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ON IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN THE TOPOLOGY INDUCED BY A FUZZY 2-NORM

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Abstract. In this paper we introduce and investigate $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence, $\mathcal{I}_2$-limit points, and $\mathcal{I}_2$-cluster points of a double sequence in a fuzzy 2-normed linear space. We prove a decomposition theorem for $\mathcal{I}_2$-convergence of double sequences. The notions of $\mathcal{I}_2$-double Cauchy and $\mathcal{I}_2^*$-double Cauchy sequence are defined, and some of their properties are studied.

Keywords: fuzzy number, fuzzy normed space, ideal convergence, ideal Cauchy sequence.

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1. Introduction

In 1965, Zadeh [41] introduced the notion of fuzzy sets and since then fuzzy set theory found very useful applications in various fields of mathematics and many other sciences. In particular, a number of papers deals with fuzzy real numbers introduced in [8]. In this paper we are interested in ideal convergence of double sequences in fuzzy 2-normed linear spaces.

The concept of 2-normed spaces was introduced by Gähler [20] in the 1960's, and then this concept has been studied by many authors [7, 11, 12]; for more information see [37].

The idea of fuzzy norm was initiated by Katsaras [27], and Matloka [32] introduced convergence of sequences of fuzzy numbers. After that a big number of works dealing with fuzzy norms and fuzzy numbers, in particular with convergence of sequences of fuzzy numbers, appeared in the literature (see, for example, [15, 21, 22, 31, 35, 38]). By using fuzzy numbers Felbin [18] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [24]. Cheng and Mordeson [9], and also Bag and Samanta [4] introduced a fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Michalek type [30]. In [5], a comparative study of the fuzzy norms defined by Katsaras [30], Felbin [18], and Bag and Samanta [4] was given.

Using the concept of ideal, Kostyrko et al. [28] introduced the notion of ideal convergence which is a common generalization of ordinary convergence and statistical convergence [17, 19, 29, 40, 14] and provides a general framework for study of various kinds of convergence. Ideal and statistical convergence were studied in the fuzzy set theory context from different points of view (see [1, 2, 3, 6, 16, 23, 25, 26, 33, 36, 39]).

This paper is organized as follows: In the second section, we present some preliminary definitions and results related to fuzzy numbers, fuzzy normed spaces and ideal convergence. In the third section, we introduce the notions of $\mathcal{I}_2^E$-convergence and $\mathcal{I}_2^{*E}$-convergence of double sequences in a fuzzy 2-normed space E and prove some basic results in this connection. We

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also study the concepts of $\mathcal{I}_2^E$-limit points and $\mathcal{I}_2^E$-cluster points of double sequences in fuzzy 2-normed spaces. In fourth section, we introduce the notions of $\mathcal{I}_2^E$-double Cauchy and $\mathcal{I}_2^{*E}$-double Cauchy sequences in a fuzzy 2-normed space.

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and real numbers, respectively. J denotes the closed unit interval $[0,1] \subset \mathbb{R}$, and $\mathcal{I}_2$ is an ideal on $\mathbb{N} \times \mathbb{N}$.

2. Definitions and preliminaries

In this section we recall some basic definitions and notions related to 2-normed spaces, fuzzy numbers, fuzzy normed and fuzzy 2-normed spaces, and ideal convergence.

Definition 2.1. ([20]) *Let X be a real vector space of dimension d, $2 \le d < \infty$. A 2-*norm *on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies:*

(i) $\|x,y\| = 0$ *if and only if x and y are linearly dependent;*
(ii) $\|x,y\| = \|y,x\|$ *for all $x,y \in X$;*
(iii) $\|cx,y\| = |c|\|x,y\|$ *for all $x,y \in X$ and $c \in \mathbb{R}$;*
(iv) $\|x+y,z\| \le \|x,z\| + \|y,z\|$ *for all $x,y,z \in X$.*

*The pair $(X, \|.,.\|)$ is called a 2-*normed space.

Definition 2.2. ([8], [18], [24]) *A* fuzzy real number*, or simply* fuzzy number*, is a fuzzy set $X : \mathbb{R} \to [0,1]$ having the following properties:*

(a) *X is normal (i.e. there exists a $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$);*
(b) *X is fuzzy convex (i.e. for $r,s \in \mathbb{R}$ and $\lambda \in J = [0,1]$, $X(\lambda r+(1-\lambda)s) \ge \min\{X(r), X(s)\}$);*
(c) *X is upper semi-continuous (i.e. $X^{\leftarrow}([0,t+\varepsilon))$ is open in $\mathbb{R}$ for each $t \in J$ and each $\varepsilon > 0$);*
(d) *The closure of the set $[X]_0 := \{t \in \mathbb{R} : X(t) > 0\}$ is compact.*

Let $\mathcal{F}(\mathbb{R})$ be the set of all fuzzy real numbers. For $X \in \mathcal{F}(\mathbb{R})$, the *$\alpha$-level set* of X [18] is defined as:

$$[X]_\alpha = \begin{cases} \{t \in \mathbb{R} : X(t) \ge \alpha\}, & \text{if } 0 < \alpha \le 1; \\ \mathrm{Cl}(\{t \in \mathbb{R} : X(t) > 0\}), & \text{if } \alpha = 0. \end{cases}$$

A real number x can be considered as a fuzzy number $\overline{x}$ defined by

$$\overline{x}(t) = \begin{cases} 1, & \text{if } t = x; \\ 0, & \text{if } t \ne x, \end{cases}$$

i.e., $\mathbb{R}$ can be embedded in $\mathcal{F}(\mathbb{R})$.

It is easy to show that X is a fuzzy number if and only if $[X]_\alpha$ is a nonempty bounded and closed interval for each $\alpha \in [0,1]$. We denote this interval $[X]_\alpha = [X_\alpha^-, X_\alpha^+]$ (see [21]).

Remark 2.3. *The above definition of fuzzy numbers slightly differs from that of* [18]*, where $X_\alpha^- = -\infty$ and $X_\alpha^+ = +\infty$ are also admissible, and the zero-level set is not considered.*

A fuzzy number X is called a *non-negative fuzzy number* if $X(t) = 0$ for $t < 0$. Let $\mathcal{F}^*(\mathbb{R})$ be the set of all non-negative fuzzy numbers. Clearly, $X \in \mathcal{F}^*(\mathbb{R})$ if and only if $X_\alpha^- \ge 0$ for each $\alpha \in J$, and $\overline{0} \in \mathcal{F}^*(\mathbb{R})$.

