TWMS J. Pure Appl. Math., V.5, N.2, 2014, pp.185-193

DEGREE LISTS OF K-BIPARTITE HYPERTOURNAMENTS

S. PIRZADA[1], ZHOU GUOFEI[2]

ABSTRACT. In this paper, we obtain necessary and sufficient conditions for a pair of non-decreasing sequences of non-negative integers to be the degree lists of some k-bipartite hypertournament.

Keywords: hypertournaments, bipartite hypertournaments, degree lists

AMS Subject Classification: 05C65

1. Introduction

Given two positive integers n and k with $n > k > 1$, a k-hypertournament H on n vertices is a pair (V, A), where V is a set of n vertices and A is a set of k-tuples of vertices, called arcs, such that for any k-subset W of V, A contains exactly one of the $k!$ possible k-tuples whose entries belong to W. Clearly, a 2-hypertournament is a tournament. If $e = (x_1, x_2, \cdots, x_k)$, then $\{x_1, x_2, \cdots, x_k\}$ is called the underlying vertex set of e, and is denoted by V_e. A k-hypertournament $H(V, A)$ is said to be transitive if we can label $V(H)$ by $v_1, v_2, \ldots, v_n$ in such a way that: if $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, then $(v_{i_1}, v_{i_2}, \cdots, v_{i_k})$ is an arc in H.

Bipartite hypergraphs are generalization of bipartite graphs. If $U = \{u_1, u_2, \cdots, u_m\}$ and $V = \{v_1, v_2, \cdots, v_n\}$ are vertex sets, then each edge of a bipartite hypergraph is a subset of the vertex sets, containing at least one vertex from U and at least one vertex from V. If each edge has exactly k vertices, the bipartite hypergraph is called a k-bipartite hypergraph. A k-bipartite hypertournament is a complete k-bipartite hypergraph with each edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. A k-bipartite hypertournament is said to be transitive, if it is obtained from a transitive k-hypertournament by deleting all the edges contained in U or V.

An oriented k-hypergraph is a k-hypergraph with each edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. Let $a = (x_1, x_2, \ldots, x_k)$ be an arc of an oriented k-hypergraph H. We call x_i the i-th entry of a; the $(i+1)$-th entry of a, x_{i+1}, is called the successor of x_i, and x_i the predecessor of x_{i+1} in a, $1 \leq i \leq k-1$. It is obvious that x_k has no successor, and x_1 has no predecessor in a. Define a function ρ on a by

$$\rho(a, x) = \begin{cases} k - i, & \text{if } x \in a \text{ and } x \text{ is the } i\text{-th entry of } a \\ 0, & \text{if } x \notin a \end{cases}$$

For $v \in V(H)$, we denote $d_H^+(v) = \sum_{a \in H} \rho(a, v)$ (or simply $d^+(v)$) the degree of v in H.

[1]Department of Mathematics, University of Kashmir,Srinagar, Kashmir, India
e-mail:pirzadasd@kashmiruniversity.ac.in

[2]Department of Mathematics, Nanjing University,Nanjing, P.R. China
e-mail: gfzhou@mail.nju.edu.cn

Manuscript received March 2014.

Let $S = (s_1, s_2, \ldots, s_n)$ be a non-decreasing sequence of non-negative integers. For $1 \le i < j \le n$, we denote $S(s_i^+, s_j^-) = (s_1, s_2, \ldots, s_i + 1, s_{i+1}, \ldots, s_{j-1}, s_j - 1, \ldots, s_n)$, and $S'(s_i^+, s_j^-) = (s'_1, s'_2, \ldots, s'_n)$ will denote a permutation of $S(s_i^+, s_j^-)$ such that $s'_1 \le s'_2 \le \cdots \le s'_n$.

Denote $S(s_i^+) = (s_1, s_2, \cdots, s_{i-1}, s_i + 1, s_{i+1}, \cdots, s_n)$ and $S'(s_i^+)$ a permutation of $S(s_i^+)$ in a non-decreasing order. Similarly, denote $S(s_i^-) = (s_1, s_2, \cdots, s_{i-1}, s_i - 1, s_{i+1}, \cdots, s_n)$ and $S'(s_i^-)$ a permutation of $S(s_i^-)$ in a non-decreasing order. The degree list of a k-hypertournament is a non-decreasing sequence of non-negative integers $[s_1, s_2, \ldots, s_n]$, where each s_i is the score of some vertex in $V(H)$.

