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APPLICATION OF MULTI-STAGE HAM-PADÉ TO SOLVE A MODEL FOR THE EVOLUTION OF COCAINE CONSUMPTION IN SPAIN

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Abstract. In this paper we obtain approximated analytical solutions for a mathematical model for the cocaine consumption in Spain using the Homotopy Analysis Method. The interest of the model is that, based on real data from Spanish Statistical Institute, it has been used to explain successfully the real evolution of the epidemic in Spain [18, 19]. First, we obtain with HAM an analytical approximation to the real solution of the model in the form of a power series of the time t. Second, we enlarge its domain of convergence applying the Padé after-treatment to the HAM solution. Finally, we apply multi-stage HAM-Padé to obtain an analytical approximate solution valid for the complete domain $t \in [0, \infty]$. This approximate solution has the form of three Padé approximants for the intervals $[0, 50]$, $[50, 150]$ and $[150, \infty]$ years.

Keywords: dynamic model, epidemic model; Homotopy analysis method, Homotopy-Padé technique, multistage homotopy analysis method, cocaine consumption model.

AMS Subject Classification: 34L30.

1. Introduction

Since epidemic models consist of a system of nonlinear differential equations, it is of great importance to have reliable methods for solving them. These models can be integrated using any standard numerical scheme. However, these algorithms may give rise to some problems: numerical instabilities, oscillations, false equilibrium states, among others. Thus, the numerical solution obtained may not correspond to the real solution of the system [10].

To avoid these kind of problems, we are interested in obtaining a continuous solution in the form of an analytical approximation to the real solution of the problem. There are different methods to do this. We have chosen the Homotopy Analysis Method (HAM), first developed by Liao [11], [12], which has been used successfully in the recent years to solve many different problems in science and engineering [13]-[17]. More similar to the present case are the solution of SIR [3], SIS models [9] and smoking model [7].

The epidemic model that describes the dynamics of the consumption of cocaine in Spain [18, 19] is a system of nonlinear ODEs without closed solution. The interest of this model is that it has been able to describe successfully the real evolution of the prevalence of cocaine consumption. It was constructed using real data for the values of the parameters and for the initial values of the subpopulations in the system. In this paper, constant population is assumed by setting equal (and nonzero) values for birth and death rates.

HAM gives an analytical approximation to the real solution of the system in the form of a power series of time t. This approximate solution reproduces the correct solution for a certain

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range of time. In general, HAM provides a set of solutions depending on four auxiliary parameters: h_1, h_2, h_3 and h_4. To simplify, we assume that they all are equal: $h_1 = h_2 = h_3 = h_4 = h$. We use the optimal convergence control parameter technique to determine the optimal value of this h in the sense of obtaining the largest domain of convergence for a fixed number of terms in the power series [16, 15]. To increase the domain of convergence we have applied the Padé after-treatment [11] to the HAM solution. Finally, we used a multi-stage HAM technique [2] to obtain an analytical approximation valid for $t \in [0, \infty]$ years. In order to significantly reduce the number of segments of the standard multi-stage protocol, we introduce an after-treatment with Padé approximants. Therefore, the proposed solution is in the form of three HAM-Padé approximants for the intervals $[0, 50]$, $[50, 150]$ and $[150, \infty]$. The final values of the HAM-Padé approximations of one interval are used as the initial values to construct the HAM-Padé of the next interval.

This paper is organized as follows. In Section 2 we explain the model for the consumption of cocaine that we are solving. Section 3 summarizes the basic ideas of the HAM applied to the cocaine model. In Section 4, we present the results of our method: the HAM solution with 20 terms, the solution obtained with the Padé after-treatment and the multi-stage HAM-Padé which is valid for $t \in [0, \infty]$. We show how the application of every after-treatment increases the domain of convergence. Finally, we present our conclusions in Section 5.

2. The model of the consumption of cocaine in Spain

This model was presented in [18, 19]. It explained successfully the real data of the consumption of cocaine in Spain in a period of time of ten years (1995 - 2005). It has been also used to make short term predictions up to 2015.

The dynamics of the different subpopulations is described by the following system of ordinary differential equations:

$$n'(t) = \mu(1 - n(t)) - \beta n(t)(o(t) + r(t) + h(t)) + \epsilon h(t), \tag{1}$$

$$o'(t) = \beta n(t)(o(t) + r(t) + h(t)) - \gamma o(t) - \mu o(t), \tag{2}$$

$$r'(t) = \gamma o(t) - \sigma r(t) - \mu r(t), \tag{3}$$

$$h'(t) = \sigma r(t) - \mu h(t) - \epsilon h(t). \tag{4}$$

These four subpopulations represent proportions of the total population. Their definitions are: $n(t)$ (non consumers) is the proportion of individuals who have never consumed cocaine, $o(t)$ (occasional consumers) is the proportion of people who have consumed sometimes in their lives, $r(t)$ (regular consumers) is the proportion of individuals who have consumed in the last year and $h(t)$ (habitual consumers) is the proportion of individuals who have consumed in the last month.

The parameters of the model mean: μ is the birth rate in Spain (it is also the death rate, since we assume constant population); β is the transmission rate due to social pressure to consume cocaine; γ is the rate at which an occasional consumer becomes a regular consumer; σ is the rate at which a regular consumer becomes a habitual consumer and ϵ is the rate at which a habitual consumer leaves cocaine consumption due to therapy programs.

Since the constant population has been normalized to unity, the variables satisfy that $n(t) + o(t) + r(t) + h(t) = 1$.