A partial order $\preceq$ on $\mathcal{F}(\mathbb{R})$ is defined by $X \preceq Y$ if and only if $X_\alpha^- \le Y_\alpha^-$ and $X_\alpha^+ \le Y_\alpha^+$, for all $\alpha \in J$. The strict inequality $\prec$ on $\mathcal{F}(\mathbb{R})$ is defined by $X \prec Y$ if and only if $X_\alpha^- < Y_\alpha^-$ and $X_\alpha^+ < Y_\alpha^+$, for all $\alpha \in J$.

Let $X, Y \in \mathcal{F}(\mathbb{R})$, define

$$\overline{d}(X,Y) = \sup_{\alpha \in [0,1]} \max\{|X_\alpha^- - Y_\alpha^-|, |X_\alpha^+ - Y_\alpha^+|\}.$$

Then $\overline{d}$ is called the *supremum metric* on $\mathcal{F}(\mathbb{R})$. It is known that $(\mathcal{F}(\mathbb{R}), \overline{d})$ is a complete metric space (for details see [24]).

Let (X_k) be a sequence in $\mathcal{F}(\mathbb{R})$ and $X_0 \in \mathcal{F}(\mathbb{R})$. We say that (X_k) *converges to* X_0 *with respect to the metric* $\overline{d}$ if $\lim\limits_{k\to\infty} \overline{d}(X_k, X_0) = 0$. In this case we write $X_k \overset{\overline{d}}{\to} X_0$ or $\overline{d} - \lim_{k\to\infty} X_k = X_0$,

Now we define the notion of fuzzy 2-normed space.

Let E be a real vector space with the zero element θ, let $\|.,.\| : E \times E \to \mathcal{F}(\mathbb{R})$, and let the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

Definition 2.4. *The quadruple* $(E, \|.,.\|, L, R)$ *is called a* fuzzy 2-normed space *and* $\|.,.\|$ *a* fuzzy 2-norm, *if the following axioms are satisfied:*

(2FN1) $\|X, Y\| = \overline{0}$ *if and only if* X *and* Y *are linearly dependent;*

(2FN2) $\|\lambda X, Y\| = |\lambda| \|X, Y\|$, $\lambda \in \mathbb{R}$;

(2FN3) *For all* $X, Y, Z \in E$,

  (i) $\|X + Y, Z\|(r + s) \geq L(\|X, Z\|(r), \|Y, Z\|(s))$, *whenever* $r \leq \|X, Z\|_1^-, s \leq \|Y, Z\|_1^-$ *and* $r + s \leq \|X + Y, Z\|_1^-$,

  (ii) $\|X + Y, Z\|(r + s) \geq R(\|X, Z\|(r), \|Y, Z\|(s))$, *whenever* $r \geq \|X, Z\|_1^-, s \geq \|Y, Z\|_1^-$ *and* $r + s \geq \|X + Y, Z\|_1^-$

In the sequel we take $L(p,q) = \min\{p,q\}$ and $R(p,q) = \max\{p,q\}$, for all $p, q \in [0,1]$ and write $(E, \|.,.\|)$ or simply E, for such L and R.

Remark 2.5. *If* $L = \min$, *then the triangle inequality* $(2FN3)(i)$ *in Definition* 2.4 *is equivalent to the triangle inequality* $\|X + Y, Z\|_\alpha^- \leq \|X, Z\|_\alpha^- + \|Y, Z\|_\alpha^-$, *for all* $X, Y, Z \in E$ *and* $\alpha \in [0,1]$, *while the inequality* $(2FN3)(ii)$, *with* $R = \max$, *is equivalent to* $\|X + Y, Z\|_\alpha^+ \leq \|X, Z\|_\alpha^+ + \|Y, Z\|_\alpha^+$, *for all* $\alpha \in [0,1]$ *and* $X, Y, Z \in E$.

In fact we have the following result.

Lemma 2.6. *For* $L = \min$ *and* $R = \max$, *we have that for each* $\alpha \in [0,1]$, $\|X, Z\|_\alpha^-$ *and* $\|X, Z\|_\alpha^+$ *are norms on* E *in the usual sense.*

The following example is similar to [21, Example 2.1] concerning fuzzy normed linear spaces.

Example 2.1. Let $(E, \|\cdot,\cdot\|_u)$ be an ordinary 2-normed linear space. Then a fuzzy 2-norm on E can be obtained as

1. $\|X, Y\| = \overline{0}$ if X and Y are linearly dependent;
2. $$\|X, Y\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a\|X, Y\|_u \text{ or } t \geq b\|X, Y\|_u; \\ \frac{t}{(1-a)\|X,Y\|_u} - \frac{a}{1-a}, & \text{if } a\|X, Y\|_u \leq t \leq \|X, Y\|; \\ \frac{1}{(1-b)\|X,Y\|_u} - \frac{b}{1-b}, & \text{if } \|X, Y\|_u \leq t \leq b\|X, Y\|_u. \end{cases}$$

if X and Y are linearly independent and $0 < a < 1$, $1 < b < \infty$;

Hence $(E, \|.,.\|)$ is a fuzzy 2-normed space. The fuzzy 2-norm considered above is called a *triangular fuzzy 2-norm.*

For $X \in E$, $\varepsilon > 0$ and $\alpha \in [0,1]$, the (ε, α)-*neighborhood of* X is the set

$$U_X(\varepsilon, \alpha) = \{Y \in E : \|X - Y, Z\|_\alpha^+ < \varepsilon, \text{ for all } Z \in E\}.$$

The (ε, α)-*neighborhood system at* X is the collection

$$U_X = \{U_X(\varepsilon, \alpha) : \varepsilon > 0, \alpha \in [0,1]\},$$

and the (ε, α)*-neighborhood system for* E is the union $U = \bigcup_{X\in E} U_X$. It is easy to see that U generates a first countable Hausdorff topology on E.

