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APPROXIMATION PROPERTIES OF SOME SYSTEMS OF ROOT FUNCTIONS GENERATED BY WELL-POSED SOLVABLE BOUNDARY VALUE PROBLEMS*

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Abstract. In this paper, we consider some properties of the systems of root functions generated by the first order differential operator. We obtain explicit form of the biorthogonal systems for the systems of root functions generated by the first order differential operator. We investigate the approximation properties of the systems of root functions of the first order differential operator. We obtain a sufficient condition for the completeness of the systems of root functions of the first order differential operator in terms of the boundary functions.

Keywords: resolvent of the operator, well-posed boundary value problems, completeness of the systems of root functions.

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1. Introduction

Let L be a differential operator in the functional space $L_2(0,1)$ such that a inverse operator L^{-1} is completely continuous. Then the spectrum of L consists of finite or countable set of isolated eigenvalues finite algebraic multiplicity without finite accumulation points according to statement in [1, p. 10]. We can set for each eigenvalue λ_n geometric multiplicity m_n of a chain of eigenfunction and associated functions of the operator L

$$E_n = \{y_{n,0}(x), y_{n,1}(x), \ldots, y_{n,m_n-1}(x)\}.$$

The system of root functions of L

$$E = \{E_n : \lambda_n \text{ is the eigenvalue of L}\}$$

is called the union of all such chains. Details are described in section 5 below. The differential operator L is the source of some system of root functions. The main problem is investigate some properties of the system of root functions generated by the first order differential operator.

In this paper, we construct explicit form of systems of root functions for first order differential operator. We claim (Theorem 1.1), the obtained systems of root functions are the minimal families [6, p. 171].

In the sequel we construct corresponding the family of biorthogonal functions in explicit form

$$\{E_k^{'} : \lambda_k \text{ is the eigenvalue of L}\},$$

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where

$$E_k^{'} = \{h_{k,m_k-1}(x), h_{k,m_k-2}(x), ..., h_{k,0}(x)\},$$

$$h_{k,m_k-1-j}(x) = -\frac{1}{(m_k-1-j)!}\lim_{\lambda\to\lambda_k}\frac{\partial^{m_k-1-j}}{\partial\lambda^{m_k-1-j}}\left(\frac{(\lambda-\lambda_k)^{m_k}M_{\overline{\lambda}}(t)}{\Delta(\lambda)}\right), j = 0, 1, ..., m_k - 1,$$

$$M_{\overline{\lambda}}(x) = \sigma(t) - i\overline{\lambda}\int\limits_x^1 e^{-i\overline{\lambda}(t-x)}\sigma(t)dt, \tag{1}$$

$$\Delta(\lambda) = 1 - \lambda\int\limits_0^1 e^{i\lambda x}\overline{\sigma(x)}dx, \sigma(x) \in L_2[0, 1]. \tag{2}$$

Similar systems of functions are considered in [14].

In [14] it is proved only the existence of biorthogonal system of functions without being explicit formulas. In this paper, we obtain in explicit form biorthogonal system of functions in terms of the boundary value problem.

Let us formulate own main results.

Theorem 1.1. *The system of functions $E^{'}$ is biorthogonal to the system of functions E, i.e.*

$$< y_{n,s}(x), h_{k,m_k-1-j}(x) >= \begin{cases} 1, & for\ (n,s) = (k,j); \\ 0, & for\ (n,s) \neq (k,j). \end{cases}$$

Theorem 1.2. *Let $\sigma(x) \in L_2[0,1]$ and the operator L has only simple eigenvalues lying in a horizontal strip. Then for the functions $f(x) \in W_2^p[0, 1]$ satisfying the conditions $U(f^{(k-2)}(x)) = = 0$, $k = 2, 3, \ldots, p$, the Fourier coefficients with respect to the system of root functions of L have the asymptotic behavior as $n \to \infty$*

$$c_{n,0}(f) = \overline{\overline{o}}\left(\frac{\lambda_n^{1-p}}{\Delta'(\lambda_n)}\right).$$

Theorem 1.3. *The system of root functions of L is complete in $L_2(0,1)$ if the following conditions hold*

$$\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\int\limits_0^\varepsilon\overline{(i+\sigma(x))}dx = \alpha_1 \neq 0,$$

$$\lim_{\varepsilon\to1-0}\frac{1}{1-\varepsilon}\int\limits_\varepsilon^1\overline{\sigma(x)}dx = \alpha_2 \neq 0,$$

where $\sigma(x) \in L_2(0, 1)$.

Completeness of the systems of root functions has a pretty rich history in the various function spaces. Completeness of the systems of root functions in the case of regular (in the sense of Birkhoff) boundary conditions for higher order differential operators is well known (see [7]). The completeness of the system of root functions of the boundary value problem with nonregular splitting boundary conditions for higher order differential equations was announced by Keldysh [4] and proved for the first time by Shkalikov [15]. Completeness of the systems of root functions of boundary value problems for second order differential equations was studied in detail. Nondegenerate boundary conditions considered in the monograph of V.A. Marchenko [10], and degenerate boundary conditions considered in the works of M.M. Malamud [8]. Completeness of the systems of root functions of the two-point boundary conditions for systems of linear differential equations studied in [9]. If the Fourier coefficients $c_{n,k}(f)$ behave as stated in Theorem 1.2, then from the work of I.S. Lomov [5], we get of the convergence of spectral decomposition.

2. Boundary value problem and auxiliary notations

In [11] it is proved the following statement

Theorem 2.1. *(Theorem M. Otelbaev)*

a) For any choice of function $\sigma(x)$ from the space $L_2(0,1)$, we consider the nonlocal boundary value problem

$$\ell(y) \equiv -iy'(x) = f(x), 0 < x < 1, \tag{3}$$

$$U(y) \equiv y(0) - \int_0^1 (-iy'(x))\overline{\sigma(x)}dx = 0 \tag{4}$$

in the space $L_2(0,1)$, which corresponds to the operator L. Then L has completely continuous inverse L^{-1}.

b) Assume that a nonhomogeneous equation (3) with some additional conditions for any right-hand side $f(x) \in L_2(0,1)$ has a unique solution $y(x)$ in the space $W_2^1[0,1]$, where $y(x)$ satisfies the a priori estimate

$$\| y \|_{L_2(0,1)} \leq c \| f \|_{L_2(0,1)}$$

Then there exists a unique function $\sigma(x)$ from the space $L_2(0,1)$ such that any additional condition is equivalent to (4).