We obtain the asymptotic behavior of eqs. (1 - 4) setting $n'(t) = 0$, $o'(t) = 0$, $r'(t) = 0$ and $h'(t) = 0$ and solving the system. The result yields two equilibrium points: CFE (cocaine free equilibrium) and CEE (cocaine endemic equilibrium). They are $CFE = (1, 0, 0, 0)$ and $CEE = (n_E, o_E, r_E, h_E)$, where

$$n_E = \frac{(\epsilon+\mu)(\gamma+\mu)(\sigma+\mu)}{\beta[\epsilon(\gamma+\mu+\sigma)+(\gamma+\mu)(\sigma+\mu)]}, \tag{5}$$

$$o_E = \frac{(\epsilon+\mu)(\sigma+\mu)[\beta(\epsilon(\gamma+\mu+\sigma)+(\gamma+\mu)(\sigma+\mu))-(\epsilon+\mu)(\gamma+\mu)(\sigma+\mu)]}{\beta[\epsilon(\gamma+\mu+\sigma)+(\gamma+\mu)(\sigma+\mu)]^2}, \tag{6}$$

$$r_E = \frac{\gamma(\epsilon+\mu)[\beta(\epsilon(\gamma+\mu+\sigma)+(\gamma+\mu)(\sigma+\mu))-(\epsilon+\mu)(\gamma+\mu)(\sigma+\mu)]}{\beta[\epsilon(\gamma+\mu+\sigma)+(\gamma+\mu)(\sigma+\mu)]^2}, \tag{7}$$

$$h_E = \frac{\gamma\sigma[\beta(\epsilon(\gamma+\mu+\sigma)+(\gamma+\mu)(\sigma+\mu))-(\epsilon+\mu)(\gamma+\mu)(\sigma+\mu)]}{\beta[\epsilon(\gamma+\mu+\sigma)+(\gamma+\mu)(\sigma+\mu)]^2}. \tag{8}$$

Analyzing the eigenvalues of the Jacobian matrix of system of eqs. (1 - 4) can be proven that one of the equilibrium points is asymptotically stable and the other one is unstable. Which point is the stable one depends on the numerical values of the parameters. In particular, for the values of the parameters obtained in [19], the CEE equilibrium point is the asymptotically stable one. Figures 6-9 below in this paper illustrate this issue.

3. Basic ideas of HAM applied to the cocaine model

Let us present a brief summary of the ideas of the Homotopy Analysis Method that we need to solve the cocaine model. For a more extensive explanation we refer the interested reader to the original works by S.J. Liao [11, 16].

Consider that $y(t)$ is the unknown solution of the system and N is the operator that represents the equations of the model

$$N[y(t)] = 0. \tag{9}$$

The main idea is to construct a homotopy with an auxiliary parameter q such as if $q = 0$ the solution of the homotopy is y_0 (the initial and constant guess of the solution) and if $q = 1$ the solution of the homotopy is $y(t)$, the solution of the original system of equations.

The homotopy is:

$$(1-q)L[\phi(t,q)-y_0] - qhH(t)N[\phi(t,q)] = \hat{H}[\phi(t,q), y_0, H(t), h, q], \tag{10}$$

where $h \neq 0$ is an auxiliary parameter, $H(t)$ is an auxiliary function and L is an auxiliary operator, in our case $L = \frac{\partial}{\partial t}$.

The zero order deformation equation is obtained by setting the right hand side of eq. (10) equal to zero:

$$(1-q)L[\phi(t,q)-y_0] = qhH(t)N[\phi(t,q)]. \tag{11}$$

Since $L(0) = 0$, we see that for $q = 0$ eq. (11) means that

$$\phi(t,0) = y_0. \tag{12}$$

On the other hand, when $q = 1$, (and $h \neq 0$, $H(t) \neq 0$) , eq.(11) is

$$\phi(t,1) = y(t). \tag{13}$$

As the parameter q increases from 0 to 1, the function $\phi(t,q)$ changes continuously from the initial y_0 to the exact solution $y(t)$. This continuous variation is called deformation in homotopy.

We can expand $\phi(t,q)$ in a power series of q

$$\phi(t,q) = y_0 + \sum_{m=1}^{\infty} y_m(t) q^m, \tag{14}$$

where

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t,q)}{\partial q^m}\bigg|_{q=0}. \tag{15}$$

The initial guess $y_0(t)$, the operator L, the parameter h and the function $H(t)$ must satisfy:

1. $\phi(t,q)$ exists for all $0 \le q \le 1$,
2. $y_m(t)$ exists for $m = 1, 2, ...$, and
3. the power series (14) of $\phi(t,q)$ is convergent for $q = 1$,

Setting $q = 1$ in eq. (14) we obtain the solution series of the system of differential equations of the model

$$\phi(t,1) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{16}$$

To obtain the terms $y_m(t)$ we differentiate m times eq. (11) with respective to q. Setting $q = 0$ in the resulting expression we obtain the so-called mth-order deformation equation

$$L[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)\mathcal{R}_m(y_{m-1}(t)), \tag{17}$$

where

$$\mathcal{R}_m(y_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t,q)]}{\partial q^{m-1}} \tag{18}$$

with $\chi_m = 0$ for $m \le 1$ and $\chi_m = 1$ for $m \ge 2$.

From eq. (17) we can obtain $y_m(t)$ from $y_{m-1}(t)$. Thus, beginning with $y_0(t)$, we can calculate $y_1(t), y_2(t), ...$ and so on.

