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EXPANSION APPROACH FOR SOLVING NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we develop and modify Taylor-series expansion method to approximate a solution of nonlinear Volterra integro-differential equations (IDEs) as well as a solution of a system of nonlinear Volterra equations. By means of the n^{th}-order Taylor-series expansion of an unknown function at an arbitrary point, a nonlinear Volterra equations can be converted approximately to a system of nonlinear equations for the unknown function itself and first n derivatives. Proposed method enables us to control truncation error by adjusting the step size used in the numerical scheme. The nth-order approximate solution is exact for a polynomial solution of degree equal to or less than n. Finally, error estimation of the proposed method is presented. Some numerical examples are provided to illustrate the accuracy of the method.

Keywords: integro-differential equations, approximate solution, nonlinear Volterra equations, error estimation, Taylor-series expansion method.

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1. Introduction

We consider the following Volterra IDEs of the form

$$D(x,y(x),y^{(1)}(x),...,y^{(\alpha)}(x)) - \lambda \int_0^x \varphi(x,t,y(t),y^{(1)}(t),...,y^{(\beta)}(t))\mathrm{d}t = f(x),\ \ x,t \in \Gamma = [0,b], \tag{1}$$

with the initial conditions

$$\sum_{j=0}^{\alpha-1} B_{ij} y^{(j)}(0) = c_i,\ \ i = 1,2,...,\alpha, \tag{2}$$

where D and φ are in the following forms

$$D(x,y(x),y^{(1)}(x),...,y^{(\alpha)}(x)) = \sum_{i=0}^{\mu_1}(p_i(x)\prod_{j=0}^{\alpha}(y^{(j)}(x))^{\alpha_{ij}}),$$

$$\varphi(x,t,y(t),y^{(1)}(t),...,y^{(\beta)}(t)) = \sum_{i=0}^{\mu_2}(k_i(x,t)\prod_{j=0}^{\beta}(y^{(j)}(t))^{\beta_{ij}}),$$

and $\alpha_{ij},\ \beta_{ij} \in \mathbb{N} \cup \{0\}$.

Also assumed that the functions $k_i(x,t),\ p_i(x)$ and $f(x)$ are polynomials, otherwise they can be approximate by theirs truncated Taylor-series expansion respect to all arguments at origin.

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Problems involving these equation arise frequently in many branch of applied science which include engineering mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc, [2, 17, 23, 27, 26, 28, 29]. Since, the analytical solutions for (1)-(2) can not be obtained always, due to this we have to use the approximation solution. In recent years, several researcher attempt to solve linear or nonlinear Volterra IDEs. Volterra integral and IDEs of convolution type have been solved in [4], by using operational matrices of piecewise constant functions. A modified homotopy perturbation method is used to solve nonlinear integral equations in [11]. The application of Sinc collection method for solution of nonlinear Volterra IDEs has been used in [31]. A combination of Laplace transform with the Adomian decomposition method has been used for solves nonlinear Volterra IDEs in [24]. The differential transformation method is used for solves (1)-(2) in Ref. [18]. The variational iteration method has been proposed by [25] for solution of Volterra IDEs. Chebyshev cardinal functions have been used to approximate the solution of the fourth-order IDEs in [14]. Ham is used to solve the high-order nonlinear Volterra and Fredholm integro-differential problems in [21]. Taylor-series expansion method is a powerful technique for solving above problems. The Taylor-series expansion approach for solving Volterra integral equations has been presented by Kanwal and Liu [13] and then this method has been extended to Volterra integral equations and differential equations by Sezer [19, 20]. A similar approach has been used to solve linear Volterra-Fredholm IDEs in [30]. In [12, 15], Taylor-series expansion approach is used to Abel integral equation. In this study, we will develop and modify Taylor-series expansion method [22] to solve nonlinear Volterra IDEs (1)-(2).

In the Taylor-series expansion method, the solution of (1)-(2) can be consider in the following form

$$y(x) \approx \sum_{j=0}^{N} e_j x^j, \tag{3}$$

where $e_j = \frac{y^{(j)}(0)}{j!}$ are known from (2) for $j = 0, 1, ..., \alpha - 1$ and for $j = \alpha,\ \alpha + 1, ..., N$, are unknown parameters which have to be determine. Substituting (3) into (1), we can obtain

$$\sum_{i=0}^{N-\alpha} (\Psi_i - \xi_i) x^i + Q(x^\tau) = 0,$$

where $Q(x^\tau)$ is a polynomial of degree greater than $N - \alpha$, Ψ_i is a nonlinear combination of $e_\alpha, e_{\alpha+1}, ..., e_N$, and ξ_i is known constant. In the above relation we need to determine the $N-\alpha$ unknown parameters which can be obtained by using recursive relations and equating to zero the sufficient terms (that is $N - \alpha$ terms). This yielding the following $N - \alpha$ nonlinear algebraic equations

$$\Psi_i = \xi_i,\ i = 0, 1, ..., N - \alpha. \tag{4}$$

By solving above nonlinear system via the powerfull Gröbner basis method, which will be introduced in the next section, we obtain the unknown parameters $e_j,\ j = \alpha, ..., N$.