The following result is due to Wang and Zhou [9],

Theorem 1.1. *Given $n \ge k > 1$, a non-decreasing sequence $S = [s_1, s_2, \ldots, s_n]$ of non-negative integers is a degree list of some k-hypertournament, if and only if for each r,*

$$\sum_{i=1}^{r} s_i \ge \binom{r}{2}\binom{n-2}{k-2},$$

with equality for $r = n$.

The degree lists of a k-bipartite hypertournament is a pair of non-decreasing sequences of non-negative integers $A = [a_1, a_2, \cdots, a_m]$ and $B = [b_1, b_2, \cdots, b_n]$, where a_i is the degree of vertex $u_i \in U$ and b_j is the degree of vertex $v_j \in V$. In this paper, we obtain necessary and sufficient conditions for a pair of non-decreasing sequences of non-negative integers to be the degree lists of some k-bipartite hypertournament. Various results on scores of hypertournaments, bipartite hypertournaments, score sets in oriented bipartite graphs and mark sets in digraphs can be found in [2, 3, 4, 5, 6, 7, 8, 10, 11].

2. Degree lists in k-bipartite hypertournaments

The following Theorem is the main result of this paper.

Theorem 2.1. *Given non-negative integers m, n and k with $m + n > k > 3$, let $A = [a_i]_1^m$ and $B = [b_j]_1^n$ be non-decreasing sequences of non-negative integers. Then A and B are the degree lists of some k-bipartite hypertournament if and only if*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \ge \binom{p+q}{2}\binom{m+n-2}{k-2} - \binom{p}{2}\binom{m-2}{k-2} - \binom{q}{2}\binom{n-2}{k-2}, \qquad (1)$$

for $1 \le p \le m$, $1 \le q \le n$ with equality when $p = m$ and $q = n$.

In order to prove Theorem 2.1, we need some lemmas and definitions as follows.

Lemma 2.1. [11, Lemma 2.2] *A non-decreasing sequence $S = [s_1, s_2, \ldots, s_n]$ of non-negative integers is a degree list of some transitive k-hypertournament if and only if $s_i = (i-1)\binom{n-2}{k-2}$, for all $1 \le i \le n$.*

Lemma 2.2. *If $A = [a_i]_1^m$ and $B = [b_j]_1^n$ are non-decreasing sequences of non-negative integers, then (A, B) is the degree list of some transitive k-bipartite hypertournament if and only if*

$$a_i = (n-1+i)\binom{n+m-2}{k-2} - (i-1)\binom{m-2}{k-2}, b_j = (j-1)\left[\binom{n+m-2}{k-2} - \binom{n-2}{k-2}\right]$$

or

$$b_j = (m-1+j)\binom{n+m-2}{k-2} - (j-1)\binom{n-2}{k-2}, a_i = (i-1)\left[\binom{n+m-2}{k-2} - \binom{m-2}{k-2}\right].$$

Proof. The proof follows immediately from Lemma 2.1 and the definition of transitive k-bipartite hypertournament.

Lemma 2.3. *Let $A = [a_i]_1^m$ and $B = [b_j]_1^n$ be non-decreasing sequences of non-negative integers satisfying conditions (1). Let t be an integer such that $a_t \neq (n-1+t)\binom{n+m-2}{k-2} - (t-1)\binom{m-2}{k-2}$ and $a_i = (n-1+i)\binom{n+m-2}{k-2} - (i-1)\binom{m-2}{k-2}$ for all $i > t$. Then there is j_0 $(1 \le j_0 \le t-1)$, such that $A' = [a_1, a_2, \cdots, a_{j_0-1}, a_{j_0} - 1, a_{j_0+1}, \cdots, a_{t-1}, a_t + 1, a_{t+1}, \cdots, a_m]$, $B' = B$ are non-decreasing and satisfy (1).*

Proof. We first show that $a_t < (n-1+t)\binom{n+m-2}{k-2} - (t-1)\binom{m-2}{k-2}$. Suppose on the contrary that $a_t > (n-1+t)\binom{n+m-2}{k-2} - (t-1)\binom{n-2}{k-2}$.

