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ON SOME NEW SEQUENCE SPACES OF NON-ABSOLUTE TYPE

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Abstract. In this paper, we define the new sequence spaces $c_0^\lambda(u)$, $c^\lambda(u)$ and $\ell_\infty^\lambda(u)$, where $\lambda = (\lambda_k)_{k=0}^\infty$ is a strictly increasing sequence of positive reals tending to ∞, $u = (u_n)$ is a sequence of complex numbers. $\Lambda-$ transforms of these spaces are in the spaces $c_0(u)$, $c(u)$ and $\ell_\infty(u)$, respectively. We also establish some inclusion relations between these spaces which are BK-spaces.

Keywords: sequence space, BK-space, matrix mapping, sequence spaces of non-absolute type.

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1. Introduction

By ω, we denote the space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space.

We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively, which are BK-spaces with the same norm given by

$$\|x\|_\infty = \sup_k |x_k|,$$

for all $k \in \mathbb{N}$.

A sequence space X with a linear topology is called a K-space provided each of the maps $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$ where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, ...\}$. A K-space X is called an FK-space provided X is a complete linear metric space. An FK-space whose topology is normable is called a BK- space.

Let X and Y be sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{A_n(x)\}$, the A- transform of x, exists and is in Y, where

$$A_n(x) = \sum_k a_{nk} x_k \qquad (n \in \mathbb{N}). \tag{1}$$

By (X, Y), we denote the class of all infinite matrices that map X into Y. Thus, $A \in (X, Y)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and $Ax \in Y$ for all $x \in X$.

For a sequence space X, the matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in \omega : Ax \in X\} \tag{2}$$

which is a sequence space.

Throughout this paper, let $\lambda = (\lambda_k)_{k=0}^\infty$ be a strictly increasing sequence of positive reals tending to ∞, that is

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$$0 < \lambda_0 < \lambda_1 < ... \quad \text{and } \lambda_k \to \infty \text{ as } k \to \infty. \tag{3}$$

We shall use the convention that any term with a negative subscript is equal to zero, e.g., $\lambda_{-1} = 0$ and $x_{-1} = 0$.

We define the infinite matrix $\Lambda = (\lambda_{nk})_{n,k=0}^{\infty}$ by

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & 0 \leq k \leq n \\ 0\ , & k > n \end{cases} \tag{4}$$

for all $k \in \mathbb{N}$. Then, for any sequence $x = (x_k) \in \omega$, the Λ- transform of x is the sequence $\Lambda(x) = \{\Lambda_n(x)\}$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \tag{5}$$

for all $n \in \mathbb{N}$. It is obvious that by (4) that the matrix $\Lambda = (\lambda_{nk})$ is a triangle, i.e., $\lambda_{nn} \neq 0$ and $\lambda_{nk} = 0$ for all $k > n$ $(n \in \mathbb{N})$.

The idea of constructing a new sequence by means of the matrix domain of a particular limitation method has recently been studied by several authors, e.g., Altay and Başar [1], Mursaleen et all [2], Mursaleen and Noman [7, 8], Malkowsky [5], Malkowsky and Savaş [6].

2. Main results

In this section we introduce some new sequence spaces, as the sets of all sequences whose $\Lambda-$ transforms are in the spaces $c(u)$, $c_0(u)$ and $\ell_\infty(u)$, that is

$$\begin{aligned} c_0^\lambda(u) &= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} u_n \left(\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) = 0 \right\}, \\ c^\lambda(u) &= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} u_n \left(\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) \text{ exists} \right\}, \\ \ell_\infty^\lambda(u) &= \left\{ x = (x_k) \in \omega : \sup_n \left| \frac{u_n}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\}, \end{aligned}$$

where $u = (u_n)$ is a sequence of complex numbers such that $u_n \neq 0$ for all $n \in \mathbb{N}$.