2. Solving nonlinear algebraic system using Gröbner basis

In this section we define the basic concepts of Gröbner basis, the details can be found in [8]. The method of Gröbner bases, allows us to solve systems of polynomial equations in an algorithmic fashion. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, x_2, \cdots, x_n]$ be the ring of all polynomials in $x_1, x_2, \cdots, x_n$, with coefficients in $\mathbb{K}$. A power product $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in R$ with nonnegative exponents, is called a term. To define Groebner basis, we first have to fix a term order. Our first example of a term order on T will be *lexicographical order*(or *Lex order* for short). In following definition, we suppose $x_1 > x_2 > \ldots > x_n$.

Definition 2.1. *Given two distinct terms $s = x_1^{i_1} \ldots x_n^{i_n}$ and $t = x_1^{j_1} \ldots x_n^{j_n}$ in T, we say that $s \prec_{lex} t$ if the first non-zero coordinate, from the left, of the vector $(j_1 - i_1, \ldots, j_n - i_n)$ is positive.*

Suppose a term order has been fixed. For any non-zero polynomial f we denote the the maximum term of f by $LT(f)$, the coefficient of $LT(f)$ by $LC(f)$. Now we can define a Gröbner basis of an ideal in R.

Definition 2.2. *A* Gröbner basis *of an ideal I in the polynomial ring R is a finite set of generators $G = \{f_1, \ldots, f_m\}$ for I whose leading terms generate the ideal of all leading terms in I, i.e. for all $f \in I$ there exists a polynomial $f_i \in G$ such that $LT(f_i)|LT(f)$. A Gröbner basis G is called a* reduced Gröbner basis *for I if for any $f_i \in G$, $LC(f_i) = 1$ and no term of f_i lies in the ideal generated by $\{LT(f_j)|1 \leq j \neq i \leq m\}$.*

The main properties of Gröbner bases are summarized in the following proposition.

Theorem 2.1. *Let I be the ideal generated by $\{h_1, \cdots, h_m\}$ in R and $f \in R$. For a given monomial order,*

(1) *I has a unique reduced Groebner basis $G = \{g_1, \cdots, g_s\}$, and we have an algorithm to compute it.*
(2) *the equations $h_1 = 0, \cdots, h_m = 0$ have no solutions in any extending field of $\mathbb{K}$ if and only if $G = \{1\}$.*
(3) *the equations $h_1 = 0, \cdots, h_m = 0$ have a finite number of solutions if and only if for each variable x_i, there is a polynomial $g \in G$ and a nonegative integer j_i such that $LT(g) = x_i^{j_i}$. In this case we say that the* **dimension** *of I is zero, and the number of solutions is at most $j_1 j_2 \cdots, j_n$, in addition if If I is a radical ideal and $\mathbb{K}$ is algebraically closed, then equality holds.*

Proof: See [7].

In 1965, B. Buchberger presented an algorithm to compute the Groebner basis of any given ideal. There are many effort to improve the Buchberger algorithm and find the new methods. But among all available methods for computing the Gröbner basis the algorithms of F_4 and F_5 which have been suggested by J.C. Faugère [9, 10], are more efficient. Many computer algebra systems implement these algorithms. In our computations, the MAPLE computer algebra system is used.

A Gröbner basis for the ideal generated by Eq.(4) in $\mathbb{K}[e_\alpha, e_{\alpha+1}, \cdots, e_N]$ with respect to lex order can be computed using Faugère's Algorithm. In general, the Gröbner basis of a zero dimensional ideal in $\mathbb{K}[e_\alpha, e_{\alpha+1}, \cdots, e_N]$ w.r.t lex order has the upper triangular structure, i.e. G basis has the following form:

$$\begin{cases} g_{1,1}(e_\alpha) \in \mathbb{K}[e_\alpha] \\ g_{2,1}(e_\alpha, e_{\alpha+1}), \ldots, g_{2,p_2}(e_\alpha, e_{\alpha+1}) \in \mathbb{K}[e_\alpha, e_{\alpha+1}] \\ \vdots \\ g_{N-\alpha+1,1}(e_\alpha, \ldots, e_N), \ldots, g_{N-\alpha+1,p_{N-\alpha+1}}(e_\alpha, \ldots, e_N) \in \mathbb{K}[e_\alpha, \ldots, e_N]. \end{cases} \tag{5}$$

So we can find the solutions of Eq.(5) same as the Gaussian elimination method for solving a linear system which has a triangular structure.

3. Modified Taylor-series method

As we know that the conventional Taylor-series approximation has the appropriate accuracy at the closed neighborhood $x = 0$, but for the points far from origin this approximation has not appropriate accuracy, to overcome such difficultly, in this section we modified Taylor-series expansion approximation at $x = h$.

