TWMS J. Pure Appl. Math. V.5, N.2, 2014, pp.210-220

SOME SUBORDINATION AND SUPERORDINATION RESULTS ASSOCIATED WITH A NEW OPERATOR*

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Abstract. In this paper, we obtain some subordination and superordination results of $p-$valent meromorphic functions associated with a new linear operator. Sandwich-type theorem for these p-valent functions is also obtained.

Keywords: p-Valent meromorphic functions, subordination, superordination, linear operator.

AMS Subject Classification: 30C45.

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

For $f, F \in H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [5] and [6]).

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{2}$$

then $p(z)$ is a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (2) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in U and if $p(z)$ satisfies the first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{3}$$

then $p(z)$ is a solution of the differential superordination (3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (3) is called

*Research and the writing of this paper were supported in part by NSF Grant ANI-031976.

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Manuscript received December 2012.

the best subordinant (see [5] and [6]).

For analytic functions $f(z) \in \sum_p$, given by (1) and $\phi(z) \in \sum_p$ given by $\phi(z) = z^{-p} + \sum\limits_{n=1}^{\infty} b_{n-p} z^{n-p}$ $(p \in \mathbb{N})$, the Hadamard product (or convolution) of $f(z)$ and $\phi(z)$, is defined by

$$(f * \phi)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} = (\phi * f)(z). \tag{4}$$

Aqlan et al. [1] defined the operator $Q^{\alpha}_{\beta,p} : \sum_p \to \sum_p$ by:

$$Q^{\alpha}_{\beta,p} f(z) = \begin{cases} z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum\limits_{n=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\alpha)} a_{n-p} z^{n-p} & (\alpha > 0; \beta > -1; p \in \mathbb{N}; f \in \Sigma_p) \\ f(z) & (\alpha = 0\,; \beta > -1; p \in \mathbb{N}; f \in \Sigma_p)\,. \end{cases} \tag{5}$$

Mostafa [7] used the Aqlan et al. operator and defined the following linear operator $H^{\alpha}_{p,\beta,\mu} : \Sigma_p \to \Sigma_p$ as follows:
First put

$$G^{\alpha}_{\beta,p}(z) = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\alpha)} z^{n-p} \quad (p \in \mathbb{N}) \tag{6}$$

and let $G^{\alpha *}_{\beta,p,\mu}$ be defined by

$$G^{\alpha}_{\beta,p}(z) * G^{\alpha *}_{\beta,p,\mu}(z) = \frac{1}{z^p (1-z)^{\mu}} \quad (\mu > 0; p \in \mathbb{N})\,. \tag{7}$$

Then

$$H^{\alpha}_{p,\beta,\mu} f(z) = G^{\alpha *}_{\beta,p}(z) * f(z) \quad (f \in \Sigma_p)\,. \tag{8}$$

Using (6) and (8), we have

$$H^{\alpha}_{p,\beta,\mu} f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta+\alpha)(\mu)_n}{\Gamma(n+\beta)(1)_n} a_{n-p} z^{n-p}\,, \tag{9}$$

where $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N})\,. \end{cases} \tag{10}$$

It is readily verified from (9) that (see [7])

$$z(H^{\alpha}_{p,\beta,\mu} f(z))' = (\alpha+\beta) H^{\alpha+1}_{p,\beta,\mu} f(z) - (\alpha+\beta+p) H^{\alpha}_{p,\beta,\mu} f(z) \tag{11}$$

and

$$z(H^{\alpha}_{p,\beta,\mu} f(z))' = \mu H^{\alpha}_{p,\beta,\mu+1} f(z) - (\mu+p) H^{\alpha}_{p,\beta,\mu} f(z). \tag{12}$$

It is noticed that, putting $\mu = 1$ in (9), we obtain the operator

$$H^{\alpha}_{p,\beta,1} f(z) = H^{\alpha}_{p,\beta} f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n+\beta)} a_{n-p} z^{n-p}. \tag{13}$$

To prove our results, we need the following definitions and lemmas.

Definition 1. [5]. *Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\bar{U}\backslash E(q)$ where*

$$E(q) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\right\}$$

and are such that $q^{'}(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(q)$. Further let the subclass of $\mathcal{F}$ for which $q(0) = a$ be denoted by $\mathcal{F}$ (a), $\mathcal{F}(0) \equiv \mathcal{F}_0$ and $\mathcal{F}(1) \equiv \mathcal{F}_1$.