Following carefully the steps described in [11, 16] we find that the operators N_i described above are defined as follows:

$$N_1[\phi_i(t,q)] = \frac{\partial \phi_i(t,q)}{\partial t} - \mu(1 - \phi_i(t,q)) + \beta\phi_i(t,q)(\phi_2(t,q) + \phi_3(t,q) + \phi_4(t,q)) - \\ - \epsilon\phi_4(t,q), \tag{19}$$

$$N_2[\phi_i(t,q)] = \frac{\partial \phi_i(t,q)}{\partial t} - \beta\phi_1(t,q)(\phi_i(t,q) + \phi_3(t,q) + \phi_4(t,q)) + (\gamma + \mu)\phi_i(t,q), \tag{20}$$

$$N_3[\phi_i(t,q)] = \frac{\partial \phi_i(t,q)}{\partial t} - \gamma\phi_2(t,q) + (\sigma + \mu)\phi_i(t,q), \tag{21}$$

$$N_4[\phi_i(t,q)] = \frac{\partial \phi_i(t,q)}{\partial t} - \sigma\phi_3(t,q) + (\epsilon + \mu)\phi_i(t,q). \tag{22}$$

These operators correspond to the system of equations of the model in eqs. (1-4).

Following [11, 16] we set $H_i(t) = 1$, for $i = 1, 2, 3, 4$. We also assume that $h_1 = h_2 = h_3 = = h_4 = h$ and then we use h to find its optimal value for the convergence of the solution.

According to eqs.(17-18), the mth-order deformation equations for $m \ge 1$ are:

$$n_m(t) = \chi_m n_{m-1}(t) + h\int_0^t d\tau \Big[n'_{m-1}(\tau) - \mu(o_{m-1}(\tau) + r_{m-1}(\tau) + h_{m-1}(\tau)) + \\ + \beta\sum_{k=0}^{m-1}(o_k(\tau) + r_k(\tau) + h_k(\tau))n_{m-1-k}(\tau) - \epsilon h_{m-1}(\tau)\Big], \tag{23}$$

$$o_m(t) = \chi_m o_{m-1}(t) + h\int_0^t d\tau \Big[o'_{m-1}(\tau) - -\beta\sum_{k=0}^{m-1}(o_k(\tau)+r_k(\tau)+h_k(\tau))n_{m-1-k}(\tau) + (\gamma+\mu)o_{m-1}(\tau)\Big], \tag{24}$$

$$r_m(t) = \chi_m r_{m-1}(t) + h\int_0^t d\tau \left[r'_{m-1}(\tau) - \gamma o_{m-1}(\tau) + (\sigma+\mu)r_{m-1}(\tau)\right], \tag{25}$$

$$h_m(t) = \chi_m h_{m-1}(t) + h\int_0^t d\tau \left[h'_{m-1}(\tau) - \sigma r_{m-1}(\tau) + (\epsilon+\mu)h_{m-1}(\tau)\right]. \tag{26}$$

These formulas allow us to calculate the functions $n_m(t), o_m(t), r_m(t)$ and $h_m(t)$ of the power series.

4. Results

To obtain the approximate analytical solution of the model in the form of power series of t, we have to solve eqs. (23-26).

Since we want to reproduce the solution for the real case of cocaine consumption studied in [18, 19] we use the same values for the parameters: $\mu = 0.01$ year^{-1}, $\beta = 0.09614$ year^{-1}, $\gamma = 0.0596$ year^{-1}, $\sigma = 0.0579$ year^{-1} and $\epsilon = 0.0000456$ year^{-1}.

We also take as initial values the real proportions for the subpopulations in Spain at the beginning of the period of time analyzed in [18, 19], in year 1995:

$$n_0 = 0.944, \qquad o_0 = 0.034, \qquad r_0 = 0.018, \qquad h_0 = 0.004. \tag{27}$$

Our final goal is to obtain an analytical approximation for the real solution valid for the whole domain of the variable $t \in [0, +\infty]$. Then, we need to know the asymptote of the solution of the model for our particular set of values of the parameters.

When we substitute the values of the parameters in eqs. (5-8), we get:

$$n_E = 0.1043608956, \tag{28}$$

$$o_E = 0.1291117367, \tag{29}$$

$$r_E = 0.1133293005, \tag{30}$$

$$h_E = 0.6531980670. \tag{31}$$

4.1. **HAM solution.** Now, to solve eqs. (23-26) we only need to determine the value of the parameter h. We use h as a control parameter. We calculate the value of h that gives the best fit to the exact solution.

To do this we first define the averaged residual error E_m:

$$E_m = \frac{1}{K}\sum_{j=0}^{K}\Bigg\{\left[N_1\left(\sum_{k=0}^{m} n_k(j\Delta t)\right)\right]^2 + \left[N_2\left(\sum_{k=0}^{m} o_k(j\Delta t)\right)\right]^2 + + \left[N_3\left(\sum_{k=0}^{m} r_k(j\Delta t)\right)\right]^2 + \left[N_4\left(\sum_{k=0}^{m} h_k(j\Delta t)\right)\right]^2\Bigg\}. \tag{32}$$

We have set $K = 20$ and $\Delta t = 1$. This means that E_m is the residual error due to the difference between the HAM solution and the exact solution in the interval $0 \le t \le 20$ years, m is the number of terms in the HAM solution.

Figure 1 shows the values of E_m for $m = 6, 8$ and 10. It seems that the minimum averaged residual error correspond approximately to $h = -1$. For more details about this technique, we refer the reader to [15].

