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NUMERICAL SOLUTION OF DELAY DIFFERENTIAL EQUATIONS VIA HAAR WAVELETS

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Abstract. In this paper, Haar wavelet benefits are applied to the delay differential equations (DDEs). A discretized form of DDEs at collocation points based on some useful properties of Haar wavelets transforms original problem into a nonlinear algebraic equations. Finally, the numerical experiments are given to demonstrate the conclusions.

Keywords: Delay differential equation, haar wavelet, collocation method, numerical solution.

AMS Subject Classification: 34K28, 42C05, 65L03.

1. Introduction

Delay differential equations (DDEs) arise in many areas of mathematical modelling, as physiological and pharmaceutical kinetics, chemical kinetics, population dynamics, the navigational control of ships and aircraft, infectious diseases, and more general control problems. Given the importance of applications of DDEs, many researchers have addresssed this issue [10, 4, 12, 7]. Since analytical solutions of the DDEs may be obtained only in very restricted cases, many methods have been proposed for numerical approximation of them. Some of these techniques include Hermite interpolation [13], θ-methods [11], Runge-Kutta methods [1, 9], Parallel continuous Runge-Kutta methods [2], Spline collocation methods [5], Collocation methods [6], linear multistep methods [9].

Recently, Haar wavelets as a useful mathematical tool, have been applied extensively for signal processing in communications, physics researches and optimal control problems [3]. Haar wavelets have the simplest orthogonal series with compact support and this characteristic introduces Haar wavelets as a good candidate for application to differential equations.
In this paper, using the benefits of Haar wavelets, a novel collocation approach is introduced to find approximate solutions of the following delay differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), x(t-\tau_2), \cdots, x(t-\tau_s)\big)\ 0 < t \leq t_f,\ x(t) = \xi(t)\ t \leq 0, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}^{(s+1)n+1} \to \mathbb{R}^n$ is a nonlinear smooth function and $\tau_i > 0$, $i = 1, \cdots, s$ are constant delays. In the next sections, first Haar wavelets and their properties are introduced. Then the approximation of a function by Haar wavelets is discussed. By introducing operational integration matrix and delay operational matrix, a discretization method is established. Finally, by some numerical examples the proficiency of the given approach is examined.

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2. Haar wavelets and its properties

The orthogonal set of Haar wavelets $\phi_i(t)$ is a group of square waves with magnitude +1 or -1 in some intervals and zeros elsewhere,

$$\phi_0(t) = 1,\ 0 \le t < 1, \tag{2}$$

$$\phi_1(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le t < 1, \end{cases} \tag{3}$$

$$\phi_i(t) = \phi_1(2^j t - k) = \begin{cases} 1 & \text{if } \frac{k}{2^j} \le t < \frac{k}{2^j} + \frac{1}{2^{j+1}}, \\ -1 & \text{if } \frac{k}{2^j} + \frac{1}{2^{j+1}} \le t < \frac{k}{2^j} + \frac{1}{2^j}, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

$$i = 2^j + k,\ j = 0, \cdots, M,\ k = 0, \cdots, 2^j - 1,$$

integer $m = 2^j, (j = 0, 1, \cdots, J)$, indicates the level of the wavelet, $k = 0, 1, \cdots, m-1$ is the translation parameter, maximal level of resolution is J and the maximal value of i is $M = 2^{J+1}$. A simple calculation shows that

$$\int_0^1 \phi_i(t)\phi_l(t)dt = \begin{cases} \frac{1}{M} & \text{if } i = l = 2^j + k, \\ 0 & \text{if } i \ne l. \end{cases} \tag{5}$$

Consequently, the functions $\phi_i(t)$ are orthogonal. This allows us to transform any function square integrable on the interval time [0,1] into Haar wavelets series.