Definition 2. [6]. *A function $L(z,t)$ $(z \in U, t \geq 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in U for all $t \geq 0, L(z,0)$ is continuously differentiable on $[0;1)$ for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.*

Lemma 1. [8]. *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + ...,$ with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ Suppose that $L(.;t)$ is analytic in U f or all $t \geq 0$, $L(z;.)$ is continuously differentiable on $[0;+\infty)$ for all $z \in U$. If $L(z;t)$ satisfies*

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in U, t \geq 0).$$

and

$$|L(z;t)| \leq K_0 |a_1(t)|, |z| < r_0 < 1, t \geq 0$$

for some positive constants K_0 and r_0, then $L(z;t)$ is a subordination chain.

Lemma 2. [3]. *Suppose that the function $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$Re\{\mathcal{H}(is;t)\} \leq 0$$

for all real s and for all $t \leq -n(1+s^2)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1+p_nz^n+p_{n+1}z^{n+1}+...$ is analytic in U and

$$Re\left\{\mathcal{H}\left(p(z); zp^{'}(z)\right)\right\} > 0 \quad (z \in U),$$

then $Re\{p(z)\} > 0$ for $z \in U$.

Lemma 3. [4]. *Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in \mathcal{H}(U)$ with $h(0) = c$. If $Re\{\kappa h(z) + \gamma\} > 0\,(z \in U)$, then the solution of the following differential equation:*

$$q(z) + \frac{zq^{'}(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in U and satisfies $Re\{\kappa q(z) + \gamma\} > 0$ for $z \in U$.

Lemma 4. [5]. *Let $p \in \mathcal{F}(a)$ and let $q(z) = a + a_nz^n + a_{n+1}z^{n+1} + ...$be analytic in U with $q(z) \neq a$ and $n \geq 1$. If q is not subordinate to p, then there exists two points $z_0 = r_0e^{i\theta} \in U$ and $\zeta_0 \in \partial U\backslash E(q)$ such that*

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0p^{'}(z_0) = m\zeta_0p'(\zeta_0) \quad (m \geq n).$$

Lemma 5. [6]. *Let $q \in \mathcal{H}[a;1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$. If $L(z,t) = \varphi\left(q(z), tzq^{'}(z)\right)$ is a subordination chain and $q \in H[a;1] \cap \mathcal{F}(a)$, then*

$$h(z) \prec \varphi\left(q(z), zq^{'}(z)\right),$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$ has a univalent solution $q \in \mathcal{F}(a)$, then q is the best subordinant.

In this paper, we investigate several properties of the linear operator $H^{\alpha}_{p,\beta,\mu}$.

2. Main results

Unless otherwise mentioned, we assume throughout this section that $\alpha \geq 0, \beta > -1, \alpha+\beta \neq 0, \mu, \gamma, \eta > 0, p \in \mathbb{N}$, $z \in U$ and all powers are understood as principle values.

Theorem 1. *Let $f, g \in \sum_p$ and let*

$$Re\left\{1+\frac{z\phi^{''}(z)}{\phi'(z)}\right\} > -\delta$$

$$\left(\phi(z) = (1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\eta} + \gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu} g(z)}{H^{\alpha}_{p,\beta,\mu} g(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\eta}; z \in U\right), \tag{14}$$

where δ is given by

$$\delta = \frac{1+\left(\frac{\eta(\alpha+\beta)}{\gamma}\right)^2 - \left|1-\left(\frac{\eta(\alpha+\beta)}{\gamma}\right)^2\right|}{4\left(\frac{\eta(\alpha+\beta)}{\gamma}\right)}. \tag{15}$$

Then the subordination condition

$$\begin{aligned} &(1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\eta} + \gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu} f(z)}{H^{\alpha}_{p,\beta,\mu} f(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\eta} \prec \\ \prec\ &(1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\eta} + \gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu} g(z)}{H^{\alpha}_{p,\beta,\mu} g(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\eta} \end{aligned} \tag{16}$$

implies that

$$\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\eta} \prec \left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\eta} \tag{17}$$

and the function $\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\eta}$ is the best dominant.

Proof. Define the functions $F(z)$ and $G(z)$ in U by

$$F(z) = \left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\eta} \quad \text{and} \quad G(z) = \left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\eta} \quad (z \in U), \tag{18}$$

and assume, without loss of generality, that $G(z)$ is analytic, univalent on $\bar{U}$ and

$$G^{'}(\zeta) \neq 0 \qquad (|\zeta| = 1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\bar{U}$, so we can use them in the proof of our theorem, the results would follow by letting $\rho \to 1$.