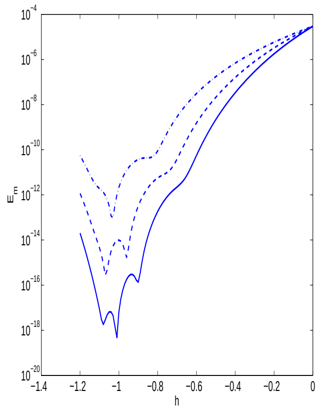

Figure 1. Averaged residual error E_m versus h. Dash-dotted line for $m = 6$, dashed line for $m = 8$ and solid line for $m = 10$. It has been calculated numerically with a h-step of 0.01.

Therefore, the solution with 20 terms given by HAM with $h = -1$ and the values of the parameters given above is:

$$\begin{aligned} n(t)^{[20]} &= 6.6867 \times 10^{-35} t^{20} - 4.1923 \times 10^{-34} t^{19} - 1.8140 \times 10^{-31} t^{18} - \\ &-1.0434 \times 10^{-29} t^{17} - 2.354401 \times 10^{-28} t^{16} + 9.389775 \times 10^{-27} t^{15} + \\ &+1.136313 \times 10^{-24} t^{14} + 4.967579 \times 10^{-23} t^{13} + 4.886670 \times 10^{-22} t^{12} - \\ &-8.445952 \times 10^{-20} t^{11} - 6.443417 \times 10^{-18} t^{10} - 2.092605 \times 10^{-16} t^{9} + \\ &+1.783075 \times 10^{-15} t^{8} + 5.967492 \times 10^{-13} t^{7} + 3.334455 \times 10^{-11} t^{6} + \\ &+7.176008 \times 10^{-10} t^{5} - 3.126322 \times 10^{-8} t^{4} - 3.625519 \times 10^{-6} t^{3} - \\ &-1.7040009 \times 10^{-4} t^{2} - 0.00452216 t + 0.944, \end{aligned} \tag{33}$$

$$\begin{aligned} o(t)^{[20]} &= -6.653245 \times 10^{-35} t^{20} - 3.605026 \times 10^{-35} t^{19} + 1.503532 \times 10^{-31} t^{18} + \\ &+9.529895 \times 10^{-30} t^{17} + 2.546789 \times 10^{-28} t^{16} - 5.772273 \times 10^{-27} t^{15} - \\ &-9.429093 \times 10^{-25} t^{14} - 4.604004 \times 10^{-23} t^{13} - 7.492358 \times 10^{-22} t^{12} + \\ &+5.706429 \times 10^{-20} t^{11} + 5.255555 \times 10^{-18} t^{10} + 2.007303 \times 10^{-16} t^{9} + \\ &+8.461225 \times 10^{-16} t^{8} - 3.879777 \times 10^{-13} t^{7} - 2.578902 \times 10^{-11} t^{6} - \\ &-7.543337 \times 10^{-10} t^{5} + 7.136147 \times 10^{-9} t^{4} + 1.908159 \times 10^{-6} t^{3} + \\ &+9.851886 \times 10^{-5} t^{2} + 0.00271594 t + 0.034, \end{aligned} \tag{34}$$

$$\begin{aligned} r(t)^{[20]} &= -1.368292 \times 10^{-36} t^{20} + 3.713880 \times 10^{-34} t^{19} + \\ &+2.805127 \times 10^{-32} t^{18} + 9.287009 \times 10^{-31} t^{17} - 8.969784 \times 10^{-30} t^{16} - \\ &-2.953033 \times 10^{-27} t^{15} - 1.752857 \times 10^{-25} t^{14} - 4.270791 \times 10^{-24} t^{13} + \\ &+1.600270 \times 10^{-22} t^{12} + 2.180718 \times 10^{-20} t^{11} + 1.080295 \times 10^{-18} t^{10} + \\ &+1.709237 \times 10^{-17} t^{9} - 1.522863 \times 10^{-15} t^{8} - 1.611275 \times 10^{-13} t^{7} - \\ &-6.025516 \times 10^{-12} t^{6} - 1.296788 \times 10^{-10} t^{5} + 1.581308 \times 10^{-8} t^{4} + \\ &+7.433577 \times 10^{-7} t^{3} + 5.363256 \times 10^{-5} t^{2} + 0.0008042 t + 0.018, \end{aligned} \tag{35}$$

$$
\begin{aligned}
h(t)^{[20]} &= 1.033029\times10^{-36}t^{20}+8.389590\times10^{-35}t^{19}+3.000984\times10^{-33}t^{18}-\\
&-2.448183\times10^{-32}t^{17}-1.026910\times10^{-29}t^{16}-6.644687\times10^{-28}t^{15}-\\
&-1.811844\times10^{-26}t^{14}+6.350433\times10^{-25}t^{13}+1.005416\times10^{-22}t^{12}+\\
&+5.588051\times10^{-19}t^{11}+1.075660\times10^{-17}t^{10}-8.562223\times10^{-16}t^{9}-\\
&-1.106334\times10^{-13}t^{8}-4.764392\times10^{-12}t^{7}-1.530018\times10^{-10}t^{6}+\\
&+1.664117\times10^{-8}t^{5}+8.313993\times10^{-7}t^{4}+9.740023\times10^{-5}t^{3}+\\
&+1.824865\times10^{-5}t^{2}+0.00100201t+0.004.
\end{aligned}
\tag{36}
$$

We see in Figures 2 - 5 that these HAM solutions (dashed line) are a good approximation to the exact solution (solid line) for a range of about 35 years. The so-called exact solution has been calculated numerically using a fourth order Runge-Kutta with a step size of $\Delta = 0.001$.

4.2. **Padé after-treatment.** Since we want to obtain an analytical approximation valid for $t \in [0, +\infty]$ we need to use the Padé after-treatment. This technique is used in literature to enlarge the domain of convergence of solutions. Some recent examples can be found in [6] and, more similar to our case, in [11, 5] and [7].