On one hand,

$$\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = \binom{m+n}{2}\binom{m+n-2}{k-2} - \binom{m}{2}\binom{m-2}{k-2} - \binom{n}{2}\binom{n-2}{k-2},$$

and on other hand,

$$\begin{aligned}
\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j &= \left(\sum_{i=1}^{t-1} a_i + \sum_{i=1}^{n} b_i\right) + \sum_{i=t}^{m} a_i > \\
&> \binom{t-1+n}{2}\binom{m+n-2}{k-2} - \binom{t-1}{2}\binom{m-2}{k-2} - \binom{n}{2}\binom{n-2}{k-2} + \\
&\quad + \sum_{i=t}^{m}\left[(n-1+i)\binom{n+m-2}{k-2} - (i-1)\binom{m-2}{k-2}\right] = \\
&= \left[\binom{t-1+n}{2}\binom{m+n-2}{k-2} + \sum_{i=t}^{m}\binom{n-1+i}{1}\binom{n+m-2}{k-2}\right] - \\
&\quad - \left[\binom{t-1}{2}\binom{m-2}{k-2} + \sum_{i=t}^{m}\binom{i-1}{1}\binom{m-2}{k-2}\right] - \binom{n}{2}\binom{n-2}{k-2} = \\
&= \binom{m+n}{2}\binom{m+n-2}{k-2} - \binom{m}{2}\binom{m-2}{k-2} - \binom{n}{2}\binom{n-2}{k-2},
\end{aligned}$$

which is a contradiction.

Let j_0 be the maximum integer such that $A' = [a_1, a_2, \cdots, a_{j_0-1}, a_{j_0} - 1, a_{j_0+1}, \cdots, a_{t-1}, a_t + 1, a_{t+1}, \cdots, a_m]$ is non-decreasing. We show that $(A', B' = B)$ satisfy (1). In fact, it suffices to show that for (A, B), the inequalities in (1) strictly hold for each $j_0 \le j \le t-1$.

Let r_0 be the least integer such that $j_0 \le r_0 \le t-1$ and the inequalities in (1) strictly hold for all $r_0 \le p \le t-1$ and $1 \le q \le n$. If $r_0 = j_0$, we are done. So we assume that $r_0 > j_0$. Since by the minimality of r_0, there is an integer q such that

$$\sum_{i=1}^{r_0-1} a_i + \sum_{j=1}^{q} b_j = \binom{r_0-1+q}{2}\binom{m+n-2}{k-2} - \binom{r_0-1}{2}\binom{m-2}{k-2} - \binom{q}{2}\binom{n-2}{k-2}$$

and

$$\sum_{i=1}^{r_0} a_i + \sum_{j=1}^{q} b_j > \binom{r_0+q}{2}\binom{m+n-2}{k-2} - \binom{r_0}{2}\binom{m-2}{k-2} - \binom{q}{2}\binom{n-2}{k-2},$$

it follows that

$$\begin{aligned}
a_{r_0} &= \sum_{i=1}^{r_0} a_i - \sum_{i=1}^{r_0-1} a_i > \\
&> \binom{r_0+q}{2}\binom{m+n-2}{k-2} - \binom{r_0}{2}\binom{m-2}{k-2} - \binom{q}{2}\binom{n-2}{k-2} - \\
&\quad -\binom{r_0-1+q}{2}\binom{m+n-2}{k-2} + \binom{r_0-1}{2}\binom{m-2}{k-2} + \binom{q}{2}\binom{n-2}{k-2} = \\
&= (r_0-1+q)\binom{m+n-2}{k-2} - (r_0-1)\binom{m-2}{k-2} \geq \\
&\geq (r_0-2+q)\binom{m+n-2}{k-2} - (r_0-2)\binom{m-2}{k-2}.
\end{aligned}$$

Thus

$$a_{r_0} > (r_0-2+q)\binom{m+n-2}{k-2} - (r_0-2)\binom{m-2}{k-2}.$$

By the maximality of j_0, we have $a_{j_0-1} < a_{j_0} = a_{j_0+1} = \cdots = a_{t-1}$. Note that $j_0 < r_0 \leq t-1$, so we have