Theorem 2.1. *The spaces $c_0^\lambda(u)$, $c^\lambda(u)$ and $\ell_\infty^\lambda(u)$ are Banach spaces with the norm*

$$\|x\| = \sup_n |u_n \Lambda_n(x)| .$$

Theorem 2.2. *The spaces $c_0^\lambda(u)$, $c^\lambda(u)$ and $\ell_\infty^\lambda(u)$ are BK$-$ spaces with the norm*

$$\|x\| = \sup_n |u_n \Lambda_n(x)| .$$

Proof. The proof follows in [8]. □

3. Some inclusion relations

In the present section, we give some inclusion relations concerning the spaces $c_0^\lambda(u)$, $c^\lambda(u)$ and $\ell_\infty^\lambda(u)$.

Lemma 3.1. *For any sequence* $x = (x_k) \in \omega$, *the equalities*

$$S_n(x) = x_n - \Lambda_n(x) \quad (n \in \mathbb{N}) \tag{6}$$

and

$$S_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} [\Lambda_n(x) - \Lambda_{n-1}(x)] \quad (n \in \mathbb{N}) \tag{7}$$

hold, where $S(x) = \{S_n(x)\}$ *is the sequence defined by*

$$S_0(x) = 0 \text{ and } S_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_{k-1}(x_k - x_{k-1}) \quad (n \geq 1).$$

Lemma 3.2. *For any sequence* $\lambda = (\lambda_k)_{k=0}^{\infty}$ *satisfying (3), we have*

(a) $\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)_{k=0}^{\infty} \notin \ell_\infty$ *if and only if* $\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} = 1$,

(b) $\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)_{k=0}^{\infty} \in \ell_\infty$ *if and only if* $\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} > 1$.

It is obvious that Lemma 3.2 still holds if the sequence $\{\lambda_k/(\lambda_k - \lambda_{k-1})\}$ is replaced by $\{\lambda_k/(\lambda_{k+1} - \lambda_k)\}$.

Theorem 3.1. *(i) If* $|u_n| \leq 1$ *for all* $n \in \mathbb{N}$, *then* $\ell_\infty \subseteq \ell_\infty^\lambda(u)$.
(ii) If $|u_n| \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\ell_\infty^\lambda(u) \subseteq \ell_\infty$.

Proof. Let $x \in \ell_\infty$. Then, there is a constant $M > 0$ such that $|x_k| \leq M$ for all $k \in \mathbb{N}$. Since $|u_n| \leq 1$ for all $n \in \mathbb{N}$, we have for every $n \in \mathbb{N}$ that

$$|u_n \Lambda_n(x)| \leq \frac{|u_n|}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k| \leq \frac{M}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = M$$

which shows that $x \in \ell_\infty^\lambda(u)$.

(ii) The proof is seen easily similiar to (i). □

Theorem 3.2. *The inclusions* $c_0^\lambda(u) \subset c^\lambda(u) \subset \ell_\infty^\lambda(u)$ *strictly hold.*

Proof. The proof is seen easily. □

Lemma 3.3. *The inclusion* $\ell_\infty^\lambda(u) \subset \ell_\infty$ *holds if and only if* $S(x) \in \ell_\infty$ *for every sequence* $x \in \ell_\infty^\lambda(u)$, *where* $\left(\frac{1}{u_n}\right) \in \ell_\infty$.

Proof. Suppose that the inclusion $\ell_\infty^\lambda(u) \subset \ell_\infty$ holds and take any $x = (x_k) \in \ell_\infty^\lambda(u)$. Then $x \in \ell_\infty$ by the hypothesis. Since $x \in \ell_\infty^\lambda(u)$ we have $\|\Lambda(x)\|_\infty < \infty$. Then

$$\|S(x)\|_\infty \leq \|x\|_\infty + \|\Lambda(x)\|_\infty < \infty.$$

So, we have $S(x) \in \ell_\infty$.

Conversely, let $x \in \ell_\infty^\lambda(u)$. Then, we have by the hypothesis that $S(x) \in \ell_\infty$. It follows by (6) that

$$\|x\|_\infty \leq \|S(x)\|_\infty + \|\Lambda(x)\|_\infty < \infty.$$

Hence, the inclusion $\ell_\infty^\lambda(u) \subset \ell_\infty$ holds and this completes the proof. □

Let $\left(\frac{1}{u_n}\right) \in \ell_\infty$. Then we have the following theorems.