Let $\triangle = \{0 = x_0, x_1, ..., x_n = b\}$, be an equidistance partition of $[0, b]$ where $s = x_{i+1} - x_i,\ i = 0, 1, ..., n-1$ is the discretization parameter of the partition. In modified Taylor series method, we need to prove the following theorem.

Theorem 3.1. *If the known functions $k_i(x,t), p_i(x)$ and $f(x)$ in (1), are sufficiently differentiable on the interval $0 \le x, t \le b$, and also $k_i(x,t)$ is a separable function, then there exist a linear combination of the independent functions $\psi_i(X)$, so that satisfying in*

$$D(X+h, Y(X), Y^{(1)}(X), ..., Y^{(\alpha)}(X)) - \lambda \int_0^X \varphi(X+h, u+h, Y(u), Y^{(1)}(u), ..., Y^{(\beta)}(u))\mathrm{d}u + \quad (6)$$

$$+\sum_{i=0}^{\gamma} c_i \psi_i(X) = f(X+h),$$

where $x = X + h,\ t = u + h$ and $Y(X) = y(X+h)$ is the exact solution.

Proof. For simplicity, the separable kernel $k_i(x,t)$ can be denote by

$$k_i(x,t) = v_i(x) w_i(t),\ i = 0, 1, ..., m. \quad (7)$$

Substitute (7) into (1) and using the change of variables $x = X + h$, $t = u + h$, we have

$$D(X+h, y(X+h), y^{(1)}(X+h), ..., y^{(\alpha)}(X+h)) - \quad (8)$$

$$\lambda \sum_{i=0}^{\mu_2} (v_i(X+h) \int_0^X w_i(u+h) \prod_{j=0}^{\beta} (y^{(j)}(u+h))^{\beta_{ij}} \mathrm{d}u) + \sum_{i=0}^{\mu_2} v_i(X+h) a_i = f(X+h),$$

where

$$a_i = \int_{-h}^{0} (w_i(u+h) \prod_{j=0}^{\beta} (y^{(j)}(u+h))^{\beta_{ij}}) \mathrm{d}u,\ i = 0, 1, ..., \mu_2, \quad (9)$$

are unknown constants.

By simplifying and classifying the last term on left hand side of (8), we can write

$$\sum_{i=0}^{m} v_i(X+h) a_i = \sum_{i=0}^{\gamma} c_i \psi_i(X), \quad (10)$$

where $\psi_i(X)$ are known linear independent functions and c_i are unknown constants. Using (10), and by denoting $Y(X) = y(X+h)$, we can rewrite (8), in the form of (6).

To determine c_i, by calculate ith derivation of relation (6) for $i = 0, 1, ..., \gamma - 1$, and then by setting X=0, the following algebraic system is obtained

$$\begin{cases} \sum\limits_{i=0}^{\gamma} c_i\psi_i(0) = f(h) - \chi(0), \\ \sum\limits_{i=0}^{\gamma} c_i\psi_i'(0) = f'(h) - \chi'(0) + \lambda\phi(0), \\ \vdots \\ \sum\limits_{i=0}^{(\gamma)} c_i\psi_i^{(\gamma-1)}(0) = f^{(\gamma-1)}(h) - \frac{d^{\gamma-1}\chi(X)}{dX^{\gamma-1}} \mid_{X=0} + \lambda\frac{d^{\gamma-2}\phi(X)}{dX^{\gamma-2}} \mid_{X=0}, \end{cases}$$

where

$\chi(X) = D(X + h, Y(X), Y^{(1)}(X), ..., Y^{(\alpha)}(X))$ and $\phi(X) = \varphi(X + h, X + h, Y, Y', ..., Y^{(\beta)})$.

By solving above algebraic system, the unknown constants can be determined as follows

$$c_i = \xi_i(h, y(h), y^{(1)}(h), ..., y^{(\alpha+\gamma-1)}(h)), \ i = 0, 1, ..., \gamma. \tag{11}$$

□

For solving nonlinear Volterra IDEs (1)-(2) by modified Taylor-series expansion method, let the solution of the converted nonlinear Volterra IDEs (6) is as follows

$$Y(X) \approx \sum_{j=0}^{N} \mathbf{e}_j X^j, \tag{12}$$

where $\mathbf{e}_j = \frac{y^{(j)}(h)}{j!}$ are known for $j = 0, 1, ..., \alpha - 1$, and for $j = \alpha, \ \alpha + 1, ..., N$, are unknown parameters which have to be determine. Substituting (12) into (6), we obtain a nonlinear algebraic system. By solving derived system, the unknown parameters (12) is obtained as follows

$$\mathbf{e}_{j+\alpha} = \phi_j(h, c_0, c_1, ..., c_\gamma, y(h), y'(h), ..., y^{(\alpha+j-1)}(h)), \ j = 0, 1, ..., N - \alpha. \tag{13}$$

At the first step, let $h = x_0$, then $c_i = 0, \ i = 0, 1, ..., \gamma$, by (9) and $\mathbf{e}_j = y^{(j)}(0)$ are known by the initial conditions (2) for $j = 0, 1, ..., \alpha - 1.$, therefore by using (13) we obtain

$$y(x) \approx \sum_{j=0}^{N} \frac{y^{(j)}(0)}{j!} x^j, \tag{14}$$

where is Taylor-series expansion approximate solution (1)-(2) at $x_0 = 0$.

