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THE SYMMETRIC P-STABLE HYBRID OBRECHKOFF METHODS FOR THE NUMERICAL SOLUTION OF SECOND ORDER IVPS

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ABSTRACT. This paper presents new two-step explicit symmetric P-stable methods, including Obrechkoff and hybrid terms, of orders four and six for solving initial value problems of second order ordinary differential equations. In this paper, we improved the method written by Xinyuan Wu and Qinghong Li [11], in a way that we could increase order and accuracy of their methods. The numerical results obtained by the new methods on some IVPs equations show the superior efficiency, accuracy, stability of the methods presented in this paper.

Keywords: hybrid methods, P-stable, off-step points, Obrechkoff methods.

AMS Subject Classification: 65l05, 65l07, 65l20.

1. INTRODUCTION

Let us consider the initial value problems of second order ordinary differential equations

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1}$$

where we presume that $f(x, y)$ is sufficiently differentiable and that the first derivative does not appear explicitly in $f(x, y)$. The numerical methods have been paid much attention to in recent years because the problems are usually encountered in celestial mechanics, quantum mechanical scattering theory, theoretical physics and chemistry, and electronics. Generally, the solution of (1) is periodic, so it is expected that the results produced by some numerical methods be of the periodicity of the analytic solution. In 1976, Lambert and Watson [10] proposed the concepts of periodicity interval and P-stability which can be used to discuss the stability of the numerical method for second order initial value problems. Although many P-stable methods have been proposed, such as linear multistep methods, high order hybrid P-stable methods, implicit Runge-Kutta-Nystrom and so on [9], these methods are implicit, so an iteration subprocess is needed in each step. The numerical integration methods for (1) can be divided into two distinct classes: (a) problems for which the solution period is known (even approximately) in advance (see [8, 12]); (b) problems for which the period is not known. There is a vast literature available for the numerical solution of this problem. Computational methods involving a parameter proposed by Sesappa Rai et al [13, 14], Shokri [15, 16] and Xiang [22] yield the numerical solution to the problem of the first class. Numerical treatment to the problems of the second class have been presented by Chawla et al [4, 5], Simos [17], Hollevoet et al [8], Ananthakrishnaiah [1, 2], and Chawla and Neta [3].

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Lambert and Watson [10], have developed linear, symmetric multistep methods of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+1-j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+1-j}, \quad k \geq 2, \tag{2}$$

where $h(>0)$ is the step length of integration and $\alpha_j = \alpha_{k-j}$, $\beta_j = \beta_{k-j}$, $j = 0(1)k$, on the discrete point set $\{x_n : x_n = nh, n = 0, 1, \ldots\}$, for finding the numerical solution of the special initial value problem (1). They derive methods for $k = 2, 4$ and 6. Motivated by the idea, we will present the new two-step explicit P-stable methods of orders four and six for solving (1).

In recent years a class of explicit methods at high order for stiff problems is presented by some authors (see [6, 7, 18, 19, 20, 21]) in which with the aid of a special vector operation, these methods can be extended to be vector-applicable [11]. Motivated by the idea, we have presented a class of two-step explicit symmetric P-stable methods for solving (1) [16]. This method has convergence of orders four and six.

2. Preliminaries

To investigate the stability properties of methods for solving the initial value problem (1), Lambert and Watson [10] introduced the scalar test equation

$$y'' = -\omega^2 y, \quad \omega \in \mathbb{R}. \tag{3}$$

When applying a symmetric two-step method to the test equation (3), one obtains the following difference equation of the form:

$$y_{n+1} - 2C(H)y_n + y_{n-1} = 0, \tag{4}$$

where $H = \omega h$ and h is a fixed step length, $C(H)$ is a rational polynomial with respect to H. The characteristic equation and polynomial are defined by the following respectively:

$$\xi^2 - 2C(H)\xi + 1 = 0, \tag{5}$$

$$Q(z, \xi) = Q_0(z^2)\xi^2 + Q_1(z^2)\xi + Q_2(z^2), \tag{6}$$

where $z = iwh$ and $Q_0(z^2)$, $Q_1(z^2)$, and $Q_2(z^2)$ are determined by the left side of (4) (see [6]).