We first show that, if

$$q(z) = 1 + \frac{zG^{''}(z)}{G'(z)} \qquad (z \in U), \tag{19}$$

then

$$Re\{q(z)\} > 0 \qquad (z \in U).$$

From (11) and the definition of the functions G, ϕ, we obtain that

$$\phi(z) = G(z) + \frac{\gamma}{\eta(\alpha+\beta)} zG^{'}(z). \tag{20}$$

Differentiating both side of (20) with respect to z yields

$$\phi^{'}(z) = \left(1+\frac{\gamma}{\eta(\alpha+\beta)}\right)G^{'}(z) + \frac{\gamma}{\eta(\alpha+\beta)} zG^{''}(z). \tag{21}$$

Combining (19) and (21), we easily get

$$1+\frac{z\phi''(z)}{\phi'(z)}=q(z)+\frac{zq'(z)}{q(z)+\frac{\eta(\alpha+\beta)}{\gamma}}=h(z)\qquad(z\in U). \tag{22}$$

It follows from (14) and (22) that

$$Re\left\{h(z)+\frac{\eta(\alpha+\beta)}{\gamma}\right\}>0\quad(z\in U). \tag{23}$$

Moreover, by using Lemma 3, we conclude that the differential equation (22) has a solution $q(z)\in H(U)$ with $h(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{v}{u+\frac{\eta(\alpha+\beta)}{\gamma}}+\delta,$$

where δ is given by (15). From (22) and (23), we obtain

$$Re\left\{H\left(q(z);zq'(z)\right)\right\}>0\qquad(z\in U).$$

To verify the condition

$$Re\{H(iu;v)\}\le 0\qquad\left(u\in\mathbb{R};v\le-\frac{1+u^2}{2}\right), \tag{24}$$

we proceed as follows:

$$Re\{H(iu;v)\}=Re\left\{iu+\frac{v}{iu+\frac{\eta(\alpha+\beta)}{\gamma}}+\delta\right\}=$$

$$=\frac{\frac{\eta(\alpha+\beta)}{\gamma}v}{u^2+\left(\frac{\eta(\alpha+\beta)}{\gamma}\right)^2}+\delta\le-\frac{\sigma(u,\eta,\beta,\alpha,\delta)}{2\left[u^2+\left(\frac{\eta(\alpha+\beta)}{\gamma}\right)^2\right]},$$

where

$$\sigma(u,\eta,\beta,\alpha,\delta)=\left[\frac{\eta(\alpha+\beta)}{\gamma}-2\delta\right]u^2-2\delta\left(\frac{\eta(\alpha+\beta)}{\gamma}\right)^2+\frac{\eta(\alpha+\beta)}{\gamma}. \tag{25}$$

For δ given by (15), we note that the expression $\sigma(u,\eta,\beta,\alpha,\delta)$ in (25) is positive, which implies that (24) holds. Thus, by using Lemma 2, we conclude that

$$Re\{q(z)\}>0\qquad(z\in U).$$

that is, that $G(z)$ defined by (18) is convex (univalent) in U. Next, we prove that the subordination condition (16) implies that

$$F(z)\prec G(z),$$

for the functions F and G defined by (18). Consider the function $L(z,t)$ given by

$$L(z,t)=G(z)+\frac{\gamma(1+t)}{\eta(\alpha+\beta)}zG'(z)\quad(0\le t<\infty;z\in U). \tag{26}$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0}=G'(0)\left(1+\frac{\gamma(1+t)}{\eta(\alpha+\beta)}\right)\ne 0\qquad(0\le t<\infty;z\in U).$$

This show that the function

$$L(z,t)=a_1(t)z+\ldots,$$

satisfies the condition $a_1(t)\ne 0$ for all $t\ge 0$ and $\lim_{t\to\infty}|a_1(t)|=+\infty$. From defination (26) and for all $t\ge 0$, we have

$$\frac{|L(z,t)|}{|a_1(t)|} = \frac{\left|G(z) + \frac{\gamma(1+t)}{\eta(\alpha+\beta)} zG'(z)\right|}{1 + \frac{\gamma(1+t)}{\eta(\alpha+\beta)}} \leq \frac{|G(z)| + \frac{\gamma(1+t)}{\eta(\alpha+\beta)} \left|zG'(z)\right|}{1 + \frac{\gamma(1+t)}{\eta(\alpha+\beta)}}. \tag{27}$$