Denote by L a operator corresponding to the problem of (3), (4). It follows from Theorem (M. Otelbaev) that the nonlocal boundary conditions (4) for all possible $\sigma(x) \in L_2(0,1)$ describe everything well-posed solvable boundary value problems corresponding to expression of $\ell(\cdot)$. In [2], [3] it is described classes well-posed boundary value problems for second order ordinary differential operator and for the Laplace operator.

3. Resolvent of the operator L

In this section we compute an explicit solution following of the nonlocal boundary value problem

$$-iy^{'}(x) = \lambda y(x) + f(x), 0 < x < 1, \tag{5}$$

$$y(0) - \int_0^1 (-iy^{'}(x))\overline{\sigma(x)}dx = 0, \tag{6}$$

where $\sigma(x)$ from the space $L_2(0,1)$. The solution of this nonlocal boundary value problem is called a resolvent of L. The explicit form of the resolvent has a significant meaning for the study of properties of biorthogonal systems of root functions of L.

Theorem 3.1. *A resolvent of the operator L is determined by the formula*

$$y(x,\lambda) = (L - \lambda I)^{-1} f(x) = R(\lambda) f(x) = \frac{< f(t), M_{\overline{\lambda}}(t) > e^{i\lambda x}}{\Delta(\lambda)} + i \int_0^x e^{i\lambda(x-t)} f(t)dt, \tag{7}$$

where the functions $M_{\overline{\lambda}}(t)$, $\Delta(\lambda)$ are defined by formulas (1) and (2).

Proof. We check that $y(x,\lambda)$ satisfies equation (5). To this we find the first derivative of y with respect to x

$$y^{'}(x,\lambda) = i\lambda \frac{< f(t), M_{\overline{\lambda}}(t) > e^{i\lambda x}}{\Delta(\lambda)} + if(x) - \lambda \int_0^x e^{i\lambda(x-t)} f(t)dt.$$

Now multiply by $-i$

$$-iy^{'}(x,\lambda)=\lambda\frac{< f(t), M_{\overline{\lambda}}(t) > e^{i\lambda x}}{\Delta(\lambda)}+f(x)+i\lambda\int\limits_0^x e^{i\lambda(x-t)}f(t)dt.$$

Hence

$$-iy^{'}(x,\lambda)=\lambda y(x,\lambda)+f(x).$$

Let us check boundary condition (4).

$$y(0)-\int\limits_0^1(-iy^{'}(x))\overline{\sigma(x)}dx=\frac{< f(t), M_{\overline{\lambda}}(t) >}{\Delta(\lambda)}-\int\limits_0^1(\lambda y(x,\lambda)+f(x))\overline{\sigma(x)}dx.$$

Using formula (7), we have

$$y(0)-\int\limits_0^1(-iy^{'}(x))\overline{\sigma(x)}dx=\frac{< f(t), M_{\overline{\lambda}}(t) >}{\Delta(\lambda)}-\lambda\frac{< f(t), M_{\overline{\lambda}}(t) >}{\Delta(\lambda)}\int\limits_0^1 e^{i\lambda x}\overline{\sigma(x)}dx-$$

$$-\left(\int\limits_0^1 f(t)\overline{\sigma(t)}dt+i\lambda\int\limits_0^1 f(t)\left(\int\limits_t^1 e^{i\lambda(x-t)}\overline{\sigma(x)}dx\right)dt\right).$$

Given equality (1), we obtain

$$y(0)-\int\limits_0^1(-iy^{'}(x))\overline{\sigma(x)}dx=\frac{< f(t), M_{\overline{\lambda}}(t) >}{\Delta(\lambda)}\left(1-\lambda\int\limits_0^1 e^{i\lambda x}\overline{\sigma(x)}dx\right)-< f(t), M_{\overline{\lambda}}(t) >=0.$$

The proof is complete. □

Note that in [2], [3] were presented resolvents of well-posed problems for ordinary second order differential operator and for the Laplace operator.

4. Asymptotic of eigenvalues of the operator L

It follows from Theorem 3.1 that the spectrum of L is only of eigenvalues, where eigenvalues are poles of resolvent. Poles of the resolvent are defined from the equality

$$\Delta(\lambda)=1-\lambda\int\limits_0^1 e^{i\lambda x}\overline{\sigma(x)}dx=0 \tag{8}$$

In order to compute the asymptotic of the roots of equality (8), we take

$$\sigma(x)=\frac{b(\frac{1}{2}-|x-\frac{1}{2}|)}{(\frac{1}{2}-|x-\frac{1}{2}|)^{\alpha}}k(x), \tag{9}$$

where $b(x)$ is weakly oscillating function, i.e. $b(x)$ is positive in some neighborhood of zero and for any $\delta>0$ function $x^{\delta}b(x)$ increases. Function $x^{-\delta}b(x)$ is decreasing in some right neighborhood of $x=0, 0<\alpha<\frac{1}{2}, k(x)\in V[0,1], k(0+0)\neq 0$, and $k(1-0)\neq 0$.

In the sequel $V[0,1]$ is denotes the class of functions of bounded variation on [0,1].

Theorem 4.1. *1. Let $\sigma(x)$ be given by (9). Then all eigenvalues of L lie in a horizontal strip $-h < Im\lambda < h$ such that for sufficiently large n eigenvalues are simple and the rightly asymptotic behavior*

$$\lambda_n = n\pi + arg\left((-1)^\alpha \frac{k(0+0)}{k(1-0)}\right) - i\ln\left|\frac{k(0+0)}{k(1-0)}\right| + o(1) \tag{10}$$

2. If dist $(\lambda, (\lambda_n)_{n=1}^{\infty}) \geq \delta > 0$ and $|\lambda| \geq \delta$, then the rightly estimate

$$0 < c \leq \frac{|\Delta(\lambda)|}{|\lambda^\alpha| b\left(\frac{1}{|\lambda|}\right) e^{-|Im\lambda|}} \leq c_1. \tag{11}$$

Proof. We separate the integral into two parts $(0, \frac{1}{2}), (\frac{1}{2}, 1)$ in (8). Consequently

$$\Delta(\lambda) = 1 - \lambda(\Delta_1(\lambda) + \Delta_2(\lambda)),$$

where

$$\Delta_1(\lambda) = \int_0^{\frac{1}{2}} e^{i\lambda x} b(x) \frac{1}{x^\alpha} k(x) dx, \tag{12}$$

$$\Delta_2(\lambda) = e^{i\lambda} \int_0^{\frac{1}{2}} e^{-i\lambda x} b(x) \frac{1}{x^\alpha} k(1-x) dx. \tag{13}$$