Definition 2.7. *Let* $(E, \|.,.\|)$ *be a fuzzy 2-norm space. A sequence* $\{X_k\}$ *in* E *is said to be* convergent to $X_0 \in E$ with respect to the norm on E, *and we denote this by* $X_k \to X_0$*, provided* $\overline{d} - \lim_{k\to\infty} \|X_k - X_0, Z\| = \overline{0}$ *for all* $Z \in E$*, i.e., for every* $\varepsilon > 0$ *there exists an integer* $k_0 = k_0(\varepsilon)$ *in* $\mathbb{N}$ *such that* $\overline{d}(\|X_k - X_0, Z\|, \overline{0}) < \varepsilon$*, for* $k \geq k_0$.

This is the same as to say that for every $\varepsilon > 0$ there exists an integer $k_0(\varepsilon)$ in $\mathbb{N}$ such that $\sup_{\alpha\in[0,1]} \|X_k - X_0, Z\|_\alpha^+ = \|X_k - X_0, Z\|_0^+ < \varepsilon$, for $k \geq k_0$.

In terms of neighborhoods, we have $X_k \to X_0$, provided that for every $\varepsilon > 0$ there exists an integer $k_0(\varepsilon)$ in $\mathbb{N}$ such that $X_k \in U_{X_0}(\varepsilon, 0)$ for all $k \geq k_0$ and all $Z \in E$.

Finally, we give some basic facts about classic notions ideals and filters.

Let $Y \neq \emptyset$. Then:

1. A family $\mathcal{I}$ of subsets of Y is said to be an *ideal* in Y provided the following conditions hold: (i) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$, and (ii) $A \in \mathcal{I}$ and $B \subset A$ imply $B \in \mathcal{I}$. If $Y \notin \mathcal{I}$, then $\mathcal{I}$ is called a *proper ideal.*

2. A non-empty family $\mathcal{F}$ of subsets of Y is said to be a *filter* on Y if (i) $\emptyset \notin \mathcal{F}$,(ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and (iii) $A \in \mathcal{F}$ and $A \subset B \subset Y$ imply $B \in \mathcal{F}$.

A proper ideal $\mathcal{I}$ is said to be *admissible* if $\{x\} \in \mathcal{I}$ for each $x \in Y$. An admissible ideal $\mathcal{I}$ on $\mathbb{N}$ is said to have the *property* (AP) [28] if for any sequence $\{A_1, A_2, \cdots\}$ of pairwise disjoint sets of $\mathcal{I}$, there is a sequence $\{B_1, B_2, \cdots\}$ of sets such each symmetric difference $A_i \Delta B_i (i = 1, 2, \cdots)$ is finite and $\bigcup_{i=1}^{\infty} B_i \in \mathcal{I}$.

If $\mathcal{I}$ is a proper ideal on Y, then the family

$$\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I} : M = Y \setminus A\}$$

is a filter in Y. It is called the *filter associated with the ideal* $\mathcal{I}$.

In what follows the symbol $\mathcal{I}_2$ denotes an ideal on $\mathbb{N} \times N$, and $(E, \|.,.\|)$ is a fuzzy 2-normed space.

3. Ideal convergence in fuzzy 2-normed linear spaces

In this section we introduce the notions of $\mathcal{I}_2^E$-convergence and $\mathcal{I}_2^{*E}$-convergence of a double sequence in a fuzzy 2-normed space $(E, \|.,.\|$ and present some basic results on this convergence. We also introduce the notions of $\mathcal{I}_2$-limit point and $\mathcal{I}_2$-cluster point of a double sequence in $(E, \|.,.\|$.

We begin with the following definition.

Definition 3.1. *A double sequence* $\{X_{jk}\}$ *in a fuzzy 2-normed space* $(E, \|.,.\|)$ *is said to be* E-convergent to X_0 *if for every* $\varepsilon > 0$ *and each* $Z \in E$ *there exists a positive integer* $n_0 = n_0(\varepsilon)$ *such that*

$$X_{jk}, Z \in U_{X_0}(\varepsilon, 0) \text{ for each } j, k \geq n_0.$$

In this case we write E-$\lim \|X_{jk} - X_0, Z\|_0^+ = 0$.

Definition 3.2. *Let* $(E, \|.,.\|)$ *be a fuzzy 2-normed space and* $\mathcal{I}_2$ *an ideal on* $\mathbb{N} \times \mathbb{N}$*. A double sequence* $\{X_{jk}\}$ *in* E *is said to be* $\mathcal{I}_2^E$-convergent to $X_0 \in E$ with respect to the fuzzy 2-norm on E *if for each* $\varepsilon > 0$ *and each* $Z \in E$*, the set* $A(\varepsilon) := \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\}$ *belongs to* $\mathcal{I}_2$.

In this case, we write $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. The element X_0 is called the $\mathcal{I}_2^E$-*limit* of $\{X_{jk}\}$ in E.

Remark 3.3. *(a) In terms of neighborhoods, we have* $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$, *provided that for each* $\varepsilon > 0$ *and* $Z \in E$,

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : X_{jk}, Z \notin U_{X_0}(\varepsilon, 0)\} \in \mathcal{I}_2.$$

The above definition can be expressed also in the following way:

$$X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0 \Longleftrightarrow \mathcal{I}_2^E - \lim_{j,k\to\infty} \|X_{jk} - X_0, Z\|_0^+ = 0, \text{ for all } Z \in E.$$

(b) Note that $\mathcal{I}_2^E - \lim_{j,k\to\infty} \|X_{jk} - X_0, Z\|_0^+ = 0$, *for all* $Z \in E$ *implies*

$$\mathcal{I}_2^E - \lim \|X_{jk} - X_0, Z\|_\alpha^- = \mathcal{I}_2^E - \lim \|X_{jk} - X_0, Z\|_\alpha^+$$

for each $\alpha \in [0,1]$ *and each* $Z \in E$.

(It is because $0 \leq \|X_{jk} - X_0, Z\|_\alpha^- \leq \lim \|X_{jk} - X_0, Z\|_\alpha^+ \leq \|X_{jk} - X_0, Z\|_0^+$, *holds for each* $(j,k) \in \mathbb{N}\times\mathbb{N}$, $\alpha \in [0,1]$ *and each* $Z \in E$.*)*

Example 3.1. (1) If we take $\mathcal{I}_2 = \mathcal{I}_{\text{fin}} = \{A \subset \mathbb{N}\times\mathbb{N} : A \text{ is finite}\}$, then $\mathcal{I}_{\text{fin}}$ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}$, and the corresponding convergence coincides with ordinary convergence with respect to the fuzzy 2-norm on E (Definition 3.1).