At the next step, let $h = x_1$, from (14), we have

$$\mathbf{e}_j = y^{(j)}(x_1), \ j = 0, 1, ..., \alpha + \gamma - 1. \tag{15}$$

Now by using (15) and (11), we obtain the unknown parameters in (13), therefore we have

$$y(x) \approx \sum_{j=0}^{N} \frac{y^{(j)}(x_1)}{j!} (x - x_1)^j, \tag{16}$$

where is Taylor approximate solution (1)-(2) at $x = x_1$.

By repeating the above step for $i = 2, 3, ..., n - 1$, we can obtain Taylor-series expansion approximate (1)-(2) at $h = x_i$.

4. Error estimation

In this section, an error function is obtained for the approximate solution of (1)-(2). Let $e(x) = y(x) - y_N(x)$ denoted as the error function of the conventional Taylor-series approximate $y_N(x)$, to approximate the exact solution $y(x)$. Substituting $y(x) = y_N(x) + e(x)$, in (1)-(2), then we have

$$\sum_{i=0}^{\mu_1} p_i(x) \prod_{j=0}^{\alpha} \left(y_N^{(j)}(x) + e^{(j)}(x)\right)^{\alpha_{ij}} - \lambda \int_0^x \sum_{i=0}^{\mu_2} k_i(x,t) \prod_{j=0}^{\beta} \left(y_N^{(j)}(t) + e^{(j)}(t)\right)^{\beta_{ij}} \mathrm{d}t = f(x), \qquad (17)$$

and

$$e(0) = 0, e'(0) = 0, ..., e^{(\alpha-1)}(0) = 0. \qquad (18)$$

By using

$$(y_N(t) + e(t))^p = \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) (y_N(t))^{p-k} (e(t))^k,$$

in Eq. (17), we obtain

$$\sum_{i=0}^{\mu_1} p_i(x) Q_i(e(x), e^{(1)}(x), ..., e^{(\alpha)}(x), y_N(x), y_N^{(1)}(x), ..., y_N^{(\alpha)}(x)) -$$

$$\lambda \int_0^x \sum_{i=0}^{\mu_2} (k_i(x,t) R_i(e(t), e^{(1)}(t), ..., e^{(\beta)}(t), y_N(t), y_N^{(1)}(t), ..., y_N^{(\beta)}(t)))\mathrm{d}t = f(x), \qquad (19)$$

where

$$Q_i(e(x), e^{(1)}(x), ..., e^{(\alpha)}(x), y_N(x), y_N^{(1)}(x), ..., y_N^{(\alpha)}(x)) = \prod_{j=0}^{\alpha} \varphi_{ij}(x, \alpha_{ij}),$$

$$R_i(e(t), e^{(1)}(t), ..., e^{(\beta)}(t), y_N(t), y_N^{(1)}(t), ..., y_N^{(\beta)}(t)) = \prod_{j=0}^{\beta} \varphi_{ij}(t, \beta_{ij}),$$

and

$$\varphi_{ij}(z, \mu) = \sum_{k=1}^{\mu} \left(\begin{array}{c} \mu \\ k \end{array} \right) (y_N(z))^{\mu-k} (e(z))^k.$$

Now suppose that $k_i(x,t) = v_i(x) w_i(t)$, by using theorem 1. the problem (19) converted to following nonlinear Volterra IDEs

$$\sum_{i=0}^{\mu_1} p_i(X+h) Q_i(E(X), E^{(1)}(X), ..., E^{(\alpha)}(X), Y_N(X), Y_N^{(1)}(X), ..., Y_N^{(\alpha)}(X))$$

$$-\lambda \int_0^X \sum_{i=0}^{\mu_2} (k_i(X+h, u+h) R_i(E(u), E^{(1)}(u), ..., E^{(\beta)}(u), Y_N(u), Y_N^{(1)}(u), ..., Y_N^{(\beta)}(u))\mathrm{d}u + \qquad (20)$$

$$+ \sum_{i=0}^{\gamma} d_i \psi_i(X) = f(X+h),$$

where $E(X) = e(X+h)$. Now, for determining the unknown constants $d_0, ..., d_\gamma$, we need to take ξth derivative of relation (20) for $\xi = 0, ..., \gamma - 1$, and then by setting X=0, we will obtain the following algebraic system

