from $(0,\tau_0)$ such that $mese(\varepsilon) > 0$ and

$$|u^*(t)| \le 1-\varepsilon \; for \; almost \; all \; t \in e(\varepsilon). \tag{16}$$

Consider the sequence of the numbers

$$0 < \tau_1 < \tau_2 < \tau_3 < \ldots . (\tau_k < \tau_0) \; and \; \lim_{k\to\infty} \tau_k = \tau_0. \tag{17}$$

Denote by e_n the set from $(0,\tau_n)$ such that

$$|u^*(t)| < 1 \; for \; almost \; all \; t \in e_n. \tag{18}$$

Denote the characteristic function of the set e_n by $\Phi_n(t)$, and define the control $u_n(t)$ by the following relation

$$p(x)\left[u_n(t) - u^*(t)\right] = \frac{\Phi_n(t)}{mese_n} \int_0^1 G(x,s,t-\tau_n)\left[y^*(s,\tau_0) - y^*(s,\tau_n)\right] ds. \tag{19}$$

Hence using the Cauchy - Bunyakovsky inequality, we get

$$\|p\| \cdot |u_n(t) - u^*(t)| \le \frac{\Phi_n(t)}{mese_n} \left(\int_0^1 \int_0^1 G^2(x,s,t-\tau_n)\, ds dx \right)^{\frac{1}{2}} \|y^*(\cdot,\tau_0) - y^*(\cdot,\tau_n)\|. \tag{20}$$

For any $\varepsilon > 0$ we can show such a number N that for $n \ge N$ we have

$$|u_n(t) - u^*(t)| < \varepsilon \; for \; t \in e(\varepsilon). \tag{21}$$

By condition (16) it follows that

$$|u_n(t)| \le |u_n(t) - u^*(t)| + |u^*(t)| \le 1$$

almost everywhere on e_n.

Thus, the control $u_n(t)$ for $n \ge N$ is admissible.

Assume

$$y_n(x,t) = \int_0^1 G(x,s,t)\, y_0(s)\, ds + \int_0^t \int_0^1 G(x,s,t-\tau)\, p(s)\, u_n(\tau)\, ds d\tau. \tag{22}$$

Then $y_n(x,t)$ is a solution of the problem (1)-(3) corresponding to the control $u_n(t)$.
Substituting (19) into (22), we get

$$y_n(x,\tau_n)=\int\limits_0^1 G(x,s,\tau_n)\,y_0(s)\,ds+\int\limits_0^{\tau_n}\int\limits_0^1 G(x,s,\tau_n-\sigma)\,p(s)\,u^*(\sigma)\,dsd\sigma+$$

$$+\int\limits_0^{\tau_n}\frac{\Phi_n(\sigma)}{mese_n}\int\limits_0^1\left(\int\limits_0^1 G(x,s,\tau_n-\sigma)\,G(s,\xi,\sigma-\tau_n)\,ds\right)[y^*(\xi,\tau_0)-y^*(\xi,\tau_n)]\,d\xi d\sigma.$$

Hence, using

$$\int\limits_0^1 X_k(s)\,Y_j(s)\,ds=\delta_{kj},\quad \int\limits_0^1 G(x,s,\tau_n-\sigma)\,G(s,\xi,\sigma-\tau_n)\,ds=G(x,\xi,0)\ ,$$

one may obtain

$$y_n(x,\tau_n)=\int\limits_0^1 G(x,s,\tau_n)\,y_0(s)\,ds+\int\limits_0^{\tau_n}\int\limits_0^1 G(x,s,\tau_n-\sigma)\,p(s)\,u^*(\sigma)\,dsd\sigma+$$
$$+\left(y^*(x,\tau_0)-y^*(x,\tau_n)\right). \tag{23}$$

Further, from equality (10) we have

$$y^*(x,\tau_n)=\int\limits_0^1 G(x,s,\tau_n)\,y_0(s)\,ds+\int\limits_0^{\tau_n}\int\limits_0^1 G(x,s,\tau_n-\sigma)\,p(s)\,u^*(\sigma)\,dsd\sigma.$$

Taking into account the equality in (23), we have

$$y_n(x,\tau_n)=y^*(x,\tau_0)=\varphi(x). \tag{24}$$

Since $\tau_n<\tau_0$, the equality (24) contradicts the assumption that τ_0 is an optimal time. The theorem is proved. □

5. Uniqueness

Lets prove that time optimal control is unique almost everywhere.