(2) If we take $\mathcal{I}_2 = \mathcal{I}_{\delta_2} = \{A \subset \mathbb{N}\times\mathbb{N} : \delta_2(A) = 0\}$, then $\mathcal{I}_{\delta_2}$ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}$, and the corresponding convergence coincides with statistical convergence with respect to the fuzzy 2-norm on E.

Proposition 3.4. *Let* $(E, \|.,.\|)$ *be a fuzzy 2-normed space. If a double sequence* $\{X_{jk}\}$ *is* $\mathcal{I}_2^E$*-convergent with respect to the norm on* E*, then* $\mathcal{I}_2^E$*-limit is unique.*

Proof. Let us assume that $X_{jk} \xrightarrow{I_2^E} X_0$ and $X_{jk} \xrightarrow{I_2^E} Y_0$, where $X_0 \neq Y_0$. Since $X_0 \neq Y_0$, select $\varepsilon > 0$ so that $U_{X_0}(\varepsilon, 0)$ and $U_{Y_0}(\varepsilon, 0)$ are disjoint neighborhoods of X_0 and Y_0. Since X_0 and Y_0 both are $\mathcal{I}_2^E$-limit of the sequence $\{X_{jk}\}$, we have that for each $Z \in E$ the sets

$$A = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\}$$

and

$$B = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - Y_0, Z\|_0^+ \geq \varepsilon\}$$

both belong to $\mathcal{I}_2$. This implies that the sets

$$A^c = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < \varepsilon\}$$

and

$$B^c = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - Y_0, Z\|_0^+ < \varepsilon\}$$

belong to $\mathcal{F}(\mathcal{I}_2)$. In this way we obtain a contradiction to the fact that the neighborhoods $U_{X_0}(\varepsilon, 0)$ and $U_{Y_0}(\varepsilon, 0)$ of X_0 and Y_0 are disjoint. Hence we have $X_0 = Y_0$. □

Proposition 3.5. *Let* $(E, \|.,.\|)$ *be a fuzzy 2-normed space. Then we have*

(1) *If* E-$\lim \|X_{jk} - X_0, Z\|_0^+ = 0$, *then* $\mathcal{I}_2^E$-$\lim \|X_{jk} - X_0, Z\|_0^+ = 0$;

(2) *If* $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ *and* $Y_{jk} \xrightarrow{\mathcal{I}_2^E} Y_0$, *then* $X_{jk} + Y_{jk} \xrightarrow{\mathcal{I}_2^E} X_0 + Y_0$;

(3) *If* $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ *and* $c \in \mathbb{R}$, *then* $cX_{jk} \xrightarrow{\mathcal{I}_2^E} cX_0$;

(4) *If* $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ *and* $Y_{jk} \xrightarrow{\mathcal{I}_2^E} Y_0$, *then* $X_{jk} \cdot Y_{jk} \xrightarrow{\mathcal{I}_2^E} X_0 \cdot Y_0$;

(5) *If* $X_{jk} \preceq Y_{jk} \preceq Z_{jk}$ *for all* $(j,k) \in \mathbb{N}\times\mathbb{N}$ *belonging to the set* $B \in \mathcal{F}(\mathcal{I}_2)$*, and* $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ *and* $Z_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$*, then* $Y_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$*.*

Proof. (1) Suppose that $E\text{-}\lim \|X_{jk} - X_0, Z\|_0^+ = 0$. Let $\varepsilon > 0$ and $Z \in E$ any nonzero element. Then there exists a positive integer n such that $\|X_{jk} - X_0, Z\|_0^+ < \varepsilon$ for each $j,k \geq n$. Since

$$A = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \subseteq \{1,2,\cdots,n-1\} \times \{1,2,\cdots,n-1\}$$

and the ideal $\mathcal{I}_2$ is admissible, we have $A \in \mathcal{I}_2$. This shows that $\mathcal{I}_2^E\text{-}\lim \|X_{jk} - X_0, Z\|_0^+ = 0$.

(2) Suppose that $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ and $Y_{jk} \xrightarrow{\mathcal{I}_2^E} Y_0$. Since $\|.,.\|_0^+$ is a 2-norm in the usual sense, we get

$$\|(X_{jk} + Y_{jk}) - (X_0 + Y_0), Z\|_0^+ \leq \|X_{jk} - X_0, Z\|_0^+ + \|Y_{jk} - Y_0, Z\|_0^+ \tag{1}$$

for all $(j,k) \in \mathbb{N}\times\mathbb{N}$. Put

$A(\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|(X_{jk} + Y_{jk}) - (X_0 + Y_0), Z\|_0^+ \geq \varepsilon\}$,

$A_1(\frac{\varepsilon}{2}) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \frac{\varepsilon}{2}\}$,

$A_2(\frac{\varepsilon}{2}) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|Y_{jk} - Y_0, Z\|_0^+ \geq \frac{\varepsilon}{2}\}$.

By assumption, we have that $A_1(\frac{\varepsilon}{2})$ and $A_2(\frac{\varepsilon}{2})$ belong to $\mathcal{I}_2$, and so $A_1(\frac{\varepsilon}{2}) \cup A_2(\frac{\varepsilon}{2}) \in \mathcal{I}_2$. From (1) it follows that $A(\varepsilon) \subseteq A_1(\frac{\varepsilon}{2}) \cup A_2(\frac{\varepsilon}{2})$. This implies that $A(\varepsilon) \in \mathcal{I}_2$. This proves (2).