$$\begin{cases} \sum\limits_{i=0}^{\gamma} d_i \psi_i(0) = f(h) - \sum\limits_{i=0}^{\mu_1} \chi_i^{(1)}(0), \\ \sum\limits_{i=0}^{\gamma} d_i \psi_i^{(1)}(0) = f^{(1)}(h) - \sum\limits_{i=0}^{\mu_1} \frac{d\chi_i^{(1)}(0)}{dX} \mid_{X=0} + \lambda \sum\limits_{i=0}^{\mu_2} \chi_i^{(2)}(0), \\ \vdots \\ \sum\limits_{i=0}^{\gamma} d_i \psi_i^{(\gamma-1)}(0) = f^{(\gamma-1)}(h) - \sum\limits_{i=0}^{\mu_1} \frac{d^{\gamma-1}\chi_i^{(1)}(X)}{dX^{\gamma-1}} \mid_{X=0} + \sum\limits_{i=0}^{\mu_2} \frac{d^{\gamma-2}\chi_i^{(2)}(X)}{dX^{\gamma-2}} \mid_{X=0}, \end{cases} \tag{21}$$

where

$$\chi_i^{(1)}(X) = p_i(X+h)Q_i(E(X), E^{(1)}(X), ..., E^{(\alpha)}(X), Y_N(X), Y_N^{(1)}(X), ..., Y_N^{(\alpha)}(X)),$$

$$\chi_i^{(2)}(X) =, k_i(X+h, X+h)R_i(E(X), E^{(1)}(X), ..., E^{(\beta)}(X), Y_N(X), Y_N^{(1)}(X), ..., Y_N^{(\beta)}(X)).$$

By solving the above algebraic system, the unknown constants can be determined as follows

$$d_i = \zeta_i(h, e(h), e^{(1)}(h), ..., e^{(\alpha+\gamma-1)}(h), y(h), y^{(1)}(h), ..., y^{(\alpha+\gamma-1)}(h)), \; i = 0, 1, ..., \gamma. \tag{22}$$

The nonlinear Volterra IDEs (20), can be solved by modified Taylor-series expansion method introduced in section 2., therefore Taylor-series expansion of the error function $e(x)$ can be determined at $x = x_i, \; i = 0, 1, ..., n-1$.

5. Solution system of nonlinear Volterra IDEs via Taylor-series method

In this section, we apply the modified Taylor-series method to a system of nonlinear Volterra IDEs

$$\sum_{s=1}^{\mu_1} D_{rs}(x, y_1(x), y_1^{(1)}(x), ..., y_1^{(\alpha_{rs1})}(x), ..., y_m(x), y_m^{(1)}(x), ..., y_m^{(\alpha_{rsm})}(x))$$

$$-\lambda_r \sum_{s=1}^{\mu_2} \int_0^x \varphi_{rs}(x, t, y_1(t), y_1^{(1)}(t), ..., y_1^{(\beta_{rs1})}(t), ..., y_m(t), y_m^{(1)}(t), ..., y_m^{(\beta_{rsm})}(t))\mathrm{d}t = f_r(x), \tag{23}$$

$$x, t \in \Gamma = [0, b],$$

for $r = 1, 2, ..., m$, with the initial conditions

$$\sum_{k=1}^{m} \sum_{j=0}^{\alpha_k - 1} B_{ijk} y_k^{(j)}(0) = c_i, \; i = 1, 2, ..., \alpha, \tag{24}$$

where α_k is order of $y_k(x)$, $\alpha = \sum_{k=1}^{m} \alpha_k$ and

$$D_{rs}(x, y_1(x), y_1^{(1)}(x), ..., y_1^{(\alpha_{rs1})}(x), ..., y_m(x), y_m^{(1)}(x), ..., y_m^{(\alpha_{rsm})}(x)) = p_{rs}(x) \prod_{i=1}^{m} (\prod_{j=0}^{\alpha_{rsi}} (y_i^{(j)}(x))^{\alpha_{rsij}}),$$

$$\varphi_{rs}(x, t, y_1(t), y_1^{(1)}(t), ..., y_1^{(\beta_{rs1})}(t), ..., y_m(t), y_m^{(1)}(t), ..., y_m^{(\beta_{rsm})}(t)) = k_{rs}(x, t) \prod_{i=1}^{m} (\prod_{j=0}^{\beta_{rsi}} (y_i^{(j)}(t))^{\beta_{rsij}}).$$

To obtain the Taylor-series approximation for (23)-(24), we need to state the following theorem, which is useful in our discussion.