We have by Remark 1 in [13] on each horizontal line $Im\lambda = h, (h \in R)$ for $|\lambda| \to \infty$

$$\Delta_1(\lambda) = \Gamma(1-\alpha)(-i\lambda)^{\alpha-1} b\left(\frac{1}{|\lambda|}\right)(k(0+0) + o(1)). \tag{14}$$

We get similarly for $\Delta_2(\lambda)$ on each horizontal line $Im\lambda = h, (h \in R)$ at $|\lambda| \to \infty$

$$\Delta_2(\lambda) = \Gamma(1-\alpha) e^{i\lambda} (i\lambda)^{\alpha-1} b\left(\frac{1}{|\lambda|}\right)(k(0+0) + o(1)). \tag{15}$$

We consider $\Delta(\lambda) = 0$ for $|\lambda| \to \pm\infty$ to calculate the asymptotic formula of eigenvalues. We have

$$e^{i\lambda} = \frac{(-1)^\alpha k(0+0)(1+o(1))}{k(1-0)(1+o(1))} = \frac{(-1)^\alpha k(0+0)}{k(1-0)}(1+o(1)). \tag{16}$$

So

$$\lambda = \pi n + arg\left((-1)^\alpha \frac{k(0+0)}{k(1-0)}\right) - i\ln\left|\frac{k(0+0)}{k(1-0)}\right| + o(1). \tag{17}$$

Let us prove that there are zeros of $\Delta(\lambda)$ which is defined by formula (8). Assume that

$$\lambda_n^\circ = 2\pi n + arg\left((-1)^\alpha \frac{k(0+0)}{k(1-0)}\right) - i\ln\left|\frac{k(0+0)}{k(1-0)}\right|. \tag{18}$$

The relation (17) can be rewritten as follows

$$\lambda = \lambda_n^\circ + o(1), n = 0, \pm 1, \pm 2, \ldots. \tag{19}$$

We describe around of each point λ_n^0 the circle Γ_n of same radius δ. At large enough n these circles will be located entirely in the strip $-h < Im\lambda < h$. Equation (8) is equivalent to (16) then last relation can be rewritten as

$$e^{i(\lambda - A + B)} - 1 - o(1) = 0, \tag{20}$$

where numbers A, B introduced by the equalities $k(0+0) = e^{iA}$, $k(1-0) = e^{iB}$.

Outside the circles Γ_n function

$$f = e^{i(\lambda - A + B)} - 1$$

is bounded from below by a positive number. If $\zeta = \lambda - A + B$, then $f = e^{i\zeta} - 1$. The circle Γ_n go over in a circle Γ_n' same radius δ around points $\zeta_n = 2\pi n$. Since $f = f(\zeta)$ is a periodic function with period 2π then it suffices to prove it is bounded from below in the region D, where D is bounded by direct $h = \pm\pi$ and circle Γ_0'. But in this region the function $f(\zeta)$ does not vanish at zero. At $|Im\lambda|$ sufficiently large $f(\zeta)$ is bounded from below

$$\lim_{Im\lambda\to+\infty} |f(\zeta)| = \infty, \lim_{Im\lambda\to-\infty} |f(\zeta)| = 1.$$

We have our statement. It follows that for sufficiently large $|\lambda|$ function $\Delta(\lambda)$ has not zeros outside the circles of Γ_n.

Let m be the minimum of the function $|e^{i(\lambda-A+B)} - 1|$ on Γ_n. At the same circle $|o(1)| < m$ for sufficiently large $|\lambda|$. It follows from the Rouche's theorem in [7, p. 78] that equation (18) has the inside Γ_n same roots as it there is an equation $e^{i(\lambda-A+B)} - 1 = 0$, i.e. exactly one root.

We obtain (10), solving the equation $e^{i(\lambda-A+B)} - 1 = 0$ and using (19).

We claim that for sufficiently large h out of strip $-h < Im\lambda < h$ has not zeros $\Delta(\lambda)$ and the true estimate (11). We consider the $\Delta(\lambda)$ at $Im\lambda \to +\infty$, $Im\lambda \to -\infty$ to prove (11). Hence at $Im\lambda \to +\infty$, $\Delta(\lambda) \to (-i)^{\alpha-1}\lambda^{\alpha}$, at $Im\lambda \to -\infty$, $\Delta(\lambda) \to (i)^{\alpha-1}\lambda^{\alpha}e^{i\lambda}$. This means that there are not zeros outside the strip $\Delta(\lambda)$ and true estimate (11).

The proof is complete. □

In Theorem 4.1 shows the asymptotic behavior of eigenvalues of a special choice of $\sigma(x)$ in the form of (9). We have in general, if $\sigma(x) \in L_2(0,1)$, then rightly statement

Theorem 4.2. *For any $\sigma(x) \in L_2(0,1)$ a trace of L^{-1} is defined by the formula*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \int_0^1 \overline{\sigma(x)}dx$$

Proof. At $\lambda = 0$ resolvent takes the form:

$$R(0)(\bullet) =< (\bullet), \sigma(t) > +i \int_0^x (\bullet)dt.$$

We obtain the required relation calculating the traces from the left-hand and right-hand sides of latter identity.

The proof is complete. □

Therefore series $\sum \frac{1}{\lambda_k}$ is converges conditionally. We can also write explicitly sums form $\sum\limits_{k=1}^{\infty} \frac{1}{\lambda_k^j}$ for $j \geq 1$.

5. System of root functions of L and the corresponding biorthogonal system

Let λ_n be an eigenvalue of L of multiplicity m_n. Then

$$\Delta(\lambda_n) = 0, \Delta^{'}(\lambda_n) = 0, ..., \Delta^{(m_n-1)}(\lambda_n) = 0, \Delta^{(m_n)}(\lambda_n) \neq 0.$$

In [12, p. 445] give a decomposition theorem. It follows from that for some $\delta > 0$ the projector $P_n : L_2[0,1] \to Ker(L - \lambda_n I)^{m_n}$ is a residue of the resolvent at the singular point λ_n

$$(P_n f)(x) = -\frac{1}{2\pi i} \oint_{|\lambda-\lambda_n|=\delta} (L - \lambda I)^{-1} f(x)d\lambda.$$

Let us remark that $i\int_0^x e^{i\lambda(x-t)}f(t)dt$ is an entire function in λ. Recalling of representation resolvent (7) from Theorem 3.1 form of the projector P_n can be refined

$$-\frac{1}{2\pi i}\oint_{|\lambda-\lambda_n|=\delta}(L-\lambda I)^{-1}f(x)d\lambda = -\operatorname*{res}_{\lambda_n}(L-\lambda I)^{-1}f(x).$$