Definition 2.1. *Let ξ_1, ξ_2 be the two roots of (4), the method (2) is unconditionally stable if $|\xi_1| \leq 1$, $|\xi_2| \leq 1$ for all values of wh.*

Definition 2.2. *The interval $(0, H_0^2)$ is the periodicity interval of the method (2) if the roots of (4) satisfy $\xi_1 = \overline{\xi}_2 = e^{ig(H)}$, for all $H^2 \in (0, H_0^2)$, where $g(H)$ is a real function of H.*

Definition 2.3. *The method (2) is P-stable if the periodicity interval of the method is $(0, +\infty)$.*

Definition 2.4. *The order of the root of (5) (say ξ_1) is p if ξ_1 satisfies*

$$e^z - \xi_1 = Cz^{p+1} + O(z^{p+2}), \quad z \to 0, \tag{7}$$

where C is the error constant of ξ_1.

Theorem 2.1. *Suppose (4) is the characteristic equation of some method, and $|C(H)| < 1$ for all $H^2 \in (0, H_0^2)$, then the interval of periodicity of the method is $(0, H_0^2)$.*

Proof. See [10]. □

Theorem 2.2. *Set $p \geq 1$, the root of the characteristic polynomial of some method is of order p if and only if*

$$Q(z, e^z) = C\frac{\partial^2 Q}{\partial \xi^2}(0, 1)z^{p+2} + O(z^{p+3}), \quad z \to 0. \tag{8}$$

Proof. See [7]. □

Lambert and Watson [10] have proved that the method described by (2) has a nonvanishing interval of periodicity only if it is symmetric and for P-stability the order cannot exceed 2. Further, the method is implicit. Later Chawla and Rao [5] noted that Numerov method has phase-lag error of $\frac{H^6}{480}$ and derived a Numerov type method of algebraic order four with minimal phase-lag $\frac{H^6}{12096}$ and having an periodicity interval (0, 2.71). This method is implicit and its implementation involves the computations of Jacobians and solution of nonlinear systems of equations. So subsequently many authors proposed explicit modifications of Numerov method.

3. The new nonlinear hybrid Obrechkoff methods

First of all, we introduce the our new hybrid Obrechkoff method, for solving second order initial value problem of ordinary differential equations (1) as follow

$$y_{n+1} = 2y_n \exp\left(\frac{1}{2y_n}\left(h^2 f_n + h^4 b(f''_{n+\alpha} + f''_{n-\alpha})\right)\right) - y_{n-1}, \tag{9}$$

where h is the step length and α, b are two arbitrary parameters such that $0 < \alpha < 1$ and presume $y_n \neq 0$. The above formula (9) can only be used if we know the values of the solution $y(x)$ and $y''(x)$ at two successive points. These two values will be assumed to be given. Further, this method is called an explicit or predictor formula because y_{n+1} occurs only on the left hand side of the formula. In other words, y_{n+1} can be calculated directly from the right hand side values.

Now with the difference equation (9), we can associate the difference operator L defined by next definition.

Definition 3.1. *Let the differential equation* (1) *has unique solution* $y(x)$ *on* $[a, b]$ *and that* $y(x) \in C^{(p+2)}[a, b]$ *for* $p \geq 1$. *Then the deference operator* L *for the method of* (9) *is can be written as*

$$\begin{aligned} L[y(x), h] &= y(x+h) + y(x-h) - \\ &- 2y(x) \exp\left(\frac{1}{2y(x)}\left(h^2 y''(x) + h^4 b\big(y^{(4)}(x+\alpha h) + y^{(4)}(x-\alpha h)\big)\right)\right). \end{aligned}$$