Theorem 5.1. *Let, the functions $f_i(x), k_{rs}(x,t), p_{rs}(x)$, in (23) are sufficiently differentiable on the interval $0 \leq x, t \leq b$, and $k_{rs}(x,t)$ is a separable function, then there exist linear independent functions $\psi_0(X), ..., \psi_\gamma(X)$ and constant coefficients $c_0, ..., c_\gamma$ such that, $[Y_1(X), Y_2(X), ..., Y_m(X)]^T$ is the exact solution of following nonlinear Volterra IDEs system*

$$\sum_{s=1}^{\mu_1} D_{rs}(X+h, Y_1(X), Y_1^{(1)}(X), ..., Y_1^{(\alpha_{rs1})}(X), ..., Y_m(X), Y_m^{(1)}(X), ..., Y_m^{(\alpha_{rsm})}(X))$$

$$+\sum_{k=0}^{\gamma} \delta_{rk} c_k \varphi_k(X) - \lambda_r \sum_{s=1}^{\mu_2} \int_0^X \varphi_{rs}(X+h, u+h, Y_1(u), Y_1^{(1)}(u), ..., Y_1^{(\beta_{rs1})}(u), ..., \tag{25}$$

$$Y_m(u), Y_m^{(1)}(u), ..., Y_m^{(\beta_{rsm})}(u))\mathrm{d}u = f_r(X+h), x, t \in \Gamma = [0, b],$$

for $r = 1, 2, ..., m$, where $Y_i(X) = y_i(X+h)$, $x = X+h$ and δ_{ij}, is equal 0, 1, or -1.

Proof of this theorem is same the theorem 2.

Here, to obtain the Taylor-series approximation by our proposed method, we suppose that the solution of the converted system of nonlinear Volterra IDEs (25), can be consider as

$$Y_i(X) \approx \sum_{j=0}^{N} \mathbf{e}_{ij} X^j, \ i = 1, 2, ..., m, \tag{26}$$

where for $i = 1, 2, ..., m$, $\mathbf{e}_{ij} = \frac{y_i^{(j)}(h)}{j!}$, $j = 0, 1, ..., \alpha_i - 1$ are known and $\mathbf{e}_{ij}$, $j = \alpha_i, \ \alpha_i + 1, ..., N$ are unknown parameters which have to be determine. Substituting (26) into (25), we can obtain

$$\sum_{j=0}^{N-\sigma} (\Psi_{ij} - \xi_{ij}) x^j + Q(x^{\tau_i}) = 0, \ i = 1, 2, ..., m, \tag{27}$$

where $\sigma = \min \alpha_j, \ j = 1, 2, ..., m$ and $Q(x^{\tau_i})$ is a polynomial of degree greater than $N - \alpha_i$, Ψ_{ij} is a nonlinear combination of $\mathbf{e}_{i(\alpha_i)}, \mathbf{e}_{i(\alpha_i+1)}, ..., \mathbf{e}_{i(\alpha_i+j)}$ for $i = 1, 2, ..., m$, and ξ_{ij} is known constant. In the above relation we need to determine the $N - \alpha$ unknown parameters which can be obtained by using recursive relations and equating to zero the sufficient terms (that is $N - \alpha$ terms). This yielding the following $N - \alpha$ nonlinear algebraic equations

$$\begin{cases} \Psi_{10} = \xi_{10}, \\ \vdots \\ \Psi_{m0} = \xi_{m0}, \\ \vdots \\ \Psi_{1(N-\sigma)} = \xi_{1(N-\sigma)}, \\ \vdots \\ \Psi_{m(N-\sigma)} = \xi_{m(N-\sigma)}. \end{cases} \tag{28}$$

Solving above system by using the Gröbner algorithms introduced in section 2. we can obtain, the unknown parameters $\mathbf{e}_{ij}, \ j = \alpha_i, \ \alpha_i + 1, ..., N, i = 1, 2, ..., m$, in (26)which are with respect to $h$andc_k.

In consequently, we can obtain the Taylor-series expansion approximate (23)-(24) at $x = x_i, i = 0, 1, ..., n-1$ by similar procedure which introduced in the section 3. Also we can obtain the error estimate of our method by similar manner which introduced in section 4.

6. Application

In order to illustrate the performance and accuracy of the proposed methods in the solution of nonlinear Volterra IDEs and also system of nonlinear IDEs, we consider the following Examples.
Example 1. Consider the following Volterra IDEs

$$y'''(x) + x^2y'(x) + \sin(x)y(x) = f(x) + \int_0^x (\sin(x)y(t) + (2x+t)y'(t) + (t^2 - x)y''(t))dt, \quad (29)$$

$$x, t \in \Gamma = [0, 5],$$

under the initial conditions $y(0) = y'(0) = y''(0) = 1$, that $f(x) = x + 1 + \sin(x)$, and $y(x) = e^x$ is the exact solution.