2.1. **Function approximation.** We just pointed out that a square integrable function can be expressed in terms of Haar orthogonal basis on interval $\ell \in [0, 1]$. However, before the procession to this transfer, it is necessary to unify the time interval. Using a linear transformation, the actual time t can be expressed as a function of ℓ via $t = [(t_f - t_0)\ell + t_0]$, where t_0 is the initial time and t_f is the final time in a square integrable function $f(t)$.
Any function $f(t)$ which is square integrable in the interval $[0, 1]$ can be expanded in a Haar series with an infinite number of terms

$$f(t) = \sum_{i=0}^{\infty} a_i \phi_i(t),\ i = 2^j + k,\ j \ge 0,\ 0 \le k < 2^j,\ t \in [0, 1), \tag{6}$$

where the Haar coefficients

$$a_i = 2^j \int_0^1 f(t)\phi_i(t)dt, \tag{7}$$

are determined in such a way that the integral square error

$$\varepsilon = \int_0^1 (f(t) - \sum_{i=1}^{M-1} a_i \phi_i(t))^2 dt \tag{8}$$

is minimum. Here ε is vanished when M tends to infinity. Usually, the series expansion of (5) contains an infinite number of terms for smooth $f(t)$. If $f(t)$ is a piece wise constant or may be approximated as a piecewise constant, then the summation (6) will be terminated after M terms, that is,

$$f(t) \approx \sum_{i=0}^{M-1} a_i \phi_i(t) = A^T \Psi_M(t), \tag{9}$$

where the coefficient vector $A = [a_0, a_1, \cdots, a_{M-1}]^T$ and $\Psi_M(t) = [\phi_0, \phi_1, \cdots, \phi_{M-1}]^T$.
Let us define the collocation points $t_k = (k - 0.5)/M$, $(k = 1, \cdots, M)$. With these chosen

collocation points, the function is discretized into a series of nodes with equivalent distances. Let the Haar matrix $H_{M\times M}$ be the combination of $\Psi_M(t)$ at all the collocation points. Thus,

$$H_{M\times M}=[\Psi_M(t_0),\cdots,\Psi_M(t_{M-1})]=\begin{bmatrix}\phi_0(t_0) & \phi_0(t_1) & \dots & \phi_0(t_{M-1})\\ \phi_1(t_0) & \phi_1(t_1) & \dots & \phi_1(t_{M-1})\\ \vdots & \vdots & \ddots & \vdots\\ \phi_{M-1}(t_0) & \phi_{M-1}(t_1) & \dots & \phi_{M-1}(t_{M-1})\end{bmatrix}_{M\times M}. \tag{10}$$

For example,

$$H_{2\times 2}=\begin{bmatrix}\Psi_2(t_0) & \Psi_2(t_1)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}_{2\times 2}.$$

Therefore, the function $f(t)$ may be approximated as

$$f(t_k)\approx c^T_{1\times M}H_{M\times M}, \tag{11}$$

where $c^T_{1\times M}=[c_1,c_2,\cdots,c_M]$ are the wavelet coefficients. The integration of the vector $\Psi_M(t)$ defined in (9) can be approximated by

$$\int_0^t \Psi_M(t^{'})dt^{'}=P_{M\times M}\Psi_M(t), \tag{12}$$

where $P_{M\times M}$ is $M\times M$ operational integration matrix which satisfies the following recursive formula [8],

$$P_{M\times M}=\begin{bmatrix}P_{M/2\times M/2} & \frac{-1}{2M}H_{M/2\times M/2}\\ \frac{1}{2M}H^{-1}_{M/2\times M/2} & 0\end{bmatrix},\ P_{1\times 1}=[\frac{1}{2}], \tag{13}$$

where $0_{(M/2)\times(M/2)}$ is a null matrix of order$(M/2)\times(M/2)$.

2.2. **The delay operational matrix.** The delay function $\Psi_M(t-\tau)$ is the shift of $\Psi_M(t)$ in (9) along the time axis by τ and to estimate them respect to $\Psi_M(t)$ the delay operational matrix $D(\tau)$ is defined as

$$\Psi_M(t-\tau)\approx D(\tau)\Psi_M(t),\ t>\tau,\ 0\le t<1, \tag{14}$$

where $\tau\in[0,1]$ is the delay constant. Here is shown how to generate the delay operational matrix $D(\tau)=[d_{ij}]$ for $0\le\tau\le\frac{1}{2}$ with four basis functions. These basis functions are given by $\phi_0,\ \phi_1,\ \phi_2,\ \phi_3$. By (4) and (14) we have