Using eqs. (33-36) we obtain the $[10, 10]$ Padé approximant for $n(t), o(t)$ and $r(t)$. For $h(t)$ we calculate the $[9, 9]$ Padé approximant to avoid a singularity that appears around $t = 70$ years for the $[10, 10]$ approximant.

The Padé approximants for each subpopulation are:

$$
\begin{aligned}
n(t)_{[10,10]} &= (-5.576749\times10^{-21}t^{10}+2.333510\times10^{-18}t^{9}-\\
&-4.310552\times10^{-16}t^{8}+1.819817\times10^{-14}t^{7}+4.824694\times10^{-12}t^{6}-\\
&-1.022153\times10^{-9}t^{5}+6.636199\times10^{-8}t^{4}+1.373240\times10^{-6}t^{3}-\\
&-5.480936\times10^{-4}t^{2}+0.0358696t-0.944)/(-6.815663\times10^{-21}t^{10}+\\
&+1.824669\times10^{-18}t^{9}-5.415325\times10^{-16}t^{8}+2.988108\times10^{-15}t^{7}+\\
&+3.414415\times10^{-12}t^{6}-1.119249\times10^{-9}t^{5}+5.951156\times10^{-8}t^{4}+\\
&+7.242361\times10^{-7}t^{3}-6.020403\times10^{-4}t^{2}+0.0332070t-1.0)
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
o(t)_{[10,10]} &= (6.804961\times10^{-20}t^{10}-1.237372\times10^{-17}t^{9}-6.661183\times10^{-16}t^{8}-\\
&-1.094541\times10^{-13}t^{7}+1.266174\times10^{-12}t^{6}-1.650255\times10^{-10}t^{5}-\\
&-2.098134\times10^{-8}t^{4}-1.372488\times10^{-6}t^{3}+2.149058\times10^{-5}t^{2}+\\
&+0.00183589t-0.034))/(-2.754623\times10^{-18}t^{10}+1.207033\times10^{-16}t^{9}+\\
&+1.445818\times10^{-14}t^{8}-4.148731\times10^{-12}t^{7}+1.654422\times10^{-10}t^{6}+\\
&+1.848225\times10^{-8}t^{5}-3.147884\times10^{-6}t^{4}+2.001388\times10^{-4}t^{3}-\\
&-0.00716454t^{2}+0.133877t-1.0),
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
r(t)_{[10,10]} &= (4.272355\times10^{-20}t^{10}-7.542976\times10^{-18}t^{9}+1.031839\times10^{-15}t^{8}-\\
&-1.176852\times10^{-13}t^{7}+1.030360\times10^{-11}t^{6}-5.910652\times10^{-10}t^{5}+\\
&+3.359540\times10^{-8}t^{4}-1.157012\times10^{-6}t^{3}+3.502658\times10^{-5}t^{2}-\\
&-9.424112\times10^{-5}t+0.018)/(1.442206\times10^{-20}t^{10}+9.094835\times10^{-18}t^{9}+\\
&+5.348670\times10^{-16}t^{8}-1.372052\times10^{-13}t^{7}-2.079556\times10^{-12}t^{6}+\\
&+1.973131\times10^{-9}t^{5}-5.501381\times10^{-8}t^{4}-1.030520\times10^{-5}t^{3}+\\
&+0.00119635t^{2}-0.0499133t+1.0),
\end{aligned}
\tag{39}
$$

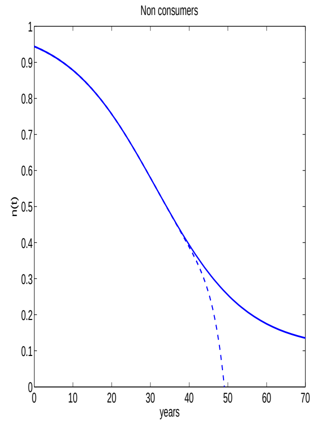

Figure 2. Comparison between $[10,10]$ Padé approximant for $n(t)$ (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dashdotted line) for $n(t)$.

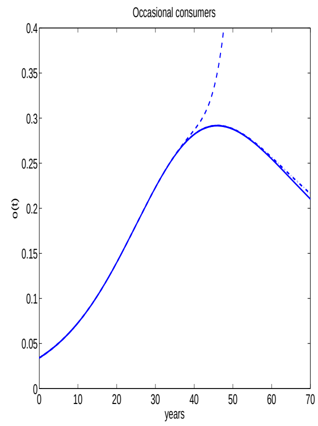

Figure 3. Comparison between $[10,10]$ Padé approximant for $o(t)$ (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dashdotted line) for $o(t)$.

$$
\begin{aligned}
h(t)_{[9,9]} = {} & (2.578423\times10^{-18}t^9 - 1.614545\times10^{-16}t^8 + 5.452005\times10^{-14}t^7 - \\
& -2.773418\times10^{-12}t^6 + 3.636500\times10^{-10}t^5 - 1.402250\times10^{-8}t^4 + \\
& +9.089968\times10^{-7}t^3 - 1.270334\times10^{-5}t^2 + \\
& +8.699086\times10^{-4}t + 0.004)/(2.175245\times10^{-18}t^9 + 6.140593\times10^{-17}t^8 - \\
& -1.341543\times10^{-14}t^7 + 5.913146\times10^{-12}t^6 + 5.358033\times10^{-10}t^5 - \\
& -5.700143\times10^{-8}t^4 + 2.880177\times10^{-7}t^3 + 5.354645\times10^{-4}t^2 - \\
& -0.0330272t + 1.0).
\end{aligned} \tag{40}
$$

We see in Figures 2-5 that the domain of convergence has been enlarged in about 20 years. The Padé approximants are similar to the *exact* solution in a range of 55-70 years depending on the subpopulation considered.