Theorem 5.1. *Let the conditions of Theorem 2 be fulfilled. Then the time optimal control is unique.*

Proof. Assume that there exist two time optimal controls $u_1(t),u_2(t)$.
Then

$$y_1(x,\tau_0)=y_2(x,\tau_0)=\varphi(x), \tag{25}$$

where $y_1(x,t)$ and $y_2(x,t)$ are the solutions of problem (1)-(3), corresponding to the controls $u_1(t)$ and $u_2(t)$, respectively.
For any $\theta\in(0,\,1)$ assume

$$u_\theta(t)=(1-\theta)\,u_1(t)+\theta u_2(t)\ .$$

Then, using this formula for the solutions $y_1(x,t)$ and $y_2(x,t)$, we get

$$y_\theta(x,\tau_0)=(1-\theta)\,y_1(x,\tau_0)+\theta y_2(x,\tau_0)=$$
$$=\int\limits_0^1 G(x,s,\tau_0)\,y_0(s)\,ds+\int\limits_0^{\tau_0}\int\limits_0^1 G(x,s,\tau_0-\sigma)\,p(s)\,u_\theta(\sigma)\,dsd\sigma.$$

From the condition (25) we have

$$y_\theta(x,\tau_0) = (1-\theta)\, y_1(x,\tau_0) + \theta y_2(x,\tau_0) = (1-\theta)\,\varphi(x) + \theta\varphi(x) = \varphi(x).$$

Thus, the control $u_\theta(t)$ is also time optimal. Therefore, the control $u_\theta(t)$ should satisfy the condition $|u_\theta(t)| = 1$ almost everywhere on $(0,\tau_0)$. But it is possible only if $u_1(t) = u_2(t)$ almost everywhere. The theorem is proved. □

6. First order necessary conditions of optimality

Let $u^*(t)$ be a time optimal control, $y^*(x,t)$ be a solution of problem (1)-(3) corresponding the control $u^*(t)$, $t \in [0,\tau_0]$, τ_0 be an optimal time. Determine the adjoint state $z(x,t)$ as a solution of the equation

$$\frac{\partial z}{\partial t} + \frac{\partial^2 z}{\partial x^2} = 0 \ in \ D_{\tau_0} = \{0 < x < 1,\ 0 < t < \tau_0\}\,, \tag{26}$$

satisfying initial

$$z(x,\tau_0) = h(x)\ ,\ 0 \le x \le 1, \tag{27}$$

and boundary conditions

$$z_x(1,t) = 0,\ \ z(0,t) = z(1,t)\ ,\ \ 0 \le t \le \tau_0, \tag{28}$$

where $h(x) \in L_2(0,1)$.

Theorem 6.1. *Let the conditions of Theorem 1 be fulfilled and $u^*(x)\,,\ \ t \in (0,\tau_0)$ be a time control. Then there exists a function $h(x) \in L_2(0,1)$ corresponding to the solution $z(x,t)$ of problem (26)-(28) such that for any $v \in [0,\ 1]$ the inequality*

$$\left(\int_0^1 z(x,t)\,p(x)\,dx\right)(v - u^*(t)) \le 0, \tag{29}$$

holds true on $(0,\tau_0)$.

Proof. Take an arbitrary admissible control $u(t)$ and denote by $y(x,t)$ the solution of problem (1)-(3) corresponding to this control. Represent the solution $y(x,t)$ in the form

$$y(x,t) = \int_0^1 G(x,s,t)\, y_0(s)\, ds + \int_0^t\int_0^1 G(x,s,t-\sigma)\, p(s)\, u(\sigma)\, ds d\sigma\,.$$

Then we have

$$y(x,\tau_0) - y^*(x,\tau_0) = \int_0^{\tau_0}\int_0^1 G(x,s,\tau_0-\sigma)\, p(s)\,(u(\sigma) - u^*(\sigma))\, ds d\sigma\,,$$

where $y^*(x,t)$ is a solution of the problem (1)-(3) corresponding to the control $u^*(t)$. Multiplying the both hand sides of this equality by the function $h(x)$ and integrating we get

$$\int_0^1 [y(x,\tau_0) - y^*(x,\tau_0)]\, h(x)\, dx = \int_0^1 \left(\int_0^{\tau_0}\int_0^1 G(x,s,\tau_0-\sigma)\, p(s)\,(u(\sigma) - u^*(\sigma))\, ds d\sigma\right) h(x)\, dx.$$

Hence, changing the integration sign, we have

$$\int_0^1 [y(x,\tau_0) - y^*(x,\tau_0)]\, h(x)\, dx =$$

$$= \int_0^{\tau_0} \int_0^1 \left(\int_0^1 G(s, x, \tau_0 - \sigma) h(s)\, ds \right) p(x)\ (u(\sigma) - u^*(\sigma))\, dx d\sigma. \tag{30}$$

We can directly verify that the function

$$z(x, t) = \int_0^1 G(s, x, \tau_0 - t) h(s)\, ds \tag{31}$$

is a solution of the problem (26)-(28).