Since the function G is convex and normalized in U, $G \in K$, the following well-known growth and distortion sharp inequalities (see [2]) are true:

$$\begin{aligned} \frac{r}{1+r} &\leq |G(z)| \leq \frac{r}{1-r}, \quad if \;\; |z| \leq r < 1, \\ \frac{1}{(1+r)^2} &\leq \left|G'(z)\right| \leq \frac{1}{(1-r)^2}, \quad if \;\; |z| \leq r < 1, \end{aligned}$$

Using the right-hand sides of these inequalities in (27), we deduce that

$$\frac{|L(z,t)|}{|a_1(t)|} = \frac{r}{(1-r)^2} \frac{\gamma(1+t) + (1-r)}{\eta(\alpha+\beta) + \gamma(1+t)} \leq \frac{r}{(1-r)^2}, \;\; |z| \leq r, t \geq 0,$$

and thus, the second assumption of Lemma 1 holds. Furthermore,

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = Re\left\{\frac{\eta(\alpha+\beta)}{\gamma} + (1+t)q(z)\right\} > 0 \quad (0 \leq t < \infty; z \in U).$$

Therefore, by using Lemma 1, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{\gamma}{\eta(\alpha+\beta)} zG'(z) = L(z,0),$$

and

$$L(z,0) \prec L(z,t) \quad (0 \leq t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(U,0) = \phi(U) \quad (0 \leq t < \infty; \zeta \in \partial U). \tag{28}$$

If F is not subordinate to G, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty). \tag{29}$$

Hence, by virtue of (16), (18), (26) and (29), we have

$$\begin{aligned} L(\zeta_0,t) &= G(\zeta_0) + \frac{\gamma(1+t)zG'(\zeta_0)}{\eta(\alpha+\beta)} = F(z_0) + \frac{\gamma z_0 F'(z_0)}{\eta(\alpha+\beta)} \\ &= (1-\gamma)\left(z_0^p H_{p,\beta,\mu}^{\alpha} f(z_0)\right)^{\eta} + \gamma\left(\frac{H_{p,\beta,\mu}^{\alpha+1} f(z_0)}{H_{p,\beta,\mu}^{\alpha} f(z_0)}\right)\left(z_0^p H_{p,\beta,\mu}^{\alpha} f(z_0)\right)^{\eta} \in \phi(U). \end{aligned}$$

This contradicts (28). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function G is the best dominant. This completes the proof of Theorem 1.

Similarly, we can prove the following theorem.

Theorem 2. *Let $f, g \in \sum_p$ and let*

$$Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\sigma$$

$$\left(\phi(z)=(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}g(z)}{H^{\alpha}_{p,\beta,\mu}g(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\lambda};z\in U\right),$$

where σ is given by

$$\sigma=\frac{1+\left(\frac{\lambda\mu}{\gamma}\right)^{2}-\left|1-\left(\frac{\lambda\mu}{\gamma}\right)^{2}\right|}{4\frac{\lambda\mu}{\gamma}}. \tag{30}$$

Then the subordination condition

$$(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}\prec$$

$$\prec\quad(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}g(z)}{H^{\alpha}_{p,\beta,\mu}g(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\lambda}$$

implies that

$$\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}\prec\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\lambda}$$

and the function $\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\lambda}$ *is the best dominant.*

We now derive the following theorem.

Theorem 3. *Let $f,g\in\sum_{p}$ and let*

$$Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\}>-\delta$$

$$\left(\phi(z)=(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\eta}+\gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu}g(z)}{H^{\alpha}_{p,\beta,\mu}g(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\eta}\right), \tag{31}$$

where δ is given by (15)*. If the function*

$$(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}+\gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}$$

is univalent in U and $\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}\in\mathcal{F}$, *then the superordination condition*

$$(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\eta}+\gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu}g(z)}{H^{\alpha}_{p,\beta,\mu}g(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\eta}\prec$$

$$\prec\quad(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}+\gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}$$

implies that

$$\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\eta}\prec\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}$$

and the function $\left(z^{p}H^{\alpha}_{p,\beta,\mu}g(z)\right)^{\eta}$ *is the best subordinant.*

Proof. Suppose that the functions F,G and q are defined by (18) and (19), respectively. By applying the similar method as in the proof of Theorem 1, we get

$$Re\{q(z)\}>0\qquad(z\in U).$$

Next, to arrive at our desired result, we show that $G\prec F$. For this, we suppose that the function $L(z,t)$ be defined by (26). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{\gamma}{\eta(\alpha+\beta)} zG'(z) = \varphi\left(G(z), zG'(z)\right)$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 3.