Applying formula (7) and properties of residue in the last equality, we have

$$-\frac{1}{2\pi i}\oint_{|\lambda-\lambda_n|=\delta}(L-\lambda I)^{-1}f(x)d\lambda =$$

$$= -\sum_{j=0}^{m_n-1}\frac{1}{(m_n-1-j)!}\lim_{\lambda\to\lambda_n}\frac{\partial^{m_n-1-j}}{\partial\lambda^{m_n-1-j}}\left(\frac{(\lambda-\lambda_n)^{m_n} < f(t), M_{\overline{\lambda}}(t) >}{\Delta(\lambda)}\right)\left(e^{i\lambda_n x}\frac{(ix)^j}{j!}\right) =$$

$$= \sum_{j=0}^{m_n-1} < f(t), -\frac{1}{(m_n-1-j)!}\lim_{\overline{\lambda}\to\overline{\lambda}_n}\frac{\partial^{m_n-1-j}}{\partial\overline{\lambda}^{m_n-1-j}}\left(\frac{(\overline{\lambda}-\overline{\lambda}_n)^{m_n}M_{\overline{\lambda}}(t)}{\overline{\Delta(\lambda)}}\right) > \left(e^{i\lambda_n x}\frac{(ix)^j}{j!}\right). \quad (21)$$

We introduce of notations:

$$y_{nj}(x) = e^{i\lambda_n x}\frac{(ix)^j}{j!}, j = 0, 1, \ldots, m_n - 1,$$

$$E_n = \{e^{i\lambda_n x}, e^{i\lambda_n x}\frac{ix}{1!}, e^{i\lambda_n x}\frac{(ix)^2}{2!}, \ldots, e^{i\lambda_n x}\frac{(ix)^{m_n-1}}{(m_n-1)!}\}.$$

It follows from [7, p. 29] that $dimE_n = m_n$.

In the sequel we investigate the properties of the system of functions E, where

$$E = \{E_n : \lambda_n \text{ is the eigenvalue of L}\}.$$

Lemma 5.1. *The elements of the chain E_n satisfy the following differential equations*

$$-iy'_{n,s}(x) = \lambda_n y_{n,s}(x) + y_{n,s-1}(x), s = 1, ..., m_n - 1, \quad (22)$$

$$-iy'_{n,0}(x) = \lambda_n y_{n,0}(x) \quad (23)$$

and nonlocal boundary condition (4).

Proof. Let $j = 0$, i.e. $y_{n,0}(x) = e^{i\lambda_n x}$. Then the justice of equation (23) and boundary condition is obvious. We have $y_{n,j}(x) = e^{i\lambda_n x}\frac{(ix)^j}{j!}$ for $m_n - 1 \geq j \geq 1$. We find derivative of the first order and multiply by $-i$:

$$-iy'_{n,j}(x) = \lambda_n e^{i\lambda_n x}\frac{(ix)^j}{j!} + e^{i\lambda_n x}\frac{(ix)^{j-1}}{(j-1)!} = \lambda_n y_{n,j}(x) + y_{n,j-1}(x).$$

We check of boundary condition (4). Denote by $U(y) = y(0) - \int_0^1(-y'(x))\overline{\sigma(x)}dx.$

The rightly following relations hold $U(y_{n,j}) = \Delta^{(j)}(\lambda_n)$ for $m_n - 1 \geq j \geq 0$. Since $\Delta^{(j)}(\lambda_n) = 0$ then boundary condition (4) for $y_{n,j}(x)$ holds.

The proof is complete. □

Lemma 5.2. *For arbitrary complex numbers λ and μ, we have the identity:*

$$< e^{i\lambda x}, M_{\overline{\mu}}(x) > \equiv -\frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu}. \quad (24)$$

Proof. We calculate for arbitrary λ and μ the following scalar product

$$\lambda < e^{i\lambda x}, M_{\overline{\mu}}(x) >=< -i\frac{\partial}{\partial x}(e^{i\lambda x}), M_{\overline{\mu}}(x) > .$$

Given formula (1), we obtain

$$\lambda < e^{i\lambda x}, M_{\overline{\mu}}(x) >=< -i\frac{\partial}{\partial x}e^{i\lambda x}, \sigma(x) > -i^2\mu\int_0^1 \frac{\partial}{\partial x}e^{i\lambda x}\left(\int_x^1 e^{i\mu(\tau-x)}\overline{\sigma(\tau)}d\tau\right)dx.$$

We transform the right-hand side of the last relation using the formula of integration by parts

$$\lambda < e^{i\lambda x}, M_{\overline{\mu}}(x) >= \Delta(\mu) - \Delta(\lambda) + \mu\int_0^1 e^{i\lambda x}\overline{\sigma(x)}dx + i\mu^2\int_0^1 e^{i\mu\tau}\overline{\sigma(\tau)}\left(\int_0^\tau e^{i\lambda x}dx\right)d\tau.$$

Once again, we use the formula for integration by parts

$$\lambda < e^{i\lambda x}, M_{\overline{\mu}}(x) >= \Delta(\mu) - \Delta(\lambda) + \mu\frac{1-\Delta(\lambda)}{\lambda} - \frac{\mu^2}{\lambda-\mu}\frac{1-\Delta(\mu)}{\mu} + \frac{\mu^2}{\lambda-\mu}\int_0^1 e^{i\lambda\tau}\overline{\sigma(\tau)}d\tau.$$

Given relation (2), we have

$$\lambda < e^{i\lambda x}, M_{\overline{\mu}}(x) >= \Delta(\mu) - \Delta(\lambda) + (1-\Delta(\lambda))\frac{\mu}{\lambda}(1+\frac{\mu}{\lambda-\mu}) - \frac{\mu}{\lambda-\mu}(1-\Delta(\mu)).$$

Hence

$$\lambda < e^{i\lambda x}, M_{\overline{\mu}}(x) >= \frac{\lambda}{\lambda-\mu}(\Delta(\mu)-\Delta(\lambda)).$$

It follows from last relation that rightly of identity (24).

The proof is complete. □

Analysis of (21) leads to the following notation:

$$E_n^{'} = \{h_{n,0}(x), h_{n,1}(x), \ldots, h_{n,m_n-1}(x)\}, \tag{25}$$

where

$$h_{n,m_n-1-j}(x) = -\frac{1}{(m_n-1-j)!}\lim_{\lambda\to\lambda_n}\frac{\partial^{m_n-1-j}}{\partial\lambda^{m_n-1-j}}\left(\frac{(\lambda-\lambda_n)^{m_n}M_{\overline{\lambda}}(x)}{\Delta(\lambda)}\right), j = 0,1,...,m_n-1. \tag{26}$$

We introduce the following family of functions

$$E^{'} = \{E_n^{'} : \lambda_n \text{is arbitrary eigenvalue of L}\}.$$

We prove main results.

