TWMS J. Pure Appl. Math., V.5, N.1, 2014, pp.59-65

STATISTICALLY CONVERGENT AND STATISTICALLY CAUCHY SEQUENCE IN A CONE METRIC SPACE

N.R. DAS[1], RINKU DEY[1], BINOD CHANDRA TRIPATHY[2]

Abstract. In this paper we have introduced the concept of statistically convergent sequence in case of cone metric space and constructed statistically convergent, Cauchy and complete cone metric space and some theorems based on them. Consequently we have generalised several results in cone metric spaces from metric spaces.

Keywords: cone metric, statistically convergent, statistically Cauchy, complete metric space.

AMS Subject Classification: 40A05, 40A35, 46B20, 54E35.

1. Introduction

The concept of statistical convergence is found in Zygmund [16]. The notion of statistical convergence was investigated by Steinhaus [9] and Fast [4] and later followed by Schoenberg [8] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [5], Rath and Tripathy [6], Salat [7], Tripathy [10], Tripathy and Baruah [12], Tripathy and Dutta [13], Tripathy and Sen ([14], [15]) and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-$\acute{C}$ech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

The notion of cone metric space has been applied by various authors in the recent past. It has been applied for introducing and investigating different new sequence spaces and studying their different algebraic and topological properties by Abdeljawad [1], Beg, Abbas and Nazir [2], Dhanorkar and Salunke [3] and many others. In this article we have investigated different properties of the notion of statically convergence in cone metric space.

2. Definitions and preliminaries

Definition 2.1. *A subset P of a real Banach space E is called a cone if and only if*

(i) P is closed, non-empty and $P \neq \{0\}$.

(ii) If $a, b \in R$, $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$.

(iii) If both $x \in P$ and $-x \in P$, then $x = 0$.

[1] Department of Mathematics, Gauhati University, Assam, India,
e-mail: nrd47@yahoo.co.in, rinkudey_math@rediffmail.com,

[2] Mathematical Sciences Division; Institute of Advanced Study in Science and Technology, Assam, India,
e-mail: tripathybc@yahoo.com, tripathybc@rediffmail.com

Manuscript received March 2013.

For a given cone $P \in E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P, x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

Definition 2.2. *A cone metric space is an ordered pair (X,d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:*
(i) $0 < d(x, y)$ for all $x, y \in X$.
(ii) $d(x, y) = 0$ if and only if $x = y$.
(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 2.1. *Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$, is a constant and $x, y \in X$. Then it is well known that (X, d) is a cone metric space.*

We provide the following example of a cone metric space involving sequence spaces.

Example 2.2. *Let $E = \ell_\infty, P = \{(x_1, x_2, x_3, \ldots) \in E : x_1, x_2, x_3, ... \geq 0\}, X = R$ and $d : X \times X \to E$ be defined by $d(x, y) = (|x - y|, \alpha_1|x - y|, \alpha_2|x - y|, ...)$, where $x, y \in X$ and $\alpha = (\alpha_n)$ is any sequence in $E = \ell_\infty$. Then (X, d) is a cone metric space.*

Throughout the article we consider cone metric space unless otherwise stated.

Definition 2.3. *Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 << c$ there exists n_0 such that for all $n > n_0, d(x_n, x) << c$, then $\{x_n\}$ is said to be convergent to $x \in X$ i.e. $\lim_{n\to\infty} x_n = x$.*

Definition 2.4. *Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 << c$ there exists n_0 such that for all $m, n > n_0, d(x_n, x_m) << c$ then $\{x_n\}$ is called Cauchy sequence in X.*

Definition 2.5. *A subset $E \subset N$ is said to have density or asymptotic density $\delta(E)$ if*

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists},$$

where χ_E is the characteristic function of E or indicator function.