$$a_{r_0-1} = a_{r_0} > (r_0-2+q)\binom{m+n-2}{k-2} - (r_0-2)\binom{m-2}{k-2}.$$

Therefore, for each $1 \leq q \leq n$

$$\begin{aligned}
\sum_{i=1}^{r_0-1} a_i + \sum_{j=1}^{q} b_j &= \left(\sum_{i=1}^{r_0-2} a_i + \sum_{j=1}^{q} b_j\right) + a_{r_0} > \\
&> \left[\binom{r_0-2+q}{2}\binom{m+n-2}{k-2} - \binom{r_0-2}{2}\binom{m-2}{k-2} - \binom{q}{2}\binom{n-2}{k-2}\right] + \\
&\quad + \left[(r_0-2+q)\binom{m+n-2}{k-2} - (r_0-2)\binom{m-2}{k-2}\right] = \\
&= \binom{r_0-1+q}{2}\binom{m+n-2}{k-2} - \binom{r_0-1}{2}\binom{m-2}{k-2} - \binom{q}{2}\binom{n-2}{k-2},
\end{aligned}$$

which contradicts the minimality of r_0. □

Note that if $a_1 = 0$, then by conditions (1), we have $b_1 > 0$ and $a_2 > 0$, so we have the following observation.

Lemma 2.4. *Let $A = [a_i]_1^m$ and $B = [b_j]_1^n$ be non-decreasing sequences of non-negative integers, where $a_1 = 0$. Then $A' = [a_1+1, a_2, \cdots, a_m]$, $B' = [b_1-1, b_2, \cdots, b_n]$, are non-decreasing and satisfy (1).*

By a similar argument as in Lemma 2.3, we get the following result.

Lemma 2.5. *Let $A = [a_i]_1^m$ and $B = [b_j]_1^n$ be non-decreasing sequences of non-negative integers, satisfying conditions (1) where $a_i = (n-1+i)\binom{n+m-2}{k-2} - (i-1)\binom{m-2}{k-2}$ for all $1 \leq i \leq m$. Let t be an integer such that $b_t \neq (t-1)\left[\binom{n+m-2}{k-2} - \binom{n-2}{k-2}\right]$ and $b_i = (i-1)\left[\binom{n+m-2}{k-2} - \binom{n-2}{k-2}\right]$ for all $i > t$. Then there is j_0 ($1 \leq j_0 \leq t-1$), such that $A' = A$, $B' = [b_1, b_2, \cdots, b_{j_0-1}, b_{j_0}-1, b_{j_0+1}, \cdots, b_{t-1}, b_t+1, b_{t+1}, \cdots, b_n]$, are non-decreasing and satisfy (1).*

Definition 2.1. *Given an oriented k-hypergraph $H(V, A)$, x and y being two distinct vertices in H. If we can choose t arcs a_1, $\cdots$, a_t (repetitions allowed) and $t-1$ distinct vertices z_1, z_2, $\ldots$, z_{t-1} which are different from x and y, such that x is the predecessor of z_1 in a_1, z_i is* *the predecessor of* z_{i+1} *in* a_{i+1}, $1 \leq i \leq t-2$, z_{t-1} *is the predecessor of* y *in* a_t, *and* $a_i \neq a_{i+1}$, $1 \leq i \leq t-1$, *then we say that there is a consecutive path from* x *to* y, *denoted by*

$$P(x,y) = xa_1z_1a_2z_2 \cdots z_{t-2}a_{t-1}z_{t-1}a_ty.$$

y *is called a reachable vertex from* x, *or simply reachable from* x. $P(x,y)$ *can be simply written as* P_y *if* x *is given. Denote all the consecutive paths from* x *to* y *in* H *by* $P_H(x,y)$.

Example 2.1. *Let* $n = 5$, $k = 4$. *Consider a* 4*-hypertournament* $H(V,A)$ *on* 5 *vertices, where* $V = \{1,2,3,4,5\}$, A *consists of* $a_1 = (1,2,3,4)$, $a_2 = (1,2,3,5)$, $a_3 = (1,2,4,5)$, $a_4 = (2,3,4,5)$, $a_5 = (4,1,5,3)$. *Then* 1 *is reachable from* 5, *and one consecutive path is* $5a_53a_14a_51$.

Given a k-hypertournament $H(V,A)$ and a vertex x of V. we introduce the following notation.