Proof of Theorem 1.1. We consider two of eigenvalues λ_s and λ_n. They correspond to pairs (s,p) and (n,j), where $p = 0,1,\ldots,m_s-1$ and $j = 0,1,\ldots,m_n-1$. Note that the scalar product

$$< y_{s,p}(x), h_{n,m_n-1-j}(x) >=$$
$$= -\lim_{\lambda\to\lambda_s}\lim_{\mu\to\lambda_n}\frac{1}{p!}\frac{\partial^p}{\partial\lambda^p}\frac{1}{(m_n-1-j)!}\frac{\partial^{m_n-1-j}}{\partial\mu^{m_n-1-j}}\left(< e^{i\lambda x}, M_{\overline{\mu}}(x) > \frac{(\mu-\lambda_n)^{m_n}}{\Delta(\mu)}\right).$$

Recalling Lemma 5.2, we obtain

$$< y_{s,p}(x), h_{n,m_n-1-j}(x) >=$$
$$= -\lim_{\lambda\to\lambda_s}\lim_{\mu\to\lambda_n}\frac{1}{p!}\frac{\partial^p}{\partial\lambda^p}\frac{1}{(m_n-1-j)!}\frac{\partial^{m_n-1-j}}{\partial\mu^{m_n-1-j}}\left(\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_n)^{m_n}}{\Delta(\mu)}\right). \tag{27}$$

We introduce the notation

$$H_{n,k}(\lambda)=\lim_{\mu\to\lambda_n}\frac{1}{k!}\frac{\partial^k}{\partial\mu^k}\left(\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_n)^{m_n}}{\Delta(\mu)}\right). \quad (28)$$

Consider the function

$$F(\mu)=\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}\frac{(\mu-\lambda_n)^{m_n}}{\Delta(\mu)}.$$

We decompose it into neighborhood of $\mu=\lambda_n$ in Taylor series. Then

$$F(\mu)=H_{n,0}(\lambda)+H_{n,1}(\lambda)(\mu-\lambda_n)+H_{n,2}(\lambda)(\mu-\lambda_n)^2+\cdots+H_{n,m_n-1}(\lambda)(\mu-\lambda_n)^{m_n-1}+\ldots.$$

$H_{n,k}(\lambda)$ is the kth coefficient in Taylor corresponding decomposition in the neighborhood of $\mu=\lambda_n$. A direct calculation of the coefficient of Taylor series for function $F(\mu)$ leads to the following formula at $k=0,1,\ldots,m_n-1$

$$H_{n,k}(\lambda)=\Delta(\lambda)\left(A_{n,m_n-1}\frac{1}{(\lambda-\lambda_n)^{k+1}}+A_{n,m_n-2}\frac{1}{(\lambda-\lambda_n)^{k}}+\cdots+A_{n,m_n-k-1}\frac{1}{\lambda-\lambda_n}\right). \quad (29)$$

The numbers $A_{n,m_n-1},\ldots,A_{n,0}$ are determined from the identity

$$\frac{1}{\Delta(\mu)}\equiv\frac{A_{n,m_n-1}}{(\mu-\lambda_n)^{m_n}}+\frac{A_{n,m_n-2}}{(\mu-\lambda_n)^{m_n-1}}+\cdots+\frac{A_{n,0}}{\mu-\lambda_n}+\sum_{q=0}^{\infty}B_{n,q}(\mu-\lambda_n)^q.$$

If $\lambda_s\neq\lambda_n$, then from (27), (28), and (29) at $p=0,1,\ldots,m_n-1$, we have

$$< y_{s,p}(x),h_{n,m_n-1-j}(x) >=\lim_{\lambda\to\lambda_s}\frac{1}{p!}\frac{\partial^p}{\partial\lambda^p}H_{n,k}(\lambda)=$$

$$=\lim_{\lambda\to\lambda_s}\frac{1}{p!}\frac{\partial^p}{\partial\lambda^p}\Delta(\lambda)\sum_{i=1}^{k+1}\frac{A_{n,m_n-k+i-2}}{(\lambda-\lambda_n)^i}=\frac{1}{p!}\Delta^{(p)}(\lambda_s)\sum_{i=1}^{k+1}\frac{A_{n,m_n-k+i-2}}{(\lambda_s-\lambda_n)^i}=0,$$

since $\Delta^{(p)}(\lambda_s)=0$.

In the sequel we consider the case $\lambda_s=\lambda_n$. We transform the right side of (29).

$$H_{n,k}(\lambda)=\Delta(\lambda)\sum_{i=1}^{k+1}\frac{A_{n,m_n-k+i-2}}{(\lambda_s-\lambda_n)^i}=\Delta(\lambda)\times$$

$$\times\left(A_{n,m_n-1}\frac{1}{(\lambda-\lambda_n)^{k+1}}+A_{n,m_n-2}\frac{1}{(\lambda-\lambda_n)^{k}}+\cdots+A_{n,m_n-k-1}\frac{1}{\lambda-\lambda_n}\right)=$$

$$=\Delta(\lambda)(\lambda-\lambda_n)^{m_n-k-1}\times$$

$$\times\left(A_{n,m_n-1}\frac{1}{(\lambda-\lambda_n)^{m_n}}+A_{n,m_n-2}\frac{1}{(\lambda-\lambda_n)^{m_n-1}}+\cdots+A_{n,m_n-k-1}\frac{1}{(\lambda-\lambda_n)^{m_n-k}}\right)=$$

$$=\Delta(\lambda)(\lambda-\lambda_n)^{m_n-k-1}\times$$

$$\times\left(\frac{1}{\Delta(\lambda)}-A_{n,m_n-2}\frac{1}{(\lambda-\lambda_n)^{m_n-k-1}}-\cdots-A_{n,0}\frac{1}{\lambda-\lambda_n}-\sum_{q=m_n}^{\infty}B_{nq}(\lambda-\lambda_n)^q\right)=$$

$$=(\lambda-\lambda_n)^{m_n-k-1}+\sum_{q=m_n}^{\infty}c_{nq}^k(\lambda-\lambda_n)^q, s=0,1.\ldots,m_n-1. \quad (30)$$

From identities (27), (28), and (29), we have

$$< y_{s,p}(x),h_{n,m_n-1-j}(x) >=\frac{1}{p!}\lim_{\lambda\to\lambda_n}\frac{\partial^p}{\partial\lambda^p}H_{n,m_n-1-j}.$$

Given of relation (30), we obtain

$$< y_{s,p}(x), h_{n,m_n-1-j}(x) >= \frac{1}{p!}\lim_{\lambda\to\lambda_n}\frac{\partial^p}{\partial\lambda^p}\left((\lambda-\lambda_n)^{m_n-k-1}+\sum_{q=0}^{\infty}c_{nq}^k(\lambda-\lambda_n)^q\right).$$

It follows from last relation that there exist a required statement at $\lambda_s = \lambda_n$.