Definition 2.6. *A sequence space E is said to be solid or normal if $\{\alpha_k x_k\} \in E$ whenever $\{x_k\} \in E$ and for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.*

Let $K = \{k_1 < k_2 < k_3, ...\} \subseteq N$ and E be a class of sequences. A *K-step set* of E is a set of sequences $\lambda_K^E = \{(x_{k_n}) \in w : (x_k) \in w$.

A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_n) \in w$, defined as follows:

$$y_n = \begin{cases} x_n, \ if \ n \in K, \\ \bar{0}, \ otherwise. \end{cases}$$

Definition 2.7. *A canonical pre-image of a step set λ_K^E is a set of canonical pre-images of all elements in λ_K^E i.e. y is in canonical pre-image λ_K^E if and only if y is canonical pre-image of some $x \in \lambda_K^E$.*

Definition 2.8. *A class of sequences E is said to be monotone if E contains the canonical pre-images of all its step sets.*

Remark 2.1. *A class of sequences E is solid $\Rightarrow$ E is monotone.*

Definition 2.9. *A sequence space E is said to be convergence free if $\{x_n\} \in E$ implies $\{y_n\} \in E$ such that $y_n = 0$ whenever $x_n = 0$.*

Definition 2.10. *A class of sequences E is said to be symmetric if $x_{\pi(n)} \in E$, whenever $(x_k) \in E$ where π is a permutation of N.*

Definition 2.11. *A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $x_k, y_k \in E$.*

In this article we introduced the following definitions.

Definition 2.12. *Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 << c$ such that $\delta(\{n \in N : d(x_n, x) >> c\}) = 0$ almost all n. Then $\{x_n\}$ is said to be statistically convergent to $x \in X$ and we denote by $x_n \xrightarrow{stat} x$.*

Note 2.1. On taking $x = \theta$, the zero element of X, in the above definition, we will get the definition of *statistically null* sequence in a cone metric space.

Definition 2.13. *A sequence $\{x_n\}$ in a cone metric space (X, d) is said to be statistically Cauchy if for any $c \in E$ with $c >> 0$ there exits a natural number $m(c)$ such that $\delta(\{n \in N : d(x_n, x_m) >> c\}) = 0$ almost all n.*

We procure the following decomposition theorem, which will be used in establishing the results of this article.

Lemma 2.1. *The following statements are equivalent:*

(i) $x_n \xrightarrow{stat} x$.

(ii) there exists $K = \{k_i : i \in N\} \subset N$ such that $\delta(K) = 1$ and $\lim\limits_{i\to\infty} x_{k_i} = L$.

(iii) there exists a stat-null sequence $\{z_k\}$ and convergent sequence $\{y_k\}$ such that $x_k = y_k + z_k$ with $\lim\limits_{k\to\infty} y_k = L$. In this case if $\{x_k\}$ is bounded, then $\{x_k\} = \{y_k\} + \{z_k\}$.

(iv) there exists a convergent sequence $\{y_k\}$ such that $x_k = y_k$ for almost all k.

We formulate the following results, which can be easily established.

Lemma 2.2. *Let (X, d) be a cone metric space. If $\{x_n\}$,$\{y_n\}$ are two sequences in X such that $x_n \to x, y_n \to y$ then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.*

Lemma 2.3. *Let (X, d) be a cone metric space and $x \in X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$.*

Lemma 2.4. *Let (X, d) be a cone metric space and $x \in X$. If $\{x_n\}$ converges to x then $d(x_n, x) \to 0$ as $n \to \infty$.*

Lemma 2.5. *Let (X, d) be a complete cone metric space. Then ℓ_∞ the set of all bounded sequences is a closed subspace of w, the set of all sequences.*

3. Main results

In this section we establish the result of this article.