4.3. **Multi-stage HAM-Padé results.** To obtain an analytical approximation of the exact solutions of $n(t), o(t), r(t)$ and $h(t)$ valid for $t \in [0,+\infty]$ we apply the technique of Multi-stage HAM-Padé [2] dividing the domain of the solution in three intervals for t: $[0,50]$, $[50,150]$ and $[150,+\infty]$.

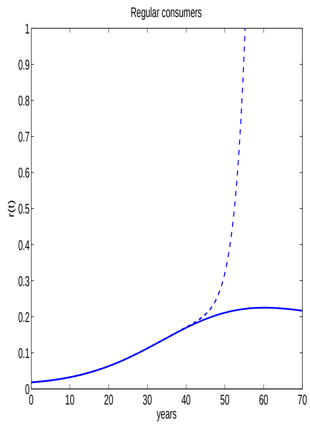

Figure 4. Comparison between $[10, 10]$ Padé approximant for $r(t)$ (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dashdotted line) for $r(t)$.

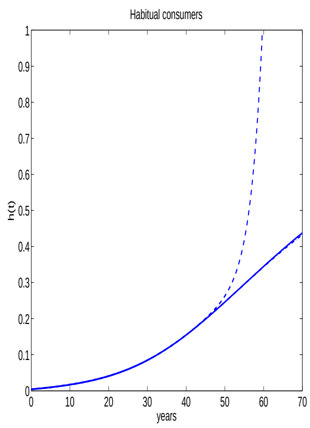

Figure 5. Comparison between $[9, 9]$ Padé approximant for $h(t)$ (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dashdotted line) for $h(t)$.

The process is divided in three stages:

(1) To obtain the 20 terms HAM solution starting from $t = 0$ and, then, the Padé approximants (see eqs.(37-40)). They are valid in $t \in [0, 50]$. We calculate the values given by the Padé approximants for $t = 50$ for each subpopulation and they are used as the initial values for the next stage.

(2) To obtain the 20 terms HAM solution starting from $t = 50$ and, then, the Padé approximants. They are valid in $t \in [50, 150]$. We calculate the values given by the Padé approximants for $t = 150$ for each subpopulation and they are used as the initial values for the next stage.

(3) To obtain the 20 terms HAM solution starting from $t = 150$ and the Padé approximants. They are valid in $t \in [150, +\infty]$.

Applying Multi-stage HAM-Padé after-treatment, we obtain the expressions for the second interval $[50, 150]$

$$\begin{aligned}
n(t)^{(2)}_{[10,10]} &= (2.011862\times10^{-21}t^{10} - 1.384324\times10^{-18}t^{9} + 7.426272\times10^{-16}t^{8} -\\
&-2.285260\times10^{-13}t^{7} + 4.926243\times10^{-11}t^{6} - 7.290338\times10^{-9}t^{5} +\\
&+7.801530\times10^{-7}t^{4} - 5.810934\times10^{-5}t^{3} + 0.00297688t^{2} - 0.0966085t +\\
&+1.691270)/(8.573118\times10^{-21}t^{10} - 1.222873\times10^{-18}t^{9} +\\
&+8.584942\times10^{-16}t^{8} - 2.125673\times10^{-13}t^{7} + 5.161492\times10^{-11}t^{6} -\\
&-7.041883\times10^{-9}t^{5} + 7.987776\times10^{-7}t^{4} - 5.710453\times10^{-5}t^{3} +\\
&+0.00302825t^{2} - 0.0937751t + 1.792012),
\end{aligned} \tag{41}$$

$$\begin{aligned}
o(t)^{(2)}_{[10,10]} &= (3.370291\times10^{-20}t^{10} - 4.935414\times10^{-18}t^{9} + 2.399295\times10^{-15}t^{8} -\\
&-3.862394\times10^{-13}t^{7} + 3.584550\times10^{-11}t^{6} - 1.756452\times10^{-9}t^{5} +\\
&+7.953822\times10^{-8}t^{4} - 1.369713\times10^{-6}t^{3} + 3.485674\times10^{-5}t^{2} +\\
&+0.00148980t + 0.00334892)/(-4.944626\times10^{-20}t^{10} + 1.826889\times10^{-16}t^{9} -\\
&-4.795985\times10^{-14}t^{8} + 8.045308\times10^{-12}t^{7} - 8.479973\times10^{-10}t^{6} +\\
&+6.468230\times10^{-8}t^{5} - 3.416117\times10^{-6}t^{4} + 1.283486\times10^{-4}t^{3} -\\
&-0.00296195t^{2} + 0.0371992t + 0.124878),
\end{aligned} \tag{42}$$

$$\begin{aligned}
r(t)^{(2)}_{[10,10]} &= (5.229988\times10^{-19}t^{10} + 4.983314\times10^{-17}t^{9} - 4.824407\times10^{-14}t^{8} +\\
&+9.546158\times10^{-12}t^{7} - 9.606764\times10^{-10}t^{6} + 6.390354\times10^{-8}t^{5} -\\
&-2.966500\times10^{-6}t^{4} + 9.606253\times10^{-5}t^{3} - 0.00218130t^{2} +\\
&+0.0287912t - 0.345775)/(1.035110\times10^{-17}t^{10} - 3.380140\times10^{-15}t^{9} +\\
&+6.284766\times10^{-13}t^{8} - 7.264348\times10^{-11}t^{7} + 5.683851\times10^{-9}t^{6} -\\
&-2.883816\times10^{-7}t^{5} + 7.807429\times10^{-6}t^{4} + 5.504524\times10^{-5}t^{3} -\\
&-0.0142067t^{2} + 0.541570t - 9.135778),
\end{aligned} \tag{43}$$