The proof is complete.

It follows from Lemma 5.1 and Theorem 1.1 that the system of functions E is the system of root functions of L and system of functions E' is the biorthogonal to the system E. Consequently the system of functions E is a minimal system of functions [6, p. 171].

In [14] it is proved only the existence of biorthogonal systems of functions without being explicit formulas. In Theorem 1.1 in source terms of boundary value problem explicitly written biorthogonal system of functions.

6. Functional series generated by the system of root functions

Since the spectrum of L is discrete then there exist indefinitely increasing sequence $\{R_N\}$ of radii such that points of spectrum of operator do not lie on appropriate the circles $|\lambda| = R_N$. Let $A_N = \{\lambda \in C : |\lambda| = R_N\}$ and $\sigma(L) = \{\lambda_1, \lambda_2, \dots\}$. Further on assume that R_N are chosen so that $dist(A_N, \sigma(L)) > \delta > 0$ for all N. We consider of the subsequence of partial sums corresponding to the selected circles

$$(S_N f)(x) = -\frac{1}{2\pi i}\oint_{|\lambda|=R_N}(L-\lambda I)^{-1}f(x)d\lambda \tag{31}$$

for any function $f(\cdot)$ from the space $L_2(0,1)$. Substitute in (31) relation (6), according to the residue theorem and taking into account relation (21), we write the partial sum in the form

$$(S_N f)(x) = \sum_{|\lambda_n|<R_N}\sum_{j=0}^{m_n-1} < f, h_{m_n-1-j} > y_{n,j}(x). \tag{32}$$

According to the right-hand side of (32) the sequence of Fourier coefficients for $f \in L_2(0,1)$ with respect to the system E is defined by

$$c(f) = \{c_{n,k}(f) =< f, h_{n,k} >, k = 0, 1, \dots, m_n - 1, \lambda_n \quad \text{is arbitrary eigenvalue of L}\}.$$

The question arises about the convergence and a summability of subsequence $\{S_N f\}$ with respect to the norm $L_2(0,1)$, the behavior of the Fourier coefficients $c_{n,k}$, and so on - questions suggested by theory trigonometric of Fourier series [5], [6], [14].

We need the following lemma to prove Theorem 1.2.

Lemma 6.1. *Let $\sigma(x) \in L_2[0,1]$ be and the operator L has only simple eigenvalues lying in a horizontal strip. Then for the functions $f(x) \in W_2^p[0,\ 1]$ are the Fourier coefficients with respect to the system of root functions of L have the following representation*

$$c_{n,0}(f) = \frac{f(0)}{\lambda_n\Delta'(\lambda_n)} + \frac{1}{\Delta'(\lambda_n)}\int_0^1 e^{i\lambda_n\tau}\left(\int_\tau^1 f'(\xi-\tau)\overline{\sigma(\xi)}d\xi\right)d\tau,\ \textit{for } p = 1,$$

$$c_{n,0}(f) = \frac{1}{\Delta'(\lambda_n)}\left(\sum_{k=2}^{p}\frac{U(f^{(k-2)}(x))}{i^{k-2}\lambda_n^{k-1}} + (-1)^p\frac{f^{(p)}(0)}{i^{p-1}\lambda_n^p} + \frac{1}{(i\lambda_n)^p}\int_0^1 e^{i\lambda_n\tau}\left(\int_\tau^1 f^{(p)}(\xi-\tau)\overline{\sigma(\xi)}d\xi\right)d\tau\right).$$

Proof. Subject to the conditions of Lemma 6.1, the Fourier coefficients have in the following form

$$c_{n,0}(f) =< f(x), h_{n,0}(x) > .$$

We transform the latter identity according to formula (26) to the form

$$c_{n,0}(f) = -\lim_{\lambda\to\lambda_n} \frac{\lambda-\lambda_n}{\Delta(\lambda)} < f(x),\, M_{\overline{\lambda}}(x) > .$$

Recalling formula (1), the last relation reduces to the form:

$$c_{n,0}(f) = -\lim_{\lambda\to\lambda_n} \frac{1}{\Delta'(\lambda_n)} \left[\int_0^1 f(x)\overline{\sigma(x)}dx + i\lambda \int_0^1 f(x) \left(\int_x^1 e^{i\lambda(t-x)}\overline{\sigma(t)}dt \right) dx \right].$$

We change variables in the second integral of the last ratio

$$c_{n,0}(f) = -\lim_{\lambda\to\lambda_n} \frac{1}{\Delta'(\lambda_n)} \left[\int_0^1 f(x)\overline{\sigma(x)}dx + i\lambda \int_0^1 e^{i\,\lambda\,\tau} \left(\int_\tau^1 f(\xi-\tau)\overline{\sigma(\xi)}d\xi \right) d\tau \right].$$

We use the formula of integration by parts in the second integral

$$c_{n,0}(f) = -\frac{f(0)}{\Delta'(\lambda_n)} \int_0^1 e^{i\lambda_n\tau}\overline{\sigma(\tau)}d\tau - \frac{1}{\Delta'(\lambda_n)} \lim_{\lambda\to\lambda_n} \int_0^1 e^{i\lambda\tau} \left(\int_\tau^1 f'(\xi-\tau)\overline{\sigma(\xi)}d\xi \right) d\tau.$$

We have

$$c_{n,0}(f) = \frac{f(0)}{\lambda_n\Delta'(\lambda_n)} + \frac{1}{\Delta'(\lambda_n)} \lim_{\lambda\to\lambda_n} \int_0^1 e^{i\lambda\tau} \left(\int_\tau^1 f'(\xi-\tau)\overline{\sigma(\xi)}d\xi \right) d\tau,$$

since

$$\int_0^1 e^{i\lambda_n\tau}\overline{\sigma(\tau)}d\tau = \frac{1}{\lambda_n}. \tag{33}$$