(3) Since $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$, we have

$$A(1) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < 1\} \in \mathcal{F}(\mathcal{I}_2).$$

Now $\|.,.\|_0^+$ is a 2-norm in the usual sense, so that

$$\|X_{jk}Y_{jk} - X_0Y_0, Z\|_0^+ \leq \|X_{jk}, Z\|_0^+\|Y_{jk} - Y_0, Z\|_0^+ + \|Y_0, Z\|_0^+\|X_{jk} - X_0, Z\|_0^+.$$

For $(j,k) \in A(1)$, we have $\|X_{jk}, Z\|_0^+ \leq \|X_0, Z\|_0^+ + 1$ and it follows that

$$\|X_{jk}Y_{jk} - X_0Y_0, Z\|_0^+ \leq (\|X_0, Z\|_0^+ + 1)\|Y_{jk} - Y_0, Z\|_0^+ + \|Y_0, Z\|_0^+\|X_{jk} - X_0, Z\|_0^+. \tag{2}$$

Let $\varepsilon > 0$ be given. Choose $\lambda > 0$ such that

$$0 < 2\lambda < \frac{\varepsilon}{\|Y_0, Z\|_0^+ + \|X_0, Z\|_0^+ + 1} \tag{3}$$

Since $X_{jk} \xrightarrow{I_2^E} X_0$ and $Y_{jk} \xrightarrow{\mathcal{I}_2^E} Y_0$, the sets

$$A_1(\lambda) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < \lambda\}$$

and

$$A_2(\lambda) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|Y_{jk} - Y_0, Z\|_0^+ < \lambda\}$$

belong to $\mathcal{F}(\mathcal{I}_2)$.

Obviously, $A(1) \cap A_1(\lambda) \cap A_2(\lambda) \in \mathcal{F}(\mathcal{I}_2)$ and for each $(j,k) \in A(1) \cap A_1(\lambda) \cap A_2(\lambda)$, we have from (2) and (3),

$$\|X_{jk}Y_{jk} - X_0Y_0\|_0^+ < \varepsilon.$$

This implies that $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} \cdot Y_{jk} - X_0 \cdot Y_0\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2$, i.e., $X_{jk} \cdot Y_{jk} \xrightarrow{\mathcal{I}_2^E} X_0 \cdot Y_0$.

(4) Let $c \in \mathbb{R}$. If $c = 0$, we have nothing to prove, so we assume that $c \neq 0$. Let $\varepsilon > 0$ be given. Since $\|.,.\|_0^+$ is a 2-norm in usual sense, $\|cX_{jk}, Z\|_0^+ = |c|\|X_{jk}, Z\|_0^+$.

Since $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$, we have

$$A(\varepsilon) = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2.$$

Let $A_1(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|cX_{jk} - cX_0, Z\|_0^+ \geq \varepsilon\}$. We need to show that $A_1(\varepsilon)$ is contained in $A(\varepsilon_1)$. Let $(t,s) \in A_1(\varepsilon)$, then $\varepsilon \leq \|cX_{ts} - cX_0\|_0^+ = |c|\|X_{ts} - X_0\|_0^+$. This implies that $\|X_{ts} - X_0\|_0^+ \geq \frac{\varepsilon}{|c|} = \varepsilon_1$. Therefore $(t,s) \in A(\varepsilon_1)$. Then we have $A_1(\varepsilon) \subset A(\varepsilon_1)$. By the definition of the ideal, we get $A_1(\varepsilon) \in \mathcal{I}_2$ which proves (4).

(5) Let $\varepsilon > 0$ and $W \in E$ be given. From $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ it follows

$$A_1(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2,$$

and from $Z_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ it follows

$$A_2(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|Z_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2.$$

We shall prove

$$C := \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|Y_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \subset A_1(\varepsilon) \cup A_2(\varepsilon) \cup (\mathbb{N}^2 \setminus B).$$

Let $(p,q) \in C$. If $(p,q) \in \mathbb{N}^2 \setminus B$, then $(p,q) \in A_1(\varepsilon) \cup A_2(\varepsilon) \cup (\mathbb{N}^2 \setminus B)$. Assume now $(p,q) \in B$. Then $\|Y_{pq} - X_0, W\|_0^+ \geq \varepsilon$. Since $Z_{pq} \succeq Y_{pq}$ we have $\|Z_{pq} - X_0, W\|_0^+ \geq \varepsilon$, hence $(p,q) \in A_2(\varepsilon)$. Therefore, $(p,q) \in A_1(\varepsilon) \cup A_2(\varepsilon) \cup (\mathbb{N}^2 \setminus B)$. Because the last set is in $\mathcal{I}_2$, we get $C \in \mathcal{I}_2$, i.e. $Y_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. □

Lemma 3.6. *Let $\mathcal{I}_2$ be an admissible ideal with the property* (AP). *If $\{P_j\}_{j=1}^{\infty}$ is a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $P_j \in \mathcal{F}(\mathcal{I}_2)$ for each j, then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$ and the set $P \setminus P_j$ is finite for all j.*

Proof. Let $A_1 = \mathbb{N}^2 \setminus P_1$, $A_m = (\mathbb{N}^2 \setminus P_m) \setminus (A_1 \cup A_2 \cdots A_{m-1})$, $m = 2, 3, \cdots$. Evidently, $A_i \in \mathcal{I}_2$ for each i, and $A_i \cap A_j = \emptyset$ when $i \neq j$. Then, by property (AP) of $\mathcal{I}_2$, we conclude that there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$. Put $P = \mathbb{N}^2 \setminus B$. It is clear that $P \in \mathcal{F}(\mathcal{I}_2)$.

Now we prove that the set $P \setminus P_i$ is finite for each i. Let $j_0 \in \mathbb{N}$ be given. Since each $A_j \Delta B_j$ $(j = 1, \cdots, j_0)$ is a finite set, there exists $(n_0, m_0) \in \mathbb{N} \times \mathbb{N}$ such that

$$\bigcup_{j=1}^{j_0} B_j \cap \{(n,m) \in \mathbb{N}^2 : n > n_0, m > m_0\} = \bigcup_{j=1}^{j_0} A_j \cap \{(n,m) \in \mathbb{N}^2 : n > n_0, m > m_0\}. \tag{4}$$