Expanding $y(x+h)$, $y(x-h)$, $y^{(4)}(x+\alpha h)$, $y^{(4)}(x-\alpha h)$ and exp(..) in Taylor's series, by simple calculation, we have

$$\begin{aligned} L[y(x), h] &= \left(\frac{1}{12} - 2b\right) h^4 y^{(4)}(x_n) + \left(\frac{1}{360} - b\alpha^2\right) h^6 y^{(6)}(x_n) + \\ &+ \left(\frac{1}{20160} - \frac{1}{12} b\alpha^4\right) h^8 y^{(8)}(x_n) + O(h^{10}). \end{aligned} \tag{10}$$

Then we get

$$L[y(x), h] = C_0 y(x_n) + C_1 h y^{(1)}(x_n) + \cdots + C_p h^q y^{(p)}(x_n) + \cdots,$$

where $C_0 = C_1 = C_2 = C_3 = 0$, $C_4 = \frac{1}{12} - 2b$, $C_5 = 0$, $C_6 = \frac{1}{360} - b\alpha^2$, $C_7 = 0$ and $C_8 = \frac{1}{20160} - \frac{1}{12} b\alpha^4$ and etc.

Definition 3.2. *The hybrid Obrechkoff method* (9) *are said to be of order* p *if,*

$$C_0 = C_1 = C_2 = \cdots = C_{p+1} = 0 \quad , \quad C_{p+2} \neq 0,$$

thus for any function $y(x) \in C^{(p+2)}$ *and for some nonzero constant* C_{p+1}, *we have*

$$L[y(x), h] = C_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3}), \tag{11}$$

where C_{p+2} is called the error constant.

In particular, $L[y(x), h]$ vanishes identically when $y(x)$ is polynomial whose degree is less than or equal to p.

Applying (9) to the scalar test equation (3), one gets its characteristic equation $C(H)$, where $H^2 = (\omega h)^2$ and $C(H) = \exp(-\frac{1}{2}H^2 + bH^4 - \frac{1}{2}b\alpha^2H^6)$.

Theorem 3.1. *The method presented in* (9)*, for all* $\sqrt{b} < \alpha < 1$ *is P-stable.*

Proof. In order to prove the above theorem we must provide conditions such that $\mid C(H) \mid < 1$ for every H^2. Therefore we discuss for behavior of $-\frac{1}{2}H^2 + bH^4 - \frac{1}{2}b\alpha^2H^6 < 0$, considering α and b. That is, the limitation of α and b should be calculated in a way that P-stable is warranted. Then we have

$$-\frac{1}{2}H^2 + bH^4 - \frac{1}{2}b\alpha^2H^6 = H^2\bigg(-\frac{1}{2} + bH^2 - \frac{1}{2}b\alpha^2H^4\bigg) < 0, \tag{12}$$

so

$$-\frac{1}{2} + bH^2 - \frac{1}{2}b\alpha^2H^4 < 0.$$

For this propose by assuming $H^2 = x$, the coefficient of x^2 and Δ from quadratic polynomial $\varphi(x) = -\frac{1}{2} + bx - \frac{1}{2}b\alpha^2x^2$ are negative. Then we have $-\frac{1}{2} + bx - \frac{1}{2}b\alpha^2x^2 < 0$, so then $-\frac{1}{2}b\alpha^2 < 0$, and $\Delta < 0$, where $\Delta = b^2 - b\alpha^2 < 0$ and this means that $\alpha^2 > b$ for all $0 < \alpha < 1$ and $-\frac{1}{2}b\alpha^2 < 0$. So $b > 0$ and then $\alpha > \sqrt{b}$. Therefore $\sqrt{b} < \alpha < 1$ and will have $-\frac{1}{2} + bx - \frac{1}{2}b\alpha^2x^2 < 0$. That is say $\mid C(H) \mid < 1$, which warranties the P-stability of the method and completes the proof. □

Theorem 3.2. *Method* (9) *is of order* 4 *if* $b = \frac{1}{24}$ *and* $\alpha \neq \sqrt{\frac{24}{360}}$ *and it is of order* 6 *if* $b = \frac{1}{24}$ *and* $\alpha = \sqrt{\frac{24}{360}}$.