We twice applying the formula of integration by parts to the last limit relation and using (33), respectively, we obtain

$$c_{n,0}(f) = \frac{1}{\Delta'(\lambda_n)} \left(\frac{1}{\lambda_n} \left(f(0) - \int_0^1 (-if'(_\xi))\overline{\sigma(\xi)}d\xi \right) + \frac{f'(0)}{i\lambda_n^2} \right) +$$

$$+\frac{1}{\Delta'(\lambda_n)} \lim_{\lambda\to\lambda_n} \frac{1}{i\lambda} \int_0^1 e^{i\lambda\tau} \left(\int_\tau^1 f''(\xi-\tau)\overline{\sigma(\xi)} \right) d\tau =$$

$$= \frac{1}{\Delta'(\lambda_n)} \left(\frac{1}{\lambda_n} U(f(\xi)) + \frac{f'(0)}{i\lambda_n^2} \right) + \frac{1}{\Delta'(\lambda_n)} \lim_{\lambda\to\lambda_n} \frac{1}{i\lambda} \int_0^1 e^{i\lambda\tau} \left(\int_\tau^1 f''(\xi-\tau)\overline{\sigma(\xi)}d\xi \right) d\tau;$$

$$c_{n,0}(f) = \frac{1}{\Delta'(\lambda_n)} \left(\frac{1}{\lambda_n} U(f(\xi)) + \frac{1}{i\lambda_n^2} U(f'(\xi)) - \frac{f''(0)}{i^2\lambda_n^3} \right) +$$

$$+\frac{1}{\Delta'(\lambda_n)} \lim_{\lambda\to\lambda_n} \frac{1}{(i\lambda)^2} \int_0^1 e^{i\lambda\tau} \left(\int_\tau^1 f^{(3)}(\xi-\tau)\overline{\sigma(\xi)}d\xi \right) d\tau.$$

We applying the formula for integration by parts to the last limit relation is p times, we obtain the required representation.

The proof is complete. □

Proof of Theorem 1.2. Since $k = 2, 3, \dots, p$, $U(f^{(k-2)}(x)) = 0$ and by the Riemann-Lebesgue Lemma for $n \to \infty$, $\int\limits_0^1 e^{i\lambda_n \tau} \left(\int\limits_\tau^1 f^{(p)}(\xi - \tau)\overline{\sigma(\xi)}d\xi \right) d\tau \to 0$, we have the proff of Theorem 1.2.

The proof is complete.

7. Completeness of system of root functions

In this section, we study of the completeness with respect to E in the functional space $L_2(0,1)$. We need the following lemma to prove Theorem 1.3.

Lemma 7.1. *Suppose there exist non-zero limits*

$$\lim_{\varepsilon \to +0} \frac{1}{\varepsilon} \int\limits_0^\varepsilon (i + \sigma(x))dx = \alpha_1,$$

$$\lim_{\varepsilon \to 1-0} \frac{1}{1-\varepsilon} \int\limits_\varepsilon^1 \sigma(x)dx = \alpha_2.$$

Then for any complex number μ rightly the limit relations

$$\lim_{R_N \to \infty} \|e^{i\mu x} - \sum_{|\lambda_n| < R_N} \sum_{j=0}^{m_n - 1} < e^{i\mu t}, h_{n, m_n - 1 - j}(t) > y_{n,j}(x)\| = 0.$$

Proof. The set of functions

$$D = \{e^{i\mu x} : \mu \text{ is arbitrary complex number}\}$$

is dense in $L_2(0,1)$ as the system $\{e^{in\pi x} : n = 0, \pm 1, \pm 2, \dots\}$ is an orthonormal basis in $L_2(0,1)$. Therefore it suffices to approximate arbitrary element of D with the desired degree of accuracy by linear combinations from E. We consider to this

$$S_N e^{i\mu x} = -\frac{1}{2\pi i} \oint_{|\lambda| = R_N} \frac{< e^{i\mu t}, M_{\overline{\lambda}}(t) >}{\Delta(\lambda)} e^{i\lambda x} d\lambda.$$

Given of Lemma 5.2, we rewrite the last relation

$$S_N e^{i\mu x} = \frac{1}{2\pi i} \oint_{|\lambda| = R_N} \frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu} \frac{e^{i\lambda x}}{\Delta(\lambda)} d\lambda = \frac{1}{2\pi i} \oint_{|\lambda| = R_N} \frac{e^{i\lambda x}}{\lambda - \mu} d\lambda - \frac{1}{2\pi i} \oint_{|\lambda| = R_N} \frac{\Delta(\mu)}{\Delta(\lambda)} \frac{e^{i\lambda x}}{\lambda - \mu} d\lambda.$$

Using the Cauchy integral formula, we obtain

$$S_N e^{i\mu x} = e^{i\mu x} - \frac{1}{2\pi i} \oint_{|\lambda| = R_N} \frac{\Delta(\mu)}{\Delta(\lambda)} \frac{e^{i\lambda x}}{\lambda - \mu} d\lambda.$$

Denote by $Q_N(x, \mu)$ the error of $S_N e^{i\mu x} - e^{i\mu x}$. We have the integral representation for the error

$$Q_N(x, \mu) = \frac{1}{2\pi i} \oint_{|\lambda| = R_N} \frac{\Delta(\mu)}{\Delta(\lambda)} \frac{e^{i\lambda x}}{\lambda - \mu} d\lambda.$$

We find sufficient conditions on $\sigma(x) \in L_2(0,1)$ such that

$$\lim_{R_N \to \infty} \|Q_N(x, \mu)\| = 0,$$

where $\| \cdot \|$ is norm in the sense of $L_2(0,1)$.

We consider the norm of error

$$\|Q_N(x, \mu)\| = \sup_{\|g\| = 1} | < Q_N(\cdot, \mu); g(\cdot) > |.$$

Given integral representation for the error, we have

$$\|Q_N(x,\mu)\| = \sup_{\|g\|=1}\left|\frac{1}{2\pi i}\oint_{|\lambda|=R_N}\frac{\Delta(\mu)}{\Delta(\lambda)(\lambda-\mu)}\left(\int_0^1 e^{i\lambda x}\overline{g(x)}dx\right)d\lambda\right|.$$

We transform of the last equality

$$\|Q_N(x,\mu)\| = \sup_{\|g\|=1}\left|\frac{1}{2\pi i}\oint_{|\lambda|=R_N}\frac{\Delta(\mu)e^{\varphi(\lambda)}}{\Delta(\lambda)(\lambda-\mu)}\left(e^{-\varphi(\lambda)}\left(\int_0^1 e^{i\lambda x}\overline{g(x)}dx\right)d\lambda\right)\right|,$$

where $\varphi(\lambda)$ will be chosen bellow.