If $n > n_0, m > m_0$ and $(n,m) \notin B$, then $(n,m) \notin \bigcup_{j=1}^{j_0} B_j$ and, by (4), $(n,m) \notin \bigcup_{j=1}^{j_0} A_j$. Since $A_{j_0} = (\mathbb{N}^2 \setminus P_{j_0}) \setminus \bigcup_{j=1}^{j_0-1} A_j$ and $(n,m) \notin A_{j_0}$, we have $(n,m) \notin \bigcup_{j=1}^{j_0-1} A_j$, and thus $(n,m) \in P_{j_0}$ for all n and m with $n > n_0$, $m > m_0$. Therefore, we get $(n,m) \in P$ and $(n,m) \in P_{j_0}$ for all $(n,m) \in \mathbb{N}^2$ with $n > n_0$, $m > m_0$. This shows that the set $P \setminus P_{j_0}$ is finite and the lemma is proved. □

Theorem 3.7. *Let $\mathcal{I}_2$ be an admissible ideal with the property* (AP). *Let $(E, \|.,.\|)$ be a fuzzy 2-normed space and $\{X_{jk}\}$ be a double sequence in E. Then $\{X_{jk}\}$ is an $\mathcal{I}_2^E$-convergent sequence in E if and only if there is an E-convergent double sequence $\{Y_{jk}\}$ such that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : X_{jk} \neq Y_{jk}\} \in \mathcal{I}_2$.*

Proof. Suppose $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. For each $n \in \mathbb{N}$ and a non-zero $Z \in E$, let

$$A_n = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < \frac{1}{n}\}.$$

Then $A_n \in \mathcal{F}(\mathcal{I}_2)$ for each $n \in \mathbb{N}$.

Since $\mathcal{I}_2$ is admissible ideal with the property (AP), by Lemma 3.6 there exists $A \subset \mathbb{N} \times \mathbb{N}$ such that $A \in \mathcal{F}(\mathcal{I})$ and the set $A \setminus A_n$ is finite for each n. Observe that $X_{jk} \to_{(A)} X_0$, i.e., for each $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $j, k \geq n_0$ and $(j,k) \in A$ implies $\|X_{jk} - X_0, Z\|_0^+ < \varepsilon$.

Define a sequence $\{Y_{jk}\}$ in E as

$$Y_{jk} = \begin{cases} X_{jk}, & \text{for } (j,k) \in A; \\ X_0, & \text{for } (j,k) \in (\mathbb{N} \times \mathbb{N}) \setminus A. \end{cases}$$

The sequence $\{Y_{jk}\}$ is E-convergent to X_0 with respect to the fuzzy norm on E. Thus we have $\{(j,k) \in \mathbb{N} \times \mathbb{N} : X_{jk} \neq Y_{jk}\} \in \mathcal{I}_2$.

Next suppose that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : X_{jk} \neq Y_{jk}\} \in \mathcal{I}_2$ and $Y_{jk} \to X_0$. Let $\varepsilon > 0$ be given. Then for each n and a non-zero $Z \in E$, we can write

$$\{j,k \leq n : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \subseteq \{j,k \leq n : X_{jk} \neq Y_{jk}\} \cup \{j,k \leq n : \|X_{jk} - X_0, Z\|_0^+ > \varepsilon\}. \tag{5}$$

Since first set on the right side of (5) belongs to $\mathcal{I}_2$, and the second set contain in a fixed number of integers and thus belongs to $\mathcal{I}_2$, we conclude that $\{(j,k) : j,k \leq n, \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\}$ belongs to $\mathcal{I}_2$. This achieves the proof. □

Now we prove a decomposition theorem for $\mathcal{I}_2^E$-convergent sequences.

Theorem 3.8. *Let $\{X_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(E, \|.,.\|)$ and $\mathcal{I}_2$ be an admissible ideal. If there exist two sequences $\{Y_{jk}\}$ and $\{Z_{jk}\}$ in E such that $X_{jk} = Y_{jk} + Z_{jk}$; Y_{jk} E-converges to X_0 and $\mathrm{supp}(Z_{jk}) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : Z_{jk} \neq \theta\} \in \mathcal{I}_2$, then $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$.*

Proof. Let $\{Y_{jk}\}$ and $\{Z_{jk}\}$ be double sequences in E as in the statement of the theorem and $H = \mathrm{supp}(Z_{jk})$. Let $\varepsilon > 0$ and $W \in E$ be given. Since $A_1 = \{(j,k) \in \mathbb{N}^2 : \|Z_{jk} - \overline{0}, W\|_0^+ \geq \varepsilon/2\} \subset \mathrm{supp}(Z_{jk}) = H$, we have $A_1 \in \mathcal{I}_2$. Further,

$$\|X_{jk} - X_0, W\|_0^+ = \|Y_{jk} + Z_{jk} - \overline{0} - X_0, W\|_0^+ \leq \|Y_{jk} - X_0, W\|_0^+ + \|Z_{jk} - \overline{0}, W\|_0^+$$

implies

$$\{(j,k) \in \mathbb{N}^2 : \|X_{jk} - X_0, W\|_0^+ < \varepsilon\} \supset \{(j,k) \in \mathbb{N}^2 : \|Y_{jk} - X_0, W\|_0^+ < \varepsilon/2\}$$
$$\cap \{(j,k) \in \mathbb{N}^2 : \|Z_{jk} - \overline{0}\|_0^+ < \varepsilon/2\}.$$

The sets on the right side are both in $\mathcal{F}(\mathcal{I}_2)$, so that the set on the left side ia also in $\mathcal{F}(\mathcal{I}_2)$. Therefore, $\{(j,k) \in \mathbb{N}^2 : \|X_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2$, i.e. $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. □

Definition 3.9. *Let $(E, \|.,.\|$ be a fuzzy 2-normed space. We say that a double sequence $\{X_{jk}\}$ in E is $\mathcal{I}_2^{*E}$-*convergent *to $X_0 \in E$ with respect to the 2-norm on E if there exists a subset*

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $E\text{-}\lim_{m\to\infty} \|X_{j_m k_m} - X_0, Z\| = 0$ for each non-zero $Z \in E$.

In this case we write $X_{jk} \xrightarrow{\mathcal{I}_2^{*E}} X_0$.

Theorem 3.10. *Let* $(E, \|.,.\|$ *be a fuzzy 2-normed space and* $\mathcal{I}_2$ *be an admissible ideal. If* $X_{jk} \xrightarrow{\mathcal{I}_2^{*E}} X_0$*, then* $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$.