$$\begin{bmatrix}\phi_0(t-\tau)\\ \phi_1(t-\tau)\\ \phi_3(t-\tau)\\ \phi_4(t-\tau)\end{bmatrix}=[d_{ij}(\tau)]\begin{bmatrix}\phi_0(t)\\ \phi_1(t)\\ \phi_3(t)\\ \phi_4(t)\end{bmatrix},\ i,j=1,2,3,4,$$

where $\phi_0(t-\tau)=1,\ \tau\le t<1$, and

$$\phi_1(t-\tau)=\begin{cases}1 & \text{if}\tau\le t<\frac{1}{2}+\tau,\\ -1 & \text{if}\frac{1}{2}+\tau\le t<1,\end{cases}$$

$$\phi_2(t-\tau)=\begin{cases}1 & \text{if}\tau\le t<\frac{1}{4}+\tau,\\ -1 & \text{if}\frac{1}{4}+\tau\le t<\frac{1}{2}+\tau,\end{cases}$$

and

$$\phi_3(t-\tau)=\begin{cases}1 & \text{if}\frac{1}{2}+\tau\le t<\frac{3}{4}+\tau,\\ -1 & \text{if}\frac{3}{4}+\tau\le t<1,\end{cases}$$

To find the entries $d_{ij}(\tau)$, $i,j=1,2,3,4$, we use the inner product. For example if $\tau=0.1$, we have $d_{11} =< \phi_0(t-\tau),\ \phi_0(t) >= \int_0^1 \phi_0(t-\tau)\phi_0(t)dt = 0.9$, $d_{31} =< \phi_3(t-\tau),\ \phi_0(t) >= \int_0^1 \phi_3(t-\tau)\phi_0(t)dt = 0$. If we calculate all $d_{ij}(\tau)$ as d_{11} and d_{31}, the 4×4 operational matrix $D(\tau)$ will be obtained as

$$D_{4\times 4}(0.1) = \begin{bmatrix} 0.9 & -0.1 & -0.1 & 0 \\ 0.1 & 0.7 & -0.1 & 0.2 \\ 0 & 0.2 & 0.2 & -0.1 \\ 0.1 & -0.1 & 0 & 0.2 \end{bmatrix}_{4\times 4},$$

In a similar manner if we use the vector function $\phi(t)$ with dimension $2^{n+1}\times 1$, then $2^{n+1}\times 2^{n+1}$ delay matrix $D(\tau)$ with $0\le\tau\le\frac{1}{2^n}$ can be obtained. Note that for any dimension if $\tau=0$ then matrix is diagonal and we have

$$d_{il} = \int_0^1 \phi_i(t)\phi_l(t)dt = \begin{cases} 2^{-j} & \text{if } i=l=2^j+k, \\ 0 & \text{if} i\neq l, \end{cases}$$

3. The collocation approach

We discretize the functions $\phi_i(t)$ in (4) by dividing the interval $\ell\in[0,1]$, to M equidistance intervals with distance parameter $\triangle t = 1/M$ and introduce the collocation points $\ell_k=(k-0.5)/M,\ k=1,...,M$, where M is the number of nodes used in the discretization and also is the maximum wavelet index number.

Also we approximate state variables $\dot{x}(\ell)$ by Haar wavelets with M collocation points, i.e.,

$$\dot{x}(\ell) \approx c_x^T \Psi_M(\ell), \tag{15}$$

where Haar coefficient vector c_x^T are defined as $c_x^T=[c_{x_1},\cdots,c_{x_M}]$. Using the operational integration matrix $P_{M\times M}$ defined in (12),

$$x(\ell) = \int_0^\ell \dot{x}(\ell')d\ell' + x(0) = \int_0^\ell c_x^T\Psi_M(\ell')d\ell' + x(0) = c_x^T P_{M\times M}\Psi_M(\ell)+x(0). \tag{16}$$