Similarly, we can prove the following theorem.

Theorem 4. *Let* $f, g \in \sum_p$ *and let*

$$Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\sigma$$

$$\left(\phi(z) = (1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\lambda} + \gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1} g(z)}{H^{\alpha}_{p,\beta,\mu} g(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\lambda}\right),$$

where σ *is given by* (30). *If the function*

$$(1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\lambda} + \gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1} f(z)}{H^{\alpha}_{p,\beta,\mu} f(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\lambda}$$

is univalent in U *and* $\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\lambda} \in \mathcal{F}$, *then the superordination condition*

$$(1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\lambda} + \gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1} g(z)}{H^{\alpha}_{p,\beta,\mu} g(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\lambda} \prec$$
$$\prec \quad (1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\lambda} + \gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1} f(z)}{H^{\alpha}_{p,\beta,\mu} f(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\lambda}$$

implies that

$$\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\lambda} \prec \left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\lambda}$$

and the function $\left(z^p H^{\alpha}_{p,\beta,\mu} g(z)\right)^{\lambda}$ *is the best subordinant.*

Combining Theorem 1 and Theorem 3, we get the following "sandwich-type result".

Theorem 5. *Let* $f, g_i \in \sum_p (j = 1, 2)$ and let

$$Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta$$

$$\left(\phi_j(z) = (1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} g_i(z)\right)^{\eta} + \gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu} g_i(z)}{H^{\alpha}_{p,\beta,\mu} g_i(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} g_i(z)\right)^{\eta} (j = 1, 2)\right),$$

where δ *is given by* (15). *If the function*

$$(1-\gamma)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\eta} + \gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu} f(z)}{H^{\alpha}_{p,\beta,\mu} f(z)}\right)\left(z^p H^{\alpha}_{p,\beta,\mu} f(z)\right)^{\eta}$$

is univalent in U *and* $\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}\in\mathcal{F}$, *then the condition*

$$(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\eta}+\gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu}g_1(z)}{H^{\alpha}_{p,\beta,\mu}g_1(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\eta}\prec$$

$$\prec\quad(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}+\gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}\prec$$

$$\prec\quad(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\eta}+\gamma\left(\frac{H^{\alpha+1}_{p,\beta,\mu}g_2(z)}{H^{\alpha}_{p,\beta,\mu}g_2(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\eta}$$

implies that

$$\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\eta}\prec\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}\prec\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\eta}$$

and the functions $\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\eta}$ *and* $\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\eta}$ *are , respectively, the best subordinant and the best dominant.*

Combining Theorem 2 and Theorem 4, we get the following "sandwich-type result".

Theorem 6. *Let* $f,g_i\in\sum_p(j=1,2)$ and let

$$Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\}>-\sigma$$

$$\left(\phi_j(z)=(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_i(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}g_i(z)}{H^{\alpha}_{p,\beta,\mu}g_i(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_i(z)\right)^{\lambda}(j=1,2)\right),$$

where σ *is given by* (30). *If the function*

$$(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}$$

is univalent in U *and* $\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}\in\mathcal{F}$, *then the condition*

$$(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}g_1(z)}{H^{\alpha}_{p,\beta,\mu}g_1(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\lambda}\prec$$

$$\prec\quad(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}f(z)}{H^{\alpha}_{p,\beta,\mu}f(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\lambda}\prec$$

$$\prec\quad(1-\gamma)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\lambda}+\gamma\left(\frac{H^{\alpha}_{p,\beta,\mu+1}g_2(z)}{H^{\alpha}_{p,\beta,\mu}g_2(z)}\right)\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\lambda}$$

implies that

$$\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\eta}\prec\left(z^{p}H^{\alpha}_{p,\beta,\mu}f(z)\right)^{\eta}\prec\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\eta}$$

and the functions $\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_1(z)\right)^{\lambda}$ *and* $\left(z^{p}H^{\alpha}_{p,\beta,\mu}g_2(z)\right)^{\lambda}$ *are , respectively, the best subordinant and the best dominant.*

Remark 1. *Putting* $\mu=1$ *, in Theorems 1, 3 and 5, we obtain the corresponding results for the operator* $H^{\alpha}_{p,\beta}$ *defined in (13).*

3. Acknowledgement

The authors would like to thank the referees of the paper for their helpful suggestions.

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