Proof. Suppose that $X_{jk} \xrightarrow{\mathcal{I}_2^{*E}} X_0$. Then by definition, there exists

$$K = \{(j_m, k_m) \in \mathbb{N} \times \mathbb{N} : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_2)$$

such that $E\text{-}\lim_{m\to\infty} \|X_{j_m k_m} - X_0, Z\| = 0$. Let $\varepsilon > 0$ and non-zero $Z \in E$ be given. Since $E\text{-}\lim_{m\to\infty} \|X_{j_m k_m} - X_0, Z\| = 0$, there exists $n_0 \in \mathbb{N}$ such that $\|X_{j_m k_m} - X_0, Z\|_0^+ < \varepsilon$ for every $m \geq n_0$. Since

$$A = \{(j_m, k_m) \in K : \|X_{j_m k_m} - X_0, Z\|_0^+ \geq \varepsilon\}$$

is contained in

$$B = \{j_1, j_2, \cdots, j_{n_0-1}; k_1, k_2, \cdots, k_{n_0-1}\}$$

and the ideal $\mathcal{I}_2$ is admissible, we have $A \in \mathcal{I}_2$. Hence

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{j_m k_m} - X_0, Z\|_0^+ \geq \varepsilon\} \subseteq K \cup B \in \mathcal{I}_2$$

for $\varepsilon > 0$ and nonzero $Z \in E$. Therefore, we conclude that

$$X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0.$$

□

Theorem 3.11. *Let* $\mathcal{I}_2$ *be an admissible ideal with the property* (AP) *and* $(E, \|.,.\|$ *be fuzzy 2-normed space and* $\{X_{jk}\}$ *be a double sequence in* E*. Then* $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$ *implies* $X_{jk} \xrightarrow{\mathcal{I}_2^{*E}} X_0$.

Proof. Let $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. Then by definition, for every $\varepsilon > 0$ and a non-zero $Z \in E$, there exists an integer $n = n(\varepsilon)$ such that the set

$$B(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2.$$

For $m \in \mathbb{N}$, we define the set P_m as follows:

$$P_1 = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq 1\}$$

and

$$P_m = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m} \leq \|X_{jk} - X_0, Z\|_0^+ < \frac{1}{m-1}\}, \text{ for } m \geq 2 \text{ in } \mathbb{N}.$$

It is clear that $\{P_1, P_2, \cdots\}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}_2$, then by the property (AP) of $\mathcal{I}_2$, there is a countable family of sets $\{Q_1, Q2, \cdots\}$ in $\mathcal{I}_2$ such that $P_j \Delta Q_j$ is a finite set for each $j \in \mathbb{N}$ and $Q = \bigcup_{j=1}^{\infty} Q_j \in \mathcal{I}_2$. Since $Q \in \mathcal{I}_2$, so there a set $B = \mathbb{N} \setminus Q$. To prove the result it is sufficient to show that $X_{jk} \to_{(B)} X_0$. Let $\xi > 0$ be given. Choose an integer p such that $\xi > \frac{1}{p+1}$. Thus, we have

$$\begin{aligned}\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \xi\} &\subseteq \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \frac{1}{p+1}\} \\ &= \bigcup_{m=1}^{p+1} P_m. \qquad (6)\end{aligned}$$

Since $P_m \cap Q_m$ is a finite set for each $m = 1, \cdots, p+1$, there exists $(j_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that

$$\left(\bigcup_{m=1}^{p+1} Q_m\right) \cap \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \geq j_0, k \geq k_0\}$$
$$= \bigcup_{m=1}^{p+1} P_m \cap \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \geq j_0, k \geq k_0\}.$$

If $j \geq j_0$ and $k \geq k_0$ and $(j,k) \in Q$. This implies that $(j,k) \notin \bigcup_{m=1}^{p+1} Q_m$ and so $(j,k) \notin \bigcup_{m=1}^{p+1} P_m$. Thus for every $j \geq j_0$ and $k \geq k_0$ and $(j,k) \in B$, from (6), we get $\|X_{jk} - X_0, Z\|_0^+ < \xi$. This shows $X_{jk} \to_{(B)} X_0$. This completes the proof. □

3.1. **$\mathcal{I}_2$-limit points and $\mathcal{I}_2$-cluster points.** In this subsection we introduce and consider the notions of $\mathcal{I}_2$-limit points and $\mathcal{I}_2$-cluster points of sequences in a fuzzy 2-normed space.

Definition 3.12. *Let $\{X_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(E, \|.,.\|)$ and $\mathcal{I}_2$ an ideal on $\mathbb{N} \times \mathbb{N}$. Then:*

(1) *an element $W \in E$ is said to be an $\mathcal{I}_2$-*limit point *of $\{X_{jk}\}$ provided that there is a set $K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$ such that $K \notin \mathcal{I}_2$ and* $\lim_m \|X_{j_m k_m} - W, Z\|_0^+ = 0$ *for each a non-zero $Z \in E$;*

(2) *an element $Y \in E$ is said to be an $\mathcal{I}_2$-*cluster point *of X_{jk} if for each $\varepsilon > 0$ and a non-zero $Z \in E$, the set $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - Y, Z\|_0^+ < \varepsilon\} \notin I_2$.*

We denote by $\mathcal{L}^E_{\mathcal{I}_2}(X_{jk})$ and $\mathcal{C}^E_{\mathcal{I}_2}(X_{jk})$ the set of all $\mathcal{I}_2$-limit points and $\mathcal{I}_2$-cluster points of a sequence $\{X_{jk}\}$ in $(E, \|.,.\|)$.

Theorem 3.13. *Let $\mathcal{I}_2$ be an admissible ideal on $\mathbb{N} \times \mathbb{N}$. Then for any sequence $\{X_{jk}\}$ in a fuzzy 2-normed space $(E, \|.,.\|)$, we have $\mathcal{L}^E_{\mathcal{I}_2}(X_{jk}) \subset \mathcal{C}^E_{\mathcal{I}_2}(X_{jk})$.*

Proof. Assume that $W \in \mathcal{L}^E_{\mathcal{I}_2}(X_{jk})$. Then by definition there is a set $K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$ such that $K \notin \mathcal{I}_2$ and

$$\lim_m \|X_{j_m k_m} - W, Z\|_0^+ = 0 \text{ for each non-zero } Z \in E. \tag{7}$$

Let $\varepsilon > 0$ and non-zero $Z \in E$ be given. According to (7), there exists an integer $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for each $m \geq n_0$, we get $\|X_{j_m k_m} - W, Z\|_0^+ < \varepsilon$.