Applying the Cauchy-Schwarz inequality, we obtain

$$\|Q_N(x,\mu)\| \le \frac{|\Delta(\mu)|}{2\pi}\left(\oint_{|\lambda|=R_N}\left|\frac{e^{\varphi(\lambda)}}{\Delta(\lambda)(\lambda-\mu)}\right|^2|d\lambda|\right)^{\frac{1}{2}}\times$$

$$\times\sup_{\|g\|=1}\left(\oint_{|\lambda|=R_N}\left|e^{-\varphi(\lambda)}\int_0^1 e^{i\lambda x}\overline{g(x)}\right|^2|d\lambda|\right)^{\frac{1}{2}}. \tag{34}$$

In order to select the function $\varphi(\lambda)$ is necessary to estimate $|\Delta(\lambda)|$ on the circle $|\lambda| = R_N$ for $R_N \to \infty$.

Suppose that there exist the limit

$$\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\int_0^{\varepsilon}\overline{(i+\sigma(x))}dx = \alpha_1 \ne 0$$

and $0 < \theta < \pi$. For an entire function $\Delta(\lambda)$ for$\lambda = R_N e^{i\theta}$ rightly the estimate from below

$$|\Delta(\lambda)| \ge c_1 > 0. \tag{35}$$

Suppose there exists

$$\lim_{\varepsilon\to1-0}\frac{1}{1-\varepsilon}\int_{\varepsilon}^{1}\overline{\sigma(x)}dx = \alpha_2 \ne 0$$

and $-\pi < \theta < 0$. For the entire function $\Delta(\lambda)$ at $\lambda = |\lambda|e^{i\theta}$ rightly the estimate from below

$$|\Delta(\lambda)| \ge c_2 e^{-R_N sin\theta}. \tag{36}$$

We choose $\varphi(\lambda) = \begin{cases} 0, & \text{for } 0<\theta<\pi; \\ |Im\lambda|, & \text{for } -\pi<\theta<0, \end{cases}$
where $\lambda = |\lambda|e^{i\theta}$. We consider a function

$$\Phi(\mu) = \oint_{|\lambda|=R_N}\left|\frac{e^{\varphi(\lambda)}}{\Delta(\lambda)(\lambda-\mu)}\right|^2|d\lambda|,$$

$$\Phi(\mu) = R_N\int_0^{\pi}\left|\frac{1}{\Delta(R_Ne^{i\theta})(R_Ne^{i\theta}-\mu)}\right|^2d\theta + R_N\int_{-\pi}^{0}\left|\frac{e^{-R_N sin\theta}}{\Delta(R_Ne^{i\theta})(R_Ne^{i\theta}-\mu)}\right|^2d\theta.$$

Using estimates (35) and (36), we obtain the inequalities

$$\Phi(\mu) \leq \frac{R_N}{c_1^2} \int\limits_0^{\pi} \frac{1}{|R_N e^{i\theta} - \mu|^2} d\theta + \frac{R_N}{c_2^2} \int\limits_{-\pi}^{0} \frac{1}{|R_N e^{i\theta} - \mu|^2} d\theta \leq$$

$$\frac{R_N}{c_1^2} \int\limits_0^{\pi} \frac{1}{(R_N - |\mu|)^2} d\theta + \frac{R_N}{c_2^2} \int\limits_{-\pi}^{0} \frac{1}{(R_N - |\mu|)^2} d\theta = \frac{1}{R_N c_1^2} \int\limits_0^{\pi} \frac{1}{(1 - \frac{|\mu|}{R_N})^2} d\theta +$$

$$+ \frac{1}{R_N c_2^2} \int\limits_{-\pi}^{0} \frac{1}{(1 - \frac{|\mu|}{R_N})^2} d\theta \to 0, for R_N \to \infty. \tag{37}$$

We introduce a new function

$$F(\lambda) = e^{-\varphi(\lambda)} \int\limits_0^1 e^{i\lambda x} \overline{g(x)} dx$$

then $Im\lambda > 0$ function $F(\lambda)$ belongs to H_+^2. It follows from Lemma 1.2.5 in [13], we have

$$|\lambda| \int\limits_0^{\pi} \left| F(|\lambda| e^{i\theta}) \right|^2 d\theta \leq c_3 \sup_{Im\lambda > 0} \left(\int\limits_{-\infty}^{\infty} |F(Re\lambda + iIm\lambda)|^2 dRe\lambda \right) \tag{38}$$

where c_3 is independent of $|\lambda|$ and $g(x)$.

Since $g \in L_2(0,1)$, at $Im\lambda < 0$ the function $F(\lambda)$ belongs to H_-^2. It follows from Lemma 1.2.5 [13], we have

$$|\lambda| \int\limits_{-\pi}^{0} |F(|\lambda| e^{i\theta})|^2 d\theta < c_4 \left(\int\limits_{-\infty}^{\infty} |F(Re\lambda + iIm\lambda)|^2 dRe\lambda \right), \tag{39}$$

where c_4 is independent of $|\lambda|$ and $g(x)$.

It follows from (37), (38), (39), and (34) that the limit relation

$$\lim_{R_N \to \infty} \|Q_N(\cdot, \mu)\| = 0.$$

The proof is complete. □

Proof of Theorem 1.3. From Lemma 7.1 and the fact that $\overline{D} = L_2(0,1)$ implies the proof of Theorem 1.3.

The proof is compete.

8. Conclusions

In this paper, we investigate of the system of root functions generated by first order differential operator with nonlocal boundary condition. Let us prove minimality of the obtained of system in $L_2(0,1)$. The results can spread for second order differential operator

$$\widetilde{\ell}(y) \equiv -y''(x) + q(x)y(x), 0 < x < 1,$$

$$V_\nu(y) \equiv y^{(\nu-1)}(0) = \int\limits_0^1 (-y''(x) + q(x)y(x)) \overline{\sigma_\nu(x)} dx, \nu = 1, 2,$$

where $q(x)$ is a continuous function, $\sigma_\nu(x) \in L_2(0,1),\ \nu = 1, 2.$

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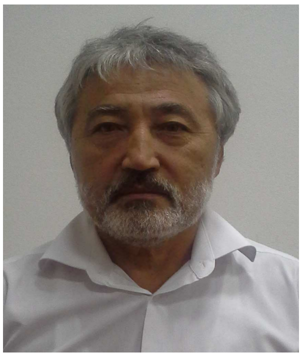

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