By using theorem 2. nonlinear Volterra IDEs (29) converts into the following form

$$\begin{aligned} Y'''(X) + (X+h)^2Y'(X) + \sin(X+h)Y(X) &= f(X+h) + c_0\sin(X+h) + c_1X + c_2 \\ &+ \int_0^X \big(\sin(X+h)Y(u) + \big(2X+u+h\big)Y'(u) + \big((u+h)^2 - (X+h)\big)Y''(u)\big)\mathrm{d}u, \end{aligned} \quad (30)$$

where c_0, c_1, c_2 are as follows

$$\begin{cases} c_0 = \csc(h)(-\sin(h) - (2\cos(h) + \sin(h))y(h) + (-3 + 2\cos(h) \\ \quad - \sin(h))y'(h) + (2 - h + \sin(h))y''(h) + hy^{(3)}(h) + y^{(5)}(h)), \\ c_1 = -1 - \cos(h) - c_1\cos(h) + (\cos(h) - \sin(h))y(h) \\ \quad +(-h + \sin(h))y'(h) + hy''(h) + y^{(4)}(h), \\ c_2 = -1 - h - \sin(h) - c_1\sin(h)) + \sin(h)y(h) + h^2y'(h) + y^{(3)}(h). \end{cases} \quad (31)$$

First of all, we expand $\sin(X + h)$ with Taylor's expansion at $X = 0$ in (30). In the same procedure given in the section (2),(4) for $N = 15, s = 0.1$, we obtain the approximate solution and error estimation in the interval $[0, 10]$. The numerical results are given in Table 1.

Table 1. Results for Example 1.

x_i	Absolute error	Error estimation
1.0	5.424823E-21	5.424821E-21
2.0	4.939478E-19	4.939475E-19
3.0	5.945844E-18	5.945841E-18
4.0	3.176782E-17	3.176780E-17
5.0	1.134140E-16	1.134140E-16
6.0	3.540182E-16	3.540180E-16
7.0	1.083947E-15	1.083946E-15
8.0	3.311980E-15	3.311988E-15
9.0	1.082753E-14	1.082753E-14
10	3.463834E-14	3.463832E-14

Results show that our modified Taylor-series methods and theirs error estimation can be used for broad intervals.
Example 2. Consider the following nonlinear Volterra IDEs [16, 5, 6]

$$y'(x) + \int_0^x 3\cos(x-t)y^2(t)\mathrm{d}t = 2\sin x\cos x, \quad (32)$$

with the initial condition $y(0) = 1$, and the exact solution $y(x) = \cos x$.

By using theorem 2., we can convert (32) to the following nonlinear Volterra IDEs

$$Y'(X) + c_0 \cos X + c_1 \sin X + \int_0^X 3\cos(X-u)Y^2(u)\mathrm{d}u = 2\sin(X+h)\cos(X+h), \tag{33}$$

where

$$\begin{cases} c_0 = 2\sin(h)\cos(h) - y'(h), \\ c_1 = 3\cos(2h) - y''(h) - 3y^2(h). \end{cases} \tag{34}$$

Table 2. shows exact error for some points using our method for $s = 0.1$, $N = 8$ and compare the results with Adomian's method, BPFs method, and also by method given in [6]. Table 2. show that our results are considerable accurate.

Table 2.Results for Example 2.

x_i	our method	Adomian's method	BPFs method(m=16)	method in [6]
0.1	0	5.32E-15	1.37E-4	1.16E-3
0.2	2.00E-15	2.36E-4	4.28E-3	1.57E-3
0.3	2.01E-1	3.49E-4	5.05E-3	1.23E-3
0.4	8.78E-15	1.04E-4	2.62E-3	2.20E-4
0.5	2.55E-13	5.35E-4	1.54E-2	1.34E-3
0.6	5.82E-13	2.74E-3	3.63E-3	7.34E-4
0.7	1.14E-12	9.51E-3	1.19E-2	7.66E-4
0.8	2.01E-12	2.90E-2	1.37E-2	1.30E-3
0.9	2.28E-12	7.44E-2	4.38E-3	2.18E-3
1.0	5.03E-12	1.76E-1	2.66E-2	2.19E-3

Example 3. Consider the following nonlinear system of IDE's

$$\begin{cases} y_1''(x) + \frac{1}{2}y_2^2(x) - \frac{1}{2}\int_0^x (y_1^2(t) + y_2^2(t))dt = 1 - \frac{1}{3}x^3, \\ y_2''(x) + xy_1(x) - \frac{1}{4}\int_0^x (y_1^2(t) - y_2^2(t))dt = -1 + x^2, \end{cases} \tag{35}$$

with the initial conditions $y_1(0) = 1, y_1'(0) = 2, y_2(0) = -1, y_2'(0) = 0$. The exact solution of this problem is given in Ref.[1] as $(y_1(x), y_2(x)) = (x + e^x, x - e^x)$. Now, by using Theorem 3., we can convert system (35) into the following nonlinear Volterra IDEs system,

$$\begin{cases} Y_1''(X) + \frac{1}{2}Y_2^2(X) + c_1 - \frac{1}{2}\int_0^X (Y_1^2(u) + Y_2^2(u))dt = 1 - \frac{1}{3}(X+h)^3, \\ Y_2''(X) + (X+h)Y_1(X) + c_2 - \frac{1}{4}\int_0^X (Y_1^2(u) - y_2^2(u))dt = -1 + (X+h)^2, \end{cases} \tag{36}$$

where

$$\begin{cases} c_0 = 1 - \frac{1}{3}h^3 - y_1''(h) - \frac{1}{2}y_2'^2(h), \\ c_1 = -1 + h^2 - hy_1(h) - y_2''(h). \end{cases} \tag{37}$$

By using proposed methods, we obtain the approximate solution and error estimation in the interval $[0,1]$. The numerical results are given in the Table 3. for $s=0.1$, and $N=9$.