Theorem 3.1. *Let (X, d) be a cone metric space. If $\{x_m\}$,$\{y_m\}$ are sequences in X such that $x_m \xrightarrow{stat} x$,$y_m \xrightarrow{stat} y$, then $d(x_m, y_m) \xrightarrow{stat} d(x, y)$ as $m \to \infty$.*

Proof. Since $x_m \xrightarrow{stat} x$ and $y_m \xrightarrow{stat} y$ so by Lemma 2.10 there exists $K_1 = \{k_i : i \in N\} \subset N$ and $K_2 = \{k_j : j \in N\} \subset N$ such that $\delta(K_1) = \delta(K_2) = 1$ and

$$\lim_{i\to\infty} x_{k_i} = x, \lim_{j\to\infty} y_{k_j} = y. \tag{1}$$

Now, $\delta(K_1) = \delta(K_2) = 1$
$\Rightarrow \delta(K_1 \cap K_2) = 1$. Let $K_1 \cap K_2 = \{m_i : i \in N\}$.
By (1) and Lemma 2.2, $d(x_m, y_m) \xrightarrow{stat} d(x, y)$. Since there exists $K_1 \cap K_2 = \{m_i : i \in N\}$ such that $\delta(K_1 \cap K_2) = 1$ and $\lim\limits_{i\to\infty} d(x_m, y_m)= d(x, y)$, so by Lemma 2.10 we get,
$d(x_m, y_m) \xrightarrow{stat} d(x, y)$.
This completes the proof. □

Theorem 3.2. *Let (X, d) be a cone metric space and $x, y \in X$. If $\{x_n\}$ is a sequence in X such that $x_n \xrightarrow{stat} x$ and $x_n \xrightarrow{stat} y$, then $x = y$.*

Proof. Suppose $x_n \xrightarrow{stat} x$ and $x_n \xrightarrow{stat} y$, so for any $c \in E$ with $0 << c$ we can find n_0 such that for any $n \geq n_0$,we have $\delta(\{n \in N : d(x_n, x) >> \frac{c}{2}\}) = 0$ and $\delta(\{n \in N : d(x_n, y) >> \frac{c}{2}\}) = 0$.
$\Rightarrow d(x_n, x) << \frac{c}{2}$ and $d(x_n, y) << \frac{c}{2}$, for almost all $n \in N$.
Now $d(x, y) = d(x, x_n) + d(x_n, y)$
$\Rightarrow d(x, y) << \frac{c}{2}+ \frac{c}{2}$
$\Rightarrow d(x, y) << c$.
Without loss of generality we can assume that
$\Rightarrow d(x, y) << \frac{c}{k}$ for all $k \geq 1$.
$\Rightarrow \frac{c}{k} - d(x, y) \in P$ for all $k \geq 1$.
Since P is closed and $\frac{c}{k} \to \infty$ as $k \to \infty$, so we have $\lim\limits_{n\to\infty} (\frac{c}{k} - d(x, y)) \in P$.

$$\Rightarrow -d(x, y) \in P. \tag{2}$$

But

$$d(x, y) \in P. \tag{3}$$

Therefore from (2) and (3) we get,
$d(x, y) = 0 \Rightarrow x = y$. □

Theorem 3.3. *Let (X, d) be a complete cone metric space. Then $m = \bar{c} \cap \ell_\infty$ that is the class of all bounded statistically convergent sequences over X is complete.*

Proof. Let $\{x^i\}$ be a Cauchy sequence in m. So for a given $c >> 0$ there exists n_0 such that $\sup\limits_k d(x^n_k, x^m_k) << \frac{c}{3}$ for all $n, m \geq n_0$

$$\Rightarrow d(x^n_k, x^m_k) << \frac{c}{3} \text{ for each fixed } k \in N. \tag{4}$$