Proof. Since $C_0 = C_1 = C_2 = C_3 = C_5 = 0$ and $C_4 = \frac{1}{12} - 2b$, if we take $b = \frac{1}{24}$ then $C_4 = 0$ and this means that our new method is of order at least 4. Furthermore, since $C_7 = 0$ and by assuming $b = \frac{1}{24}$, the amount of $\alpha = \sqrt{\frac{24}{360}}$ is the only root of C_6. Then if $\alpha \neq \sqrt{\frac{24}{360}}$, the order of the new hybrid Obrechkoff method is exactly 4 and if $\alpha = \sqrt{\frac{24}{360}}$, the order is 6 and in this case the local truncation error is

$$E_8 = 3.4171 \times 10^{-5}h^8y^{(8)}(\zeta),$$

and this completes the proof. □

Lemma 3.1. *The Phase-lag of method* (10) *is of order two.*

Proof. By a simple calculation, the proof is clear. □

By choosing $b = \frac{1}{24}$, we can write the new two-step P-stable hybrid Obrechkoff method as follow

$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{1}{2y_n}\left(h^2f_n + \frac{h^4}{24}(f''_{n+\alpha} + f''_{n-\alpha})\right)\right), \quad \frac{\sqrt{6}}{12} < \alpha < 1. \tag{13}$$

If we take $\alpha = \frac{1}{2}$, we have

$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{1}{2y_n}\left(h^2f_n + \frac{h^4}{24}\left(f''_{n+\frac{1}{2}} + f''_{n-\frac{1}{2}}\right)\right)\right), \tag{14}$$

is the explicit two-step P-stable hybrid Obrechkoff method of order 4. Moreover its local truncation error is

$$E_6 = -\frac{11}{1440} h^6 y^{(6)}(\zeta).$$

If we take $\alpha = \sqrt{\frac{24}{360}}$, we have

$$y_{n+1} + y_{n-1} = 2y_n \exp\left(\frac{1}{2y_n}\left(h^2 f_n + \frac{h^4}{24}\left(f''_{n+\sqrt{\frac{24}{360}}} + f''_{n-\sqrt{\frac{24}{360}}}\right)\right)\right), \tag{15}$$

is the explicit two-step P-stable hybrid Obrechkoff method of order 6. Moreover its local truncation error is

$$E_8 = 3.4171 \times 10^{-5} h^8 y^{(8)}(\zeta).$$

Table 1. Absolute errors for the example 4.1, with $h = \pi/200$ and $h = \pi/400$, are calculated for comparison among two methods: Li and Wu [11] and our new method (14).

	New method (14)		Wu's method	
Point	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$
5π	2.3659e-004	1.2514e-006	5.1541e-002	3.2539e-003
10π	5.1547e-004	3.2547e-006	2.0104e-001	1.3001e-002
15π	6.2689e-004	7.2546e-006	4.3307e-001	2.9117e-002
20π	8.3654e-004	9.8541e-006	7.2365e-001	5.1678e-002

4. Numerical examples

In this section, we present some numerical results obtained by our new nonlinear methods and compare them with those of other multistep methods.

Example 4.1. *Consider the scalar test equation $y'' = -\omega^2 y$, $y(0) = 1$ and $y'(0) = 0$, with the exact solution $y = \cos(\omega x)$.*

Set $\omega = 10$. Absolute errors in $y(x)$, with $h = \pi/200$, $\pi/400$, $\pi/800$ and $\pi/1600$, obtained by the new method (14), are listed in Tables 1 and 2 for comparison, where the other numerical results are from Li and Wu [11].

Example 4.2. *Consider the initial value problem $y'' = 50y^3$, $y(1) = 1/6$ and $y'(1) = -5/36$, with the exact solution $y(x) = 1/(1 + 5x)$.*

In the numerical experiment, we take the step length $h = 0.1$, 0.01, 0.001, and for simplicity, the true value at $x = 1 + h$ is taken as the second starting value. In Tables 3 and 4, we present the absolute errors at the points $x = 5$, 10, 15, 20.