As stated in (13), the expansion of the matrix $\Psi_M(\ell)$ at the M collocation points will yield the $M\times M$ Haar matrix $H_{M\times M}$ and by (16) it can be concluded that

$$\dot{x}(\ell_k) = c_x^T\Psi_M(\ell_k),\ x(\ell_k) = c_x^T P_{M\times M}\Psi_M(\ell_k)+x(0),\ k=1,\cdots,M, \tag{17}$$

Now we focus on the analysis of time-delayed systems. Choose N_i as following manner,

$$N_i = \lfloor M\tau_i + 0.5\rfloor, \tag{18}$$

$i=1,2,...,s$ and let $N_1\le N_2\le\cdots\le N_s$, (where $\lfloor . \rfloor$ denotes the bracket function). Using (12), (14) and (1),

$$x(\ell-\tau_i) = \begin{cases} \xi(\ell-\tau_i) & \text{if } 0\le\ell<\tau_i \\ C_x^T P_{M\times M} D(\tau_i)\Psi_M(\ell)+x(0) & \text{if } \tau_i\le\ell<t_f \end{cases} \quad i=1,2,\cdots,s. \tag{19}$$

By substituting $x(\ell_k)$, $\dot{x}(\ell_k)$ and $x(\ell_k-\tau_i)$, $(i=1,2,\cdots,s)$ in (1) and using (14)-(19), we have

$$c_x^T\Psi_M(\tau_k) = (t_f-t_0)\times\big\{f(\ell_k, c_x^T P_{M\times M}\Psi_M(\ell_k)+x(0), \xi(\ell_k-\tau_1), \\ \xi(\ell_k-\tau_2),\cdots,\xi(\ell_k-\tau_s))\big\},\ k=1,...,N_1, \tag{20}$$

$$c_x^T\Psi_M(\ell_k) = (t_f - t_0) \times \{f(\ell_k, c_x^T P_{M\times M}\Psi_M(\ell_k) + x(0), c_x^T P_{M\times M}D(\tau_1)\Psi_M(\ell_k) + x(0),$$
$$\xi(\ell_k - \tau_2), \cdots, \xi(\ell_k - \tau_s))\},\ k = N_1 + 1, ..., N_2, \quad (21)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_x^T\Psi_M(\ell_k) = (t_f - t_0) \times \{f(\ell_k, c_x^T P_{M\times M}\Psi_M(\ell_k) + x(0), c_x^T P_{M\times M}D(\tau_1)\Psi_M(\ell_k) + x(0),$$
$$c_x^T P_{M\times M}D(\tau_2)\Psi_M(\ell_k) + x(0), \cdots, c_x^T P_{M\times M}D(\tau_s)\Psi_M(\ell_k) + x(0))\},\ k = N_s + 1, ..., M, \quad (22)$$

In this way, the DDE problem is transformed into nonlinear algebraic equations for the coefficients c_x^T. From the equations (20)-(22), the vector unknown c_x^T using an iterative approach, e.g. Newton iterative method, may be evaluated. Since the first and last collocation points are not set as the initial and final time, the initial and final variables are calculated according to

$$x(0) = \xi(0), \quad (23)$$

$$x_f = x(M) + \dot{x}(M)/2M. \quad (24)$$

4. Numerical results

In this section, two DDE problems and the results of applying the discussed collocation approach for them is considered. Also the obtained approximate solutions have been compared with exact ones.

Example 1. We consider the equation

$$\dot{x}(t) = -x(t - 1/2) + \sin(t - 1/2) + \cos(t),\ 0 < t \leqslant 1, \quad (25)$$

$$\xi(t) = \sin(t),\ t \leq 0. \quad (26)$$

Using the Haar wavelets collocation method with $M = 16$ collocation points we have

$$\tau_1 = \frac{1}{2},\ N_1 = 8, \quad (27)$$

$$c_x^T h_k = -\sin(\ell_k - 1/2) + \sin(\ell_k - 1/2) + \cos(\ell_k),\ k = 1, 2, ..., N_1, \quad (28)$$

$$c_x^T h_k = -c_x^T P_{M\times M}D(1/2)\Psi_M(\ell_k) - x(0) + \sin(\ell_k - 1/2) + \cos(\ell_k),\ k = N_1 + 1, ..., M, \quad (29)$$

and boundary constraints

$$x(0) = \xi(0) = 0, \quad (30)$$

$$x(t_f) = c_x^T P_{M\times M}\Psi_M(\ell_M) + x(0) + c_x^T\Psi_M(\ell_M)/2M, \quad (31)$$

where c_x are all variables which must be obtained at the collocation points. The approximate solution with respect to the number of nodes with exact solution of equation are shown in Fig.1.