$\Rightarrow \{x^i_k\}$ is a Cauchy sequence in (X, d) which is complete.
Hence it converges for each $k \in N$.
Let $\lim\limits_{k\to\infty} x^n_k = x$, for $k \in N$.
Now we have to show that $\{x_k\} \in m, \delta(\{k \in N : d(x_k, L) << \frac{c}{3}\} = 1$ for some $L \in X$.
$x^i_k \xrightarrow{stat} L^i$ each $i \in N$.
$\Rightarrow \delta(\{k \in N : d(x^i_k, L^i) << \frac{c}{3}\} = 1$

$$A_i = \{k \in N : d(x^i_k, L^i) << \frac{c}{3}\}. \tag{5}$$

Let $n_0(c)$ be chosen such that for $k \in A_i \cap A_j$ for all $i, j \geq n_0$,

$d(L^i, L^j) \leq d\left(L^{2'}, x_k^{2'}\right) + d(x_k^i, x_k^j) + d(x_k^j, L^j)$

$\Rightarrow d(L^i, L^j) << c$ by (4) and (5).

Hence $\{L^i\}$ is a Cauchy sequence in (X, d), which is complete.

Let $\lim\limits_{i\to\infty} L^i = L$, say.

Now we show that $\{x_n\}$ is statistically convergent to L.

$d(x_i, L) \leq d(x_k, x_k^i) + d(x_k^i, L^i) + d(L^i, L)$

$\Rightarrow d(x_i, L) << c$ on the set $A_i \cap N \cap N = A_i$ and $\delta(A_i) = 1$. Hence m is complete cone metric space. This completes the proof. □

Theorem 3.4. *The class of all statistically convergent sequences is neither solid nor normal, where as the class of all statistically null sequences is solid and thus is monotone.*

Proof. The class of statistically null sequences is solid can be established following standard techniques. The result follows from the following example. □

Example 3.1. *Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$,$X = R$,$d : X \times X \to E$ be defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$, is a constant and $x, y \in X$. Consider the sequence (x_k) defined by*

$$x_k = \begin{cases} k, & for\ all\ k = i^2,\ i \in N, \\ 1, & otherwise. \end{cases}$$

Then clearly (x_k) is statistically convergent to with respect to 1 to the cone metric space considered. Now consider the sequence of scalars (α_k) defined by $\alpha_k = (-1)^k$, for all $k \in N$. Then it can be easily verified that the sequence $(\alpha_k x_k) = ((-1)^k x_k)$ is not statistically convergent with respect to the cone metric consider above. Hence the class of all statistically convergent sequences is not normal and hence is not solid.

Theorem 3.5. *The classes of all statistically convergent sequences and statistically null are not symmetric.*

Proof. The result follows from the following example.

Example 3.2. *Consider the cone metric space considered in Example 3.1. Consider the sequence (x_k) defined by*

$$x_k = \begin{cases} k, & for\ all\ k = i^2,\ i \in N, \\ 0, & otherwise. \end{cases}$$

Then clearly (x_k) is statistically convergent to o with respect to the cone metric space considered. Now consider the rearrangement of the sequence (x_k) defined by $(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, ...)$ Then it can be easily examined that the sequence (y_k) is not statistically convergent with respect to the cone metric consider above. Hence the class of all statistically convergent sequences is not symmetric.

□

Theorem 3.6. *The classes of all statistically convergent sequences and statistically null are not convergence free.*

Proof. The result follows from the following example.

Example 3.3. *Consider the cone metric space considered in Example 3.1. Consider the sequence (x_k) defined by*

$$x_k = \begin{cases} 1, & for\ all\ k = i^2,\ i \in N, \\ k^{-1}, & otherwise. \end{cases}$$

Then clearly (x_k) is statistically convergent to 0 with respect to the cone metric space considered. Consider the sequence (y_k) defined as follows.

$$y_k = \begin{cases} 1, & for\ all\ k = i^2,\ i \in N, \\ k, & otherwise. \end{cases}$$

Then it can be easily examined that the sequence (y_k) is not statistically convergent with respect to the cone metric consider above. Hence the class of all statistically convergent sequences is not convergence free.

We state the following result without proof, which can be established using standard techniques.