Table 3. Results for Example 3.

x_i	$e(y_1(x_i))$	$Ee(y_1(x_i))$	$e(y_2(x_i))$	$Ee(y_2(x_i))$
0.1	4.44×10^{-16}	2.53×10^{-19}	2.22×10^{-16}	2.53×10^{-16}
0.2	2.22×10^{-16}	1.71×10^{-17}	2.22×10^{-16}	1.77×10^{-17}
0.3	4.44×10^{-16}	7.94×10^{-17}	2.22×10^{-16}	8.32×10^{-17}
0.4	4.44×10^{-16}	2.15×10^{-16}	4.44×10^{-16}	2.29×10^{-16}
0.5	1.33×10^{-15}	4.47×10^{-16}	6.66×10^{-16}	4.92×10^{-16}
0.6	1.78×10^{-15}	7.89×10^{-16}	1.11×10^{-15}	9.12×10^{-16}
0.7	2.66×10^{-15}	1.24×10^{-15}	1.77×10^{-15}	1.53×10^{-15}
0.8	3.11×10^{-15}	1.75×10^{-15}	2.66×10^{-15}	2.42×10^{-15}
0.9	4.44×10^{-15}	2.24×10^{-15}	3.77×10^{-15}	3.61×10^{-15}
1.0	6.22×10^{-15}	2.54×10^{-15}	5.55×10^{-15}	5.19×10^{-15}

Example 4. Consider the following nonlinear system of IDE's

$$\begin{cases} y_1'''(x) = x - y_1'(x) - \int\limits_0^x (y_1''^2(t) + y_2''(t))dt, \\ \\ y_2'''(x) = \sin x(1+\frac{1}{2}\sin x) + \int\limits_0^x (y_1''(t)y_2(t))dt, \\ \end{cases} \tag{38}$$

with the initial conditions $y_1(0) = y_1''(0) = 0, y_2(0) = 1$ and $y_2(0) = 1, y_2'(0) = 0, y_2''(0) = -1$. The exact solution of this problem is given in Ref.[3] as $(y_1(\chi), y_2(\chi)) = (\sinh(x), \cosh(x))$.
By using theorem 3 we have

$$\begin{cases} Y_1'''(X) = X - Y_1'(X) - \int\limits_0^X (Y_1''^2(u) + Y_2''(u))du, \\ \\ Y_2'''(X) = \sin X(1+\frac{1}{2}\sin X + \int\limits_0^X (Y_1''(u)y_2(u))du, \end{cases} \tag{39}$$

where

$$\begin{cases} c_0 = y_1'''(h) - h + y_1'(h), \\ \\ c_1 = y_2'''(h) - \sin h(1+\frac{1}{2}\sin h). \end{cases}$$

The numerical results in the Table 3. are given for $s=0.1$, and $N=10$ is obtain.

Table 4. Results for Example 4.

x_i	$e(y_1(x_i))$	$Ee(y_1(x_i))$	$e(y_2(x_i))$	$Ee(y_2(x_i))$
0.5	0	1.71×10^{-18}	1.11×10^{-16}	1.64×10^{-19}
1.0	2.22×10^{-16}	2.93×10^{-17}	2.22×10^{-16}	9.71×10^{-19}
1.5	1.11×10^{-16}	1.37×10^{-16}	5.55×10^{-16}	9.81×10^{-18}
2.0	1.11×10^{-16}	4.10×10^{-16}	9.44×10^{-16}	3.57×10^{-17}
2.5	9.99×10^{-16}	1.00×10^{-15}	1.55×10^{-15}	6.13×10^{-17}
3.0	2.80×10^{-15}	2.16×10^{-15}	1.10×10^{-15}	9.96×10^{-17}
3.5	6.22×10^{-15}	4.17×10^{-15}	2.22×10^{-15}	1.85×10^{-16}
4.0	1.12×10^{-14}	7.23×10^{-15}	9.99×10^{-16}	9.77×10^{-16}
4.5	1.80×10^{-14}	1.13×10^{-14}	2.47×10^{-15}	2.94×10^{-15}
5.0	2.62×10^{-14}	1.60×10^{-14}	9.27×10^{-15}	6.91×10^{-15}

Table 4. shows that proposed method can be used for broad intervals.

7. Conclusion

In this research, Taylor-series method has been developed for approximating the solution of nonlinear Volterra IDEs. The major advantages of the modified Taylor-series method are simplification and easy-to-apply in programming, and applicable to high order of nonlinear Volterra IDEs, moreover the proposed method is a powerful procedure for solving nonlinear Volterra IDEs on broad intervals. The computed results show that our method is much accurate in comparison with Adomian's method, BPFs method, and the given method in [6].

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