$R = \{v \in V : \exists e_v \in A$ such that all consecutive paths from x to v end in the arc e_v. Here we call e_v the key arc of v, and v the key vertex of $e_v\}$,

$E = \{e \in A : e$ is the key arc of some vertex $r \in R\}$

$W = \{v \in V :$ there is no consecutive path from x to v, i.e., v is not reachable from $x\}$,

$X = V - R - W.$

Lemma 2.6. *Each arc* $e \in A \backslash E$ *can be represented as* $([W'][R'][X'])$, *where* $W' = V_e \cap W$, $R' = V_e \cap R$, $X' = V_e \cap X$, $1 \leq i \leq m$, *and* $[\cdot]$ *means optional. That is, all the vertices of* W' *precede the vertices of* R' *in* e; *and all the vertices of* R' *precede the vertices of* X'.

Proof. If some vertex from $R' \cup X'$ is followed by a vertex from W' in e, then that vertex in W can be reached by a consecutive path, a contradiction. Furthermore if some vertex from X' is followed by a vertex in R' in e then that vertex in R can be reached by a consecutive path from x ending in e, a contradiction. This completes the proof. □

Lemma 2.7. *Let* $H(U,V,A)$ *be a* k*-bipartite hypertournament and* x, y *belong to the same bipartite set. If* $m + n > k > 3$ *and there is no consecutive path from* x *to* y *in* A, *then* $d^+_H(x) < d^+_H(y)$.

Proof. First note that no edge in $e \in A \backslash E$ contains two vertices in R, since if there is such an edge, say e, then by Lemma 2.6 we may assume, without loss of generality, that r_1 precedes r_2 in e and r_1, $r_2 \in R$. Now r_2 is reachable from x by a consecutive path ending in e, a contradiction.

Let $|U| = m$ and $|V| = n$ and let $x, y \in U$. Assume $|E| \geq 2$ and let e_{r_1} and e_{r_2} be two distinct edges in E(where r_1, $r_2 \in R$). By the above arguments we know that all edges containing both r_1 and r_2 lie in E, which implies that $|R| \geq |E| \geq min\{\binom{m+n-2}{k-2} - \binom{m-2}{k-2}, \binom{m+n-2}{k-2} - \binom{n-2}{k-2}\} \geq \binom{m+n-2}{k-3} \geq m+n-2$. As $R \subseteq (U \cup V) - \{x,y\}$, the above implies equality everywhere and $R = (U \cup V) - \{x,y\}$. Therefore, if $e \in A \backslash E$ then e contains exactly x, y and some vertex from R, which implies that $k = 3$, a contradiction to the initial hypothesis that $k > 3$. Therefore $|E| \leq 1$.

Assume that $|E| = 0$, i.e., $E = \emptyset$. Let $Q \subseteq U \cup V - \{x,y\}$ be any set of $k-1$ vertices such that $Q \cap U \neq \emptyset$ and $Q \cap V \neq \emptyset$. Let e^Q_y be the edge in H with vertex set $\{y\} \cup Q$ and let e^Q_x be the edge in H with vertex set $\{x\} \cup Q$. By Lemma 2.6 (as $y \in W$ and $x \in X$) we have $\rho(e^Q_y, y) - \rho(e^Q_x, x) \geq 0$. As for every edge, e_{xy} containing both x and y, we have $\rho(e_{xy}, y) - \rho(e_{xy}, x) \geq 1$, we note that $d^+_H(y) - d^+_H(x) \geq \binom{m+n-2}{k-2} - \binom{m-2}{k-2} > 0$.

Now assume that $E \neq \emptyset$ and $E = \{e\}$. Let e' contain the same vertices as e but such that $e' = ([W'][R'][X'])$, where $W' = V_e \cap W$, $R' = V_e \cap R$, $X' = V_e \cap X$. Let $H' = H \cup e' - e$ and $r \in R$ be arbitrary. If $r \in Q$(and e^Q_y and e^Q_x are defined as above, but in H') then we note that $\rho(e_y^Q, y) - \rho(e_x^Q, x) \geq 1$ in H'. Also, for every edge e_{xy} containing both x and y, we have $\rho(e_{xy}, y) - \rho(e_{xy}, x) \geq 1$. Therefore, we get that $d^+_{H'}(y) - d^+_{H'}(x) \geq \binom{m+n-2}{k-2} - \binom{m-2}{k-2} + \binom{m+n-3}{k-2} - \binom{m-3}{k-2}$. In H we have to modify this bound by at most k (if $e = e_{xy}$ and $\rho(e_{xy}, y) - \rho(e_{xy}, x)) = 1 - k$ in H or if $e \in \{e_x^Q, e_y^Q\}$ and $\rho(e_y^Q, y) - \rho(e_x^Q, x) = 1 - k$ in H), which implies that $d^+_H(y) - d^+_H(x) \geq (m+n-2) - (m-2) + (m+n-3) - (m-3) - k > 0$.