$$\begin{aligned}
h(t)^{(2)}_{[9,9]} &= (2.276291\times10^{-17}t^{9} - 5.783620\times10^{-15}t^{8} + 9.557840\times10^{-13}t^{7} -\\
&-8.216790\times10^{-11}t^{6} + 5.428294\times10^{-9}t^{5} - 2.353707\times10^{-7}t^{4} +\\
&+8.296566\times10^{-6}t^{3} - 1.802269\times10^{-4}t^{2} + 0.00329830t -\\
&-0.0119627)/(3.394871\times10^{-17}t^{9} - 8.517749\times10^{-15}t^{8} +\\
&+1.441565\times10^{-12}t^{7} - 1.283359\times10^{-10}t^{6} + 7.606555\times10^{-9}t^{5} -\\
&-1.032623\times10^{-7}t^{4} - 1.380302\times10^{-5}t^{3} +\\
&+0.00144199t^{2} - 0.0536429t + 1.216326).
\end{aligned} \tag{44}$$

For $t > 150$ we calculate a third, and last, stage for the Padé approximant. These are the expressions for the interval $[150, +\infty]$

$$\begin{aligned}
n(t)^{(3)}_{[10,10]} &= (3.066394\times10^{-24}t^{10} - 5.652783\times10^{-22}t^{9} + 2.573802\times10^{-19}t^{8} -\\
&-2.361469\times10^{-17}t^{7} + 6.860355\times10^{-15}t^{6} - 6.217387\times10^{-13}t^{5} +\\
&+1.707713\times10^{-10}t^{4} - 2.327901\times10^{-8}t^{3} + 3.084552\times10^{-6}t^{2} -\\
&-2.209831\times10^{-4}t + 0.00774959)/(2.937589\times10^{-23}t^{10} -\\
&-5.397343\times10^{-21}t^{9} + 2.440875\times10^{-18}t^{8} - 2.059931\times10^{-16}t^{7} +\\
&+5.485722\times10^{-14}t^{6} - 1.868013\times10^{-12}t^{5} + 5.456872\times10^{-10}t^{4} -\\
&-1.944084\times10^{-8}t^{3} + 4.118797\times10^{-6}t^{2} - 2.006525\times10^{-4}t + 0.00826242),
\end{aligned} \tag{45}$$

$$o(t)^{(3)}_{[10,10]} = (1.527048 \times 10^{-23}t^{10} - 7.988698 \times 10^{-21}t^{9} + 3.017139 \times 10^{-18}t^{8} - \\ -6.922456 \times 10^{-16}t^{7} + 1.264204 \times 10^{-13}t^{6} - 1.643907 \times 10^{-11}t^{5} + \\ +1.684722 \times 10^{-9}t^{4} - 1.126995 \times 10^{-7}t^{3} + 4.356271 \times 10^{-6}t^{2} - \\ -3.215763 \times 10^{-5}t + 0.00101223)/(1.181783 \times 10^{-22}t^{10} - \\ -6.164929 \times 10^{-20}t^{9} + 2.313217 \times 10^{-18}t^{8} - 5.218258 \times 10^{-15}t^{7} + \\ +9.245947 \times 10^{-13}t^{6} - 1.142126 \times 10^{-10}t^{5} + 1.120486 \times 10^{-8}t^{4} - \\ -7.667531 \times 10^{-7}t^{3} + 3.992385 \times 10^{-5}t^{2} - 0.00127184t + 0.0250215), \tag{46}$$

$$r(t)^{(3)}_{[10,10]} = (8.485794 \times 10^{-24}t^{10} - 4.944191 \times 10^{-21}t^{9} + 2.099703 \times 10^{-18}t^{8} - \\ -5.641152 \times 10^{-16}t^{7} + 1.284396 \times 10^{-13}t^{6} - 2.270029 \times 10^{-11}t^{5} + \\ +3.325213 \times 10^{-9}t^{4} - 3.371784 \times 10^{-7}t^{3} + 2.116259 \times 10^{-5}t^{2} - \\ -5.641431 \times 10^{-4}t + 0.00738523)/(7.461562 \times 10^{-23}t^{10} - \\ -4.291235 \times 10^{-20}t^{9} + 1.765718 \times 10^{-17}t^{8} - 4.358252 \times 10^{-15}t^{7} + \\ +8.506055 \times 10^{-13}t^{6} - 1.150698 \times 10^{-10}t^{5} + 1.257267 \times 10^{-8}t^{4} - \\ -9.460249 \times 10^{-7}t^{3} + 5.625197 \times 10^{-5}t^{2} - 0.00197422t + 0.0481363), \tag{47}$$

$$h(t)^{(3)}_{[9,9]} = (1.456022 \times 10^{-20}t^{9} - 7.620439 \times 10^{-18}t^{8} + 2.736731 \times 10^{-15}t^{7} - \\ -5.932607 \times 10^{-13}t^{6} + 9.905223 \times 10^{-11}t^{5} - 1.133551 \times 10^{-8}t^{4} + \\ +9.604742 \times 10^{-7}t^{3} - 4.767449 \times 10^{-5}t^{2} + 0.00124123t - \\ -0.0142619)/(2.226115 \times 10^{-20}t^{9} - 1.160506 \times 10^{-17}t^{8} + \\ +4.134347 \times 10^{-15} - t^{7} - 8.801044 \times 10^{-13}t^{6} + 1.430636 \times 10^{-10}t^{5} - \\ -1.583420 \times 10^{-8}t^{4} + 1.345481 \times 10^{-6}t^{3} - 7.519404 \times 10^{-5}t^{2} + \\ +0.00291876t - 0.0553057). \tag{48}$$

Figures 6-9 show a comparison between the exact solutions for the model of cocaine consumption in Spain and the analytical Multi-stage HAM-Padé approximations for $n(t), o(t), r(t)$ and $h(t)$. We see that our analytical approximations are almost on top of the exact solutions up to $t = 500$ years.