Theorem 3.3. *The equality* $\ell_\infty = \ell_\infty^\lambda(u)$ *holds if and only if* $S(x) \in \ell_\infty$ *for every sequence* $x \in \ell_\infty^\lambda(u)$.

Proof. Suppose that the equality $\ell_\infty = \ell_\infty^\lambda(u)$ holds. Then, the inclusion $\ell_\infty^\lambda(u) \subset \ell_\infty$ holds which leads us with Lemma 3.5 to the consequence that $S(x) \in \ell_\infty$ for every $x \in \ell_\infty^\lambda(u)$.

Conversely, suppose that $S(x) \in \ell_\infty$ for every $x \in \ell_\infty^\lambda(u)$. Then, we have by Lemma 3.1 that the inclusion $\ell_\infty^\lambda(u) \subset \ell_\infty$ holds. Combining this with the inclusion by Theorem 3.3 we get the equality $\ell_\infty = \ell_\infty^\lambda(u)$. □

Theorem 3.4. *The inclusion* $\ell_\infty \subset \ell_\infty^\lambda(u)$ *strictly holds if and only if* $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$.

Proof. Suppose that the inclusion $\ell_\infty \subset \ell_\infty^\lambda(u)$ strict. Theorem 3.3 implies the existence of a sequence $x \in \ell_\infty^\lambda(u)$ such that $S(x) = \{S_n(x)\} \notin \ell_\infty$. Since $x \in \ell_\infty^\lambda(u)$, we have $\Lambda(x) = \{\Lambda_n(x)\} \in \ell_\infty$ and hence $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_\infty$. Combining this with the fact that $\{S_n(x)\} \notin \ell_\infty$, we have $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$ from (7) and hence $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$. Hence we have by Lemma 3.2 (a) that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$ which shows the necessity of the theorem. □

Conversely, suppose that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$. Then, we have by Lemma 3.2 (a) that $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$. Consider the sequence $x = (x_k)$ defined by $x_k = (-1)^k \lambda_k/(\lambda_k - \lambda_{k-1})$ for all $k \in \mathbb{N}$. It is obvious that $x = (x_k) \notin \ell_\infty$. On the other hand, we have for every $n \in \mathbb{N}$

$$|\Lambda_n(x)| = \frac{1}{\lambda_n}\left|\sum_{k=0}^{n}(-1)^k \lambda_k\right| \le \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1}) = 1.$$

Hence $x \in \ell_\infty^\lambda$ and since $\left(\frac{1}{u_n}\right) \in \ell_\infty$ we have $x \in \ell_\infty^\lambda(u)$. Thus, by combining this with the inclusion $\ell_\infty \subset \ell_\infty^\lambda(u)$, we deduce that this inclusion is strict. This completes the proof.

Theorem 3.5. *The equality* $\ell_\infty^\lambda(u) = \ell_\infty$ *holds if and only if* $\lim\inf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$.

Proof. The necessity follows from Theorem 3.4. Because if the equality $\ell_\infty^\lambda(u) = \ell_\infty$ holds, then $\lim\inf_{n\to\infty} \lambda_{n+1}/\lambda_n \neq 1$ and hence $\lim\inf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$.

Conversely, suppose that $\lim\inf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$. Then, by Lemma 3.2 (b) we have the bounded sequence $\{\lambda_n/(\lambda_n - \lambda_{n-1})\}$ and so $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \in \ell_\infty$. Now, let $x \in \ell_\infty^\lambda(u)$. Then, $\Lambda(x) = \{\Lambda_n(x)\} \in \ell_\infty$ and hence $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_\infty$. Thus, by (7) we have $\{S_n(x)\} \in \ell_\infty$. So, by Theorem 3.3r we have the equality $\ell_\infty^\lambda(u) = \ell_\infty$. □

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