In both of these cases, we have both $d^+_H(y) - d^+_H(x) > 0$, i.e., $d^+_H(x) < d^+_H(y)$.

The proof in case $x, y \in V$, follows by using the same arguments as above. This completes the proof of the lemma. □

If $x \in U, y \in V$, let $E_x = \{e | e \cap U = \{x\}\}$, $E_y = \{e | e \cap V = \{y\}\}$ and by replacing each x of $e \in E_x$ with y and each y of $e \in E_y$ with x, we get E_x^* and E_y^* respectively. Now let $H^*(x, y) = (U^*, V^*, A^*)$ be the bipartite hypertournament such that $U^* = U + \{y\} - \{x\}, V^* = V + \{x\} - \{y\}$ and $A^* = A - E_x - E_y + E_x^* + E_y^*$.

Lemma 2.8. *Let $H(U, V, A)$ be a k-bipartite hypertournament and let $x \in U, y \in V$. If $m + n > k > 3$ and there is no consecutive path from x to y in A then either $d^+_H(x) < d^+_H(y)$ or $d^+_{H^*}(y) < d^+_{H^*}(x)$.*

Proof. First note that no edge in $e \in A \backslash E$ contains two vertices in R, since if there is such an edge, say e, then by Lemma 2.6 we may assume, without loss of generality, that r_1 is the predecessor of r_2 in e and $r_1, r_2 \in R$. Now r_2 is reachable from x by a consecutive path ending in e, a contradiction.

Let $|U| = m$ and $|V| = n$. Let $E_x = \{e | e \cap U = \{x\}\}$ and $E_y = \{e | e \cap V = \{y\}\}$. Let $\rho_x = \sum_{e \in E_x} \rho(e, x)$ and $\rho_y = \sum_{e \in E_y} \rho(e, y)$. Without loss of generality, we assume that $\rho_y \geq \rho_x$, otherwise we consider H^*. Assume $|E| \geq 2$ and let e_{r_1} and e_{r_2} be two distinct edges in E(where $r_1, r_2 \in R$). From above observations we know that all edges containing both r_1 and r_2 lie in E, which imply that $|R| \geq |E| \geq \binom{m+n-2}{k-2} - \binom{n-2}{k-2} \geq \binom{n-1}{k-2} - \binom{n-2}{k-2} = \binom{n-2}{k-3} \geq n - 2$. As $R \subseteq U \cup V - \{x, y\}$, the above conditions imply equality everywhere and $R = U \cup V - \{x, y\}$. Therefore, if $e \in A \backslash E$ then e contains exactly x, y and some vertex from R, which implies that $k = 3$, a contradiction to $m + n > k > 3$. Therefore $|E| \leq 1$.

Assume that $|E| = 0$, i.e., $E = \emptyset$. Let $Q \subseteq V - \{x, y\}$ be any set of $k - 1$ vertices such that $Q \cap U \neq \emptyset$ and $Q \cap V \neq \emptyset$. Let e_y^Q be the edge in H with vertex set $\{y\} \cup Q$ and let e_x^Q be the edge in H with vertex set $\{x\} \cup Q$. By Lemma 2.6 (as $y \in W$ and $x \in X$) we have $\rho(e_y^Q, y) - \rho(e_x^Q, x) \geq 0$. As for every edge, e_{xy} containing both x and y we have $\rho(e_{xy}, y) - \rho(e_{xy}, x) \geq 1$.

Therefore, $d^+_H(y) - d^+_H(x) \geq \binom{n+m-2}{k-2} + \rho_y - \rho_x > 0$.