To confirm that the proposed approximation exhibits a large domain of convergence, we show in Table 1 the values of the asymptote that our approximation yields and the exact values (28-31) of the asymptote. The difference is around 0.1% resulting a good agreement with exact values.

Therefore, we conclude that we have obtained a good analytical approximation for the complete domain $[0, \infty]$ in the form of a piecewise continuous function with expressions in eqs. (37-40) for $0 \leq t \leq 50$, eqs. (41-44) for $50 \leq t \leq 150$ and eqs. (45-48) for $150 \leq t \leq \infty$.

It is important to notice that the after-treatment with Padé approximants is a key factor to reduce significantly the number of required segments for multi-stage HAM protocol, in comparison with other works [2].

TABLE 1. Exact and approximations of asymptote for model of cocaine consumption in Spain

Variable	Exact (28-31)	HAM (33-36)	HAM-Padé (37-40)	Multi-HAM-Padé (45-48)
$n(\infty)$	0.1043	$-\infty$	0.8182	0.1043
$o(\infty)$	0.1291	$+\infty$	-0.0247	0.1292
$r(\infty)$	0.1133	$+\infty$	2.9623	0.1137
$h(\infty)$	0.6531	$-\infty$	1.1853	0.6540

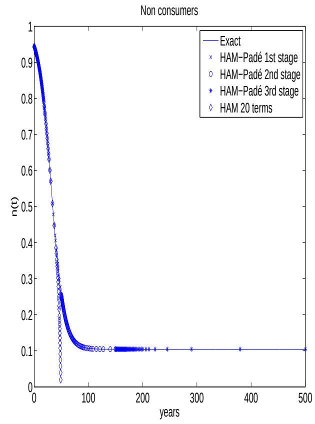

Figure 6. Solution of HAM with 20 terms for $n(t)$ (diamonds), the Multi-stage HAM-Padé solution (crosses, circles and stars) and the *exact* solution (solid line).

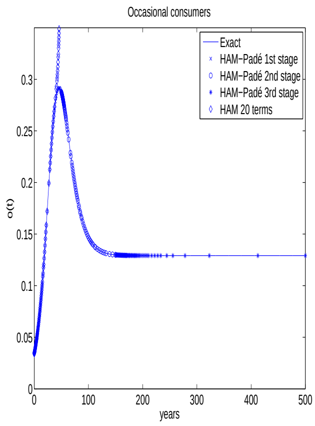

Figure 7. Solution of HAM with 20 terms for $o(t)$ (diamonds), the Multi-stage HAM-Padé solution (crosses, circles and stars) and the *exact* solution (solid line).

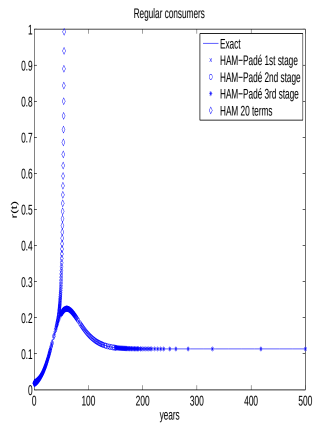

Figure 8. Solution of HAM with 20 terms for $r(t)$ (diamonds), the Multi-stage HAM-Padé solution (crosses, circles and stars) and the *exact* solution (solid line).

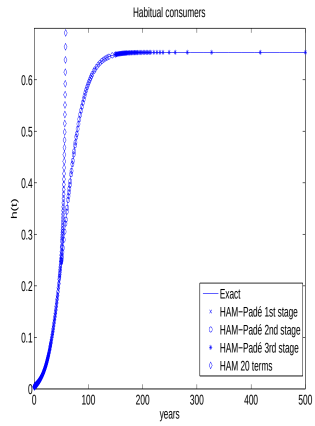

Figure 9. Solution of HAM with 20 terms for $h(t)$ (diamonds), the Multi-stage HAM-Padé solution (crosses, circles and stars) and the *exact* solution (solid line).

5. Conclusion

In this paper, the analytical methods HAM, HAM-Padé and Multi-stage Ham-Padé are applied to construct an approximate analytical solution for the model of the evolution of cocaine consumption in Spain valid for the complete domain $t \in [0, \infty]$.

The HAM method provides solutions in the form of power series whose components can be easily computed. However, even with 20 terms, the domain of convergence is only about 35 years.

Therefore, we applied Padé after-treatment to the HAM solutions to enlarge successfully the domain of convergence. The HAM-Padé solutions are valid until around $t = 70$ years.

Finally, we applied Multi-stage HAM-Padé. The HAM-Padé solution is used to obtain the initial values at $t = 70$ years. Thereupon, we construct a second HAM-Padé solution which is valid for the interval $[50, 150]$ years. And we repeat a third stage using the second solution to obtain the initial values for the third interval. This third HAM-Padé solution is a good approximation for the real solution of the system for $t \in [150, \infty]$ years, as it is shown in Table 1 comparing the exact value of the asymptote with the value given by the approximate solution.

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