Example 2. In the second example the two-dimensional DDE equation

$$\dot{x}_1(t) = 10x_2(t),$$
$$\dot{x}_2(t) = -1000x_2(t) + \frac{1}{10\exp(100)}x_2(t - 1/10) + 10x_1(t), \quad (32)$$

with delay functions $\xi_1(t) = \exp(-1000t)$, $\xi_2(t) = -100\exp(-1000t)$, for $t \leq 0$ is considered. The approximate solution with using 16 collocation points by applying the collocation approach, in comparison with the exact solution are shown in Figs. 2 and 3.

5. Conclusions

In this paper an collocation method using the profits of Haar wavelets is presented. Applying the given approach is simple and expected that along with the increase of collocation points the approximate solution converges to exact solution of DDE. Of course the approximate solution depends on the applied method of solving the nonlinear equation, which will be generated in the process of yielding the mentioned approach and so, like any iterative scheme the nonlinearity of the functional f in (1) has direct impact on the precision of the approximate solution.

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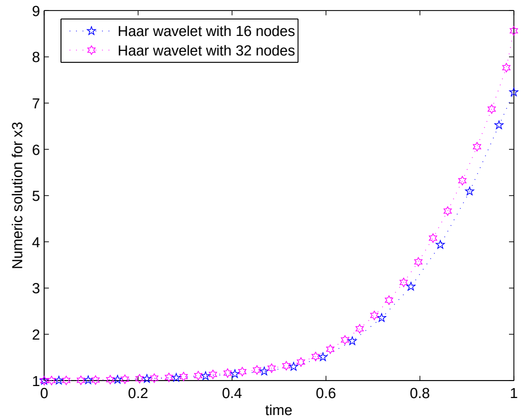

FIGURE 1. Approximate and exact solution for Example 4

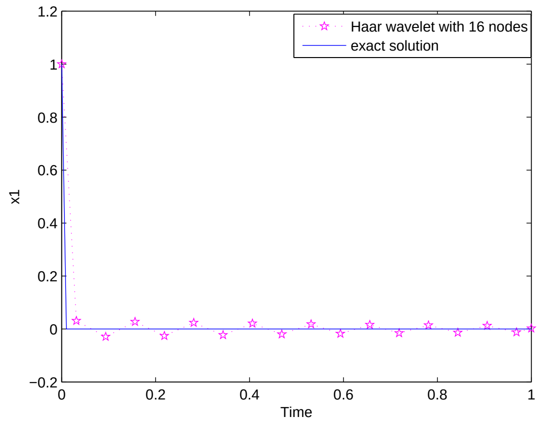

FIGURE 2. Approximate and exact solution x_1 for Example 4

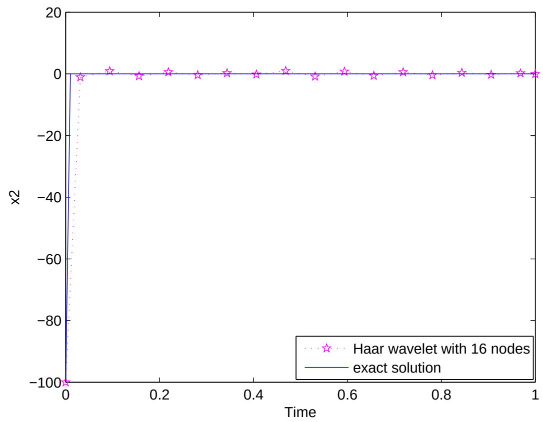

FIGURE 3. Approximate and exact solution x_2 for Example 4

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