Now assume that $E \neq \emptyset$ and $E = \{e\}$. Let e' contain the same vertices as e but such that $e' = ([W'][R'][X'])$, where $W' = V_e \cap W$, $R' = V_e \cap R$, $X' = V_e \cap X$. Let $H' = H \cup e' - e$ and $r \in R$ be arbitrary. If $r \in Q$(and e_y^Q and e_x^Q are defined as above, but in H') then we note that $\rho(e_y^Q, y) - \rho(e_x^Q, x) \geq 1$ in H'. Also, for every edge e_{xy} containing both x and y, we have $\rho(e_{xy}, y) - \rho(e_{xy}, x) \geq 1$. Therefore, we get that $d^+_{H'}(y) - d^+_{H'}(x) \geq \binom{m+n-2}{k-2} + \binom{m+n-3}{k-3} + \rho_y - \rho_x$. In H we have to modify this bound by at most k (if $e = e_{xy}$ and $\rho(e_{xy}, y) - \rho(e_{xy}, x)) = 1 - k$ in H or if $e \in \{e_x^Q, e_y^Q\}$ and $\rho(e_y^Q, y) - \rho(e_x^Q, x) = 1 - k$ in H), which implies that $d^+_H(y) - d^+_H(x) \geq (m+n-2) + (m+n-3) - k > 0$.

In both of these cases, we have both $d^+_H(y) - d^+_H(x) > 0$, i.e., $d^+_H(x) < d^+_H(y)$. □

Definition 2.2. *If $x a_{j_1} z_1 a_{j_2} z_2 \cdots z_{s-2} a_{j_{s-1}} z_{s-1} a_{j_s} y$ is a consecutive path from x to y in $H(U, V, A)$, then $H'(U', V', A')$ that consists of $U' = U, V' = V$ and*

$$A' = A - \{a_{j_1}, a_{j_2}, \cdots, a_{j_s}\} \cup \{a_{j_1}(x, z_1), a_{j_2}(z_1, z_2), \cdots, a_{j_s}(z_{s-1}, y)\},$$

is called a k-bipartite hypertournament obtained from H by reversing the consecutive path

$$xa_{j_1}z_1a_{j_2}z_2\cdots z_{s-2}a_{j_{s-1}}z_{s-1}a_{j_s}y.$$

Lemma 2.9. *Given $m+n>k>3$. Suppose that $A=[a_1,a_2,\cdots,a_m]$,$b=[b_1,b_2,\cdots,b_n]$ are degree lists of a k-bipartite hypertournament $H=(U,V,A)$. Let x,y be two vertices in H such that $d^+_H(x)<d^+_H(y)$.*

(a) If $x,y\in U$ and $x=a_i,y=a_j$, then $(A_1=A'(a_i^+,a_j^-),B_1=B)$ are degree lists of some other k-bipartite hypertournament.

(b) If $x,y\in V$ and $x=b_i,y=b_j$, then $(A_1=A,B_1=B'(b_i^+,b_j^-))$ are degree lists of some other k-bipartite hypertournament.

(c) If $x\in U,y\in V$ and $x=a_i,y=b_j$, then either $(A_1=A'(a_i^+),B_1=B'(b_j^-))$ are degree lists of some other k-bipartite hypertournament, or $(A_2=A^+(a_i,b_j),B_2=B^+(b_j,a_i))$ are degree lists of some other k-bipartite hypertournament.

Proof. (a) By Lemma 2.7, there is a consecutive path y to x. It is easy to see that the k-bipartite hypertournament obtained from H by reversing the consecutive path from y to x has the degree lists (A_1,B_1).

(b) This part follows by using the same argument as in the proof of (a).

(c) If there is a consecutive path y to x, then the k-bipartite hypertournament obtained from H by reversing the consecutive path from y to x has the degree list (A_1,B_1). If there is no such path, then by the proof of Lemma 2.8 (A_2,B_2) are the degree lists of H^*.

Proof of Theorem 2.1. Suppose first that A and B are the degree lists of some k-bipartite hypertournament, then

$$\begin{aligned}&\sum_{i=1}^{p}a_i+\sum_{j=1}^{q}b_j\geq\\ \geq\ &\sum_{i=1}^{k}\binom{p+q}{i}\binom{n+m-p-q}{k-i}\binom{i}{2}-\sum_{i=1}^{k}\binom{p}{i}\binom{m-p}{k-i}\binom{i}{2}-\sum_{i=1}^{k}\binom{q}{i}\binom{n-q}{k-i}\binom{i}{2}=\\ =\ &\binom{p+q}{2}\binom{m+n-2}{k-2}-\binom{p}{2}\binom{m-2}{k-2}-\binom{q}{2}\binom{n-2}{k-2}.\end{aligned}$$

Conversely, suppose that $A=[a_i]_1^m$ and $B=[b_j]_1^n$ are the non-decreasing sequences of non-negative integers which satisfy (1), we show that (A,B) are the degree lists of some k-bipartite hypertournament.