Taking into account (31) in (30), we get

$$\int_0^1 [y(x, \tau_0) - y^*(x, \tau_0)] h(x)\, dx = \int_0^{\tau_0} \int_0^1 z(x, t) p(x)\ (u(t) - u^*(t))\, dx dt. \tag{32}$$

Since $|u(t)| \leq 1$ and $|u^*(t)| = 1$ almost everywhere on $(0, \tau_0)$, then $u(t) - u^*(t) \leq 0$ almost everywhere on $(0, \tau_0)$. Choose $h(x)$ from $L_2(0, 1)$ so that the solution $z(x, t)$ of the problem (26)-(28) satisfy the inequality

$$\int_0^1 z(x, t) p(x)\, dx \geq 0, \ \ a.e.\ on\ \ (0, \tau_0).$$

Then from (32) we get that for any function $u(t) \in \vartheta_\partial$ the following condition is fulfilled:

$$\int_0^{\tau_0} \int_0^1 z(x, t) p(x) (u(t) - u^*(t))\, dx dt \leq 0. \tag{33}$$

Hence, taking into account the availability of local restrictions on the controls we get the inequality

$$\left(\int_0^1 z(x, t) p(x)\, dx \right) (v - u^*(t)) \leq 0\ a.e.\ on\ \ (0, \tau_0),\ and\ for\ any\ v \in [0, 1]. \tag{34}$$

The last inequality indeed is the first order necessary condition of optimality for the considered problem.

□

References

[1] Boltyansky, V.G., (1968), Mathematical Methods of the Optimal Control, M.: Nauka, 308p. (in Russian).

[2] Butkovsky, A.K., (1975), The Method of Distributed Parameter Systems Control, M.: Nauka, 568p. (in Russian).

[3] Gabasov, R., Kirillova, F.M., (1971), Quality Theory of Optimal Processes, M.: Nauka, 508p. (in Russian).

[4] Gasanov, K.K., Gasanova, A.N., (2010), Optimal control for heat conductivity equations with non-classical boundary conditions, Izvestia Pedagogicheskogo Universiteta, Baku, 5, pp.26-33 (in Russian).

[5] Egorov, A.I. (1978), Optimal Control of Heat and Diffusion Processes, M.: Nauka, 464p. (in Russian).

[6] Ionkin, N.I., (1977), Solution of the boundary value problem of theory of heat conductivity with non-classical boundary condition, Differen. Uravn. XIII(2), pp.294-304 (in Russian).

[7] Ladyzhenskaya, O.A., (1973), Boundary Value Problems of Mathematical Physics, M.: Nauka, 408p. (in Russian).

[8] Lee, E.B., Markus, L., (1967), Foundations of Optimal Control Theory, John Wiley and Sons, New York.

[9] Lions, J.L., (1972), Optimal Control of Systems Described by Partial Equations, M.: Mir, 416p.

[10] Fattorini, H.O., Russel, D.L., (1971), Exact controllability theorems for linear parabolic equations in one space dimension, Arch. For Rational Mech. And Anal. 43(4).

Kazim Hasanov was born in 1934 in Tovuz region of Azerbaijan. He graduated from Mechanics-Mathematics Faculty of Baku State University in 1958. He got his Ph.D. degree in 1961. Since 1988 he is a Professor of Mechanics-Mathematics Faculty of Baku State University. He is an author of more than 20 scientific articles and 4 books. His scientific interests are differential equations with the delayed argument, tasks of mathematical optimal administration and boundary-value problems for the differential equations.

Leyla Hasanova was born in October 1967 in Baku. She graduated from Baku State University. Presently she is an assistant professor at the Department of "Algebra and Geometry" of Baku State University. Her scientific interests are differential equations with the delayed argument, boundary-value problems for the differential equations.