□

Theorem 3.7. *The classes of all statistically convergent and statistically null sequences are sequence algebra.*

References

[1] Abdeljawad, T., (2010), Completion of cone metric spaces, Hacettepe, Jour. Math. Stat., 39(1), pp.67-74.

[2] Beg, I., Abbas, M., Nazir, T., (2010), Generalized cone metric space, J. Nonlinear Sci. Appl., 3(1), pp.21-31.

[3] Dhanorkar, G.A., Salunke, J.N., (2011), A Generalization on fixed point theorem on cone metric spaces with ω-distance, International Mathematical Forum, 6(39), pp.1915-1919.

[4] Fast, H., (1951), Sur la convergence statistique, Colloq. Math., 2, pp.241-244.

[5] Fridy, J.A., (1985), On statistical convergence, Analysis, 5, pp.301-313.

[6] Rath, D., Tripathy, B.C., (1996), Matrix maps on sequence spaces associated with sets of integers, Indian Jour. Pure Appl. Math., 27(2), pp.197-206.

[7] Salat, T., (1980), On statistically convergent sequences of real numbers, Math. Slovaca, 30, pp.139-150.

[8] Schoenberg, I.J., (1959), The integrability of certain functions and related summability methods, Amer . Math. Monthly, 66, pp.361-375.

[9] Steinhaus, H., (1951), Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2, pp.73-74.

[10] Tripathy, B.C., (1997), Matrix transformations between some class of sequences, J. Math. Anal. Appl., 206(2), pp.448-450.

[11] Tripathy, B.C., (2004), On generalized difference paranormed statistically convergent sequences, Indian Jour. Pure Appl. Math., 35(5), pp.655-663.

[12] Tripathy, B.C., Baruah, A., (2010), Lacunary statistically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers, Kyungpook Math. J., 50(4), pp.565-574.

[13] Tripathy, B.C., Dutta, H., (2012), On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunary Δ_m^n-statistical convergence, Analele Stiintifice ale Universitatii Ovidius, Seria Matematica, 20(1), pp.417-430.

[14] Tripathy, B.C., Sen, M., (2001), On generalized statistically convergent sequences, Indian Jour. Pure Appl. Math., 32(11), pp.1689-1694.

[15] Tripathy, B.C., Sen, M., (2006), Characterization of some matrix classes involving paranormed sequence spaces, Tamkang Jour. Math., 37(2), pp.155-162.

[16] Zygmund, A., (1979), Trigonometric Series, Cambridge University Press, Cambridge, UK.

Nanda Ram Das was born in 1950 at Kamrup, Assam, India. He recieved his M.Sc. degree in Pure Mathematics from the Gauhati University in 1971. He got his Ph.D. degree from Indian Institute of Technology Kanpur in the year 1983. He has published 40 research articles on functional analysis and topology, focusing on p-adic analysis, mixed topological vector space, fuzzy mathematical analysis, fuzzy topological spaces and fixed point theory. He has produced 15 Ph.D.'s. He is an Editorial Board member of the journal Assam of Academy of Mathematics.

Rinku Dey was born in 1984, Jorhat, Assam, India. He recieved his M.Sc. degree in Pure Mathematics in 2007 and M.Phil. degree in Mathematics in 2010 from the Gauhati University, India. He has taken admission into the Ph.D. programme of the Gauhati University in 2010 availing the UGC-BSR Research Fellowship.

Binod Chandra Tripathy was born in 1963 in Berhampur, Orissa, India. He got his M.Sc. degree in 1985 and Ph.D. degree in 1993 from the Berhampur University, India. He has published 134 research articles on functional analysis and topology focusing on spectra of matrix operators, sequence spaces, summability theory, mixed topological space, bitopological space, fixed point theory, fuzzy mathematical analysis, fuzzy topological spaces. He has supervised 10 Ph.D.'s. He is a Reviewer for the Mathematical Reviews and is an Editorial Board member of 11 journals from different countries.