By applying Lemma 2.3 and Lemma 2.4 repeatedly, we get sequences $A_1=[a_i']_1^m$ and $B_1=[b_j']_1^n$ such that $a_i'=(n-1+i)\binom{n+m-2}{k-2}-(i-1)\binom{m-2}{k-2}$ for all $1\leq i\leq m$. Next by applying Lemma 2.5 repeatedly, we get sequences $A'=[a_1'',a_2'',\cdots,a_m'']$, $B'=[b_1'',b_2'',\cdots,b_n'']$, such that

$$a_i''=(n-1+i)\binom{n+m-2}{k-2}-(i-1)\binom{m-2}{k-2},b_i''=(i-1)\left[\binom{n+m-2}{k-2}-\binom{n-2}{k-2}\right].$$

By Lemma 2.2, (A',B') is the degree list of a transitive k-bipartite hypertournament. Now by applying lemma 2.9 repeatedly, we finally obtain the original sequence (A,B), which are the degree lists of some k-bipartite hypertournament. □

We note that when $k=2$, the k-bipartite hypertournament becomes an ordinary bipartite tournament. Various results on scores in bipartite tournaments can be found in Beineke and Moon [1].

3. Acknowledgement

This research is supported in part by the National Science Foundation of China (No. 11371193)

References

[1] Beineke, L.W., Moon, J.W., (1980), On bipartite tournaments and scores, Proc. Fourth International Graph Theory Conference, Kalamazoo, pp.55-71.
[2] Koh, Y., Ree, S., (2001), Score sequences of hypertournament matrices, J. Korea Soc. Math. Educ. Ser. B: Pure and Appl. Math. 8(2), pp.185-191.
[3] Koh, Y. and Ree, S., (2003), On k-hypertournament matrices, Linear Algebra Appl., 373, pp.183-195.
[4] Pirzada, S., Naikoo T. A. and Chishti, T. A., (2006), Score sets in oriented bipartite graphs, Novi Sad J. Math., 36(1), pp.35-45.
[5] Pirzada, S. and Zhou, G., (2007), Score lists in (h, k)-bipartite hypertournaments, Applied Mathematics-J. of Chinese Universities, Series B, 22(4), pp.485-489.
[6] Pirzada, S., Chishti, T.A., Naikoo, T.A., (2009), Score lists in [h, k]-bipartite hyper tournaments, Discrete Mathematics and Applications, 19(3), pp.321-328.
[7] Pirzada, S., Zhou, G., (2010), On k-hypertournament losing scores, Acta Univ. Sapientiae, Informatica, 2(1), pp.5-9.
[8] Pirzada, S. and Naikoo, T. A., (2011), Mark sets in digraphs, Appl. Comput. Math., 10(2), pp.283-288.
[9] Wang, C., Zhou, G., (2008), Note on the degree sequences of k-hypertournaments, Discrete Math., 308(11), pp.2292-2296.
[10] Zhou, G. Yao Tianxing and Zhang Kemin, (2000), On score sequences of k-hypertournaments, European Journal of Combinatorics, 21, pp.993-1000.
[11] Zhou, G., Zhang Kemin, (2001), On the degree sequence of k-hypertournaments, Sica. Annal. Math., 22A, pp.115-120.

S. Pirzada was born in 1967 in Kashmir, India. He received his Ph.D. from Aligarh Muslim University, India in 1995. He is the Professor in the Department of Mathematics, University of Kashmir. He has taught at King Fahd University of Petroleum and Minerals, Saudi Arabia and National Institute of Technology, Kashmir. He has published more than 85 research papers in various international journals. He has authored two books namely Introduction to Graph theory and Analytical Geometry of three dimensions, published by Universities Press, Orient Blackswan. His research interests are Combinatorics and Graph theory which include directed graphs, tournaments, graphical sequences, signed graphs, spectral graph theory.

Guofei Zhou is a Professor of Mathematics at Nanjing University, China. He received his Ph.D. degree from Nanjing University in 2000. He is the principal investigator of several research grants from the National Natural Science Foundation and State Education Ministry of China. His main research interests include Combinatorics and Graph theory, especially include graph coloring, graphical sequences, Ramsey theory, tournaments and Hypergraph.