Example 4.3. *Consider the two-body problem*

$$\begin{cases} y_1'' = -\frac{y_1}{r^3}, & y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2'' = -\frac{y_2}{r^3}, & y_2(0) = 0, \quad y_2'(0) = 1, \end{cases}$$

where $r = \sqrt{y_1^2 + y_2^2}$.

TABLE 2. Absolute errors for the example 4.1, with $h = \pi/800$ and $h = \pi/1600$, are calculated for comparison among two methods: Li and Wu [11] and our new method (14).

	New Method (14)		Wu's Method	
Point	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$
5π	1.6532e-007	2.5567e-09	2.0362e-004	1.2731e-005
10π	3.1128e-007	5.2398e-09	8.1460e-004	5.0928e-005
15π	6.8875e-007	6.9971e-08	1.8327e-003	1.1459e-004
20π	9.5563e-006	9.2013e-08	3.2575e-003	2.0372e-004

TABLE 3. Absolute errors for the example 4.2, with $h = 0.1$, $h = 0.01$ and $h = 0.001$, are calculated for comparison among two methods: Li and Wu [11] and our new method (14).

	Method (14)		
x	$h = 0.1$	$h = 0.01$	$h = 0.001$
5	1.1325e-005	2.7745e-007	2.0025e-010
10	4.2034e-005	4.0102e-007	5.8625e-010
15	6.1478e-004	5.1436e-007	6.0369e-009
20	9.0336e-004	8.1129e-006	8.4412e-009

TABLE 4. Absolute errors for the example 4.2, with $h = 0.1$, $h = 0.01$ and $h = 0.001$, are calculated for comparison among two methods: Li and Wu [11] and our new method (14).

	Wu's Method		
x	$h = 0.1$	$h = 0.01$	$h = 0.001$
5	2.4119e-003	3.0515e-005	3.1215e-007
10	1.6102e-002	2.3060e-004	2.3630e-006
15	4.0043e-002	7.5739e-004	7.8201e-006
20	1.5401e-001	1.7428e-003	1.8350e-005

The true solution is $y_1 = \cos(x)$, $y_2 = \sin(x)$. In the numerical experiment, we take the step length $h = \pi/200$, $\pi/400$, $\pi/800$, $\pi/1600$, and for simplicity, the true value at $x = h$ is taken as the second started value. In tables 5 and 6, we gives the errors (infinite norm) at point $x = 50\pi$, 100π, 150π, 200π.

TABLE 5. Absolute errors for the example 4.3, with $h = \pi/200$ and $h = \pi/400$ are calculated for comparison among two methods: Li and Wu [11] and our new method (14).

	New Method (14)		Wu's Method	
Point	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$	$h = \frac{\pi}{200}$	$h = \frac{\pi}{400}$
$x = 50\pi$	2.3214e-5	1.2987e-7	1.2898e-002	3.3073e-003
$x = 100\pi$	4.2564e-5	3.2769e-7	2.5789e-002	4.4419e-003
$x = 150\pi$	7.1239e-5	6.1478e-7	3.8678e-002	9.5241e-003
$x = 200\pi$	9.2036e-4	8.0036e-6	5.1661e-002	1.2743e-002

TABLE 6. Absolute errors for the example 4.3, with $h = \pi/800$ and $h = \pi/1600$ are calculated for comparison among two methods: Li and Wu [11] and our new method (14).

	New Method (14)		Wu's Method	
Point	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$	$h = \frac{\pi}{800}$	$h = \frac{\pi}{1600}$
$x = 50\pi$	1.2158e-9	2.9817e-11	5.1930e-004	3.3093e-004
$x = 100\pi$	3.6987e-9	4.0065e-11	1.1391e-003	1.4533e-004
$x = 150\pi$	6.3104e-9	6.3691e-11	1.7547e-003	4.6950e-004
$x = 200\pi$	9.9812e-8	8.0651e-10	2.6830e-003	3.9078e-004

5. Conclusions

In this paper, we have presented the new symmetric P-stable nonlinear hybrid Obrechkoff method of orders 4 and 6. The details of the procedure adapted for the applications have been given in Section 3. With high derivatives in off-step points, we have improved the algebraic order of this method up to six. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.

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