Thus, we have

$$K \setminus \{(j_1, k_1), \cdots, (j_{n_0}, k_{k_0})\} \subset \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{j_m k_m} - W, Z\|_0^+ < \varepsilon\}.$$

This implies that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{j_m k_m} - W, Z\|_0^+ < \varepsilon\} \notin \mathcal{I}_2$. Hence $W \in \mathcal{C}^E_{\mathcal{I}_2}(X_{jk})$. □

Theorem 3.14. *Let $\{X_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(E, \|.,.\|)$. If $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$, then $\mathcal{L}^E_{\mathcal{I}_2}(X_{jk}) = \mathcal{C}^E_{\mathcal{I}_2}(X_{jk}) = \{X_0\}$.*

Proof. Assume that $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. Then for each $\varepsilon > 0$ and a non-zero $Z \in E$, the set

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2,$$

that is

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < \varepsilon\} \notin \mathcal{I}_2,$$

which implies that $X_0 \in \mathcal{C}^E_{\mathcal{I}}(X_{jk})$.

We assume that there exists at least one $Y_0 \in \mathcal{C}^E_{\mathcal{I}_2}(X_{jk})$ such that $Y_0 \neq X_0$. Then there exists $\varepsilon > 0$ such that

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - Y_0, Z\|_0^+ < \varepsilon\} \subset \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\}$$

holds for each non-zero $Z \in E$. But $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2$ implies that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - Y_0, Z\|_0^+ < \varepsilon\} \in \mathcal{I}_2$, which contradicts that $Y_0 \in \mathcal{C}^E_{\mathcal{I}_2}(X_{jk})$. Thus we have $\mathcal{C}^E_{\mathcal{I}_2}(X_{jk}) = \{X_0\}$.

On the other hand, from $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$, by Theorem 3.7 and Definition 3.12, we have $X_0 \in \mathcal{L}^E_{\mathcal{I}_2}$. By Theorem 3.13, we have $\mathcal{L}^E_{\mathcal{I}_2}(X_{jk}) = \mathcal{C}^E_{\mathcal{I}_2}(X_{jk}) = \{X_0\}$. □

Theorem 3.15. *Let $\mathcal{I}_2$ be an admissible ideal on $\mathbb{N} \times \mathbb{N}$. Then the set $\mathcal{C}^E_{\mathcal{I}_2}$ is closed in $(E, \|.,.\|)$, for every double sequence $\{X_{jk}\}$ in E.*

Proof. Let $W \in \overline{\mathcal{C}^E_{\mathcal{I}_2}}(X_{jk})$. Let $\varepsilon > 0$ and a non-zero $Z \in E$ be given. Then there exists an $X_0 \in \mathcal{C}^E_{\mathcal{I}_2}(X_{jk}) \cap U_W(\varepsilon, 0)$. Choose $\eta > 0$ such that $U_{X_0}(\eta, 0) \subset U_W(\varepsilon, 0)$. Obviously we have

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < \eta\} \subset \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - W, Z\|_0^+ < \varepsilon\}.$$

This implies that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - W, Z\|_0^+ < \varepsilon\} \notin \mathcal{I}_2$. Thus $W \in C^E_{\mathcal{I}_2}(\{X_{jk})$. Hence $\mathcal{C}^E_{\mathcal{I}_2}(\{X_{jk})$ is closed in E. □

4. $\mathcal{I}_2^E$- and $\mathcal{I}_2^{*E}$-double Cauchy sequences in fuzzy 2-normed spaces

In this section we study the concepts of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy double sequences in $(E, \|.,.\|)$. Moreover, we will study the relations between them. The investigation of ideal Cauchy sequences (and nets) was done in [10, 13, 34].

Definition 4.1. *Let $(E, \|.,.\|)$ be a fuzzy 2-normed space and $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. A double sequence $\{X_{jk}\}$ of elements in E is said to be*

(1) an $\mathcal{I}_2^E$-Cauchy sequence *in E if for every $\varepsilon > 0$ and a nonzero $Z \in E$, there exist $s = s(\varepsilon), t = t(\varepsilon)$ such that*

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_{st}, Z\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2.$$

(2) an $\mathcal{I}_2^{*E}$-Cauchy sequence *in E if for every $\varepsilon > 0$ and a nonzero $Z \in E$, there exists*

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $\{X_{j_m k_m}\}$ is an ordinary E-Cauchy sequence in E.

The next theorem gives a relation between $\mathcal{I}_2^E$- and $\mathcal{I}_2^{*E}$-double Cauchy sequences.

Theorem 4.2. *Let $(E, \|.,.\|)$ be a fuzzy 2-normed space and $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. If $\{X_{jk}\}$ is an $\mathcal{I}_2^{*E}$-double Cauchy sequence, then $\{X_{jk}\}$ is an $\mathcal{I}_2^E$-double Cauchy sequence.*

Proof. Since $\{X_{jk}\}$ be an $\mathcal{I}_2^{*E}$-double Cauchy sequence, for any $\varepsilon > 0$ and any non-zero $Z \in E$, there exist

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_2)$$

and a number $n_0 \in \mathbb{N}$ such that

$$\|X_{j_m k_m} - X_{j_p kp}, Z\|_0^+ < \varepsilon$$

for every $m, p \geq n_0$. Now, fix $p = j_{n_0+1}, r = k_{n_0+1}$. Then for every $\varepsilon > 0$ and a non-zero $Z \in E$, we have

$$\|X_{j_m k_m} - X_{pr}, Z\|_0^+ < \varepsilon$$

for every $m \geq n_0$. Let $H = \mathbb{N} \times \mathbb{N} \setminus K$. It is obvious that $H \in \mathcal{F}(\mathcal{I}_2)$ and

$$A(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{j_m k_m} - X_{pr}, Z\|_0^+ \geq \varepsilon\}$$
$$\subset H \cup \{j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in \mathcal{I}_2.$$

Therefore, for every $\varepsilon > 0$ and non-zero $Z \in E$, we can find $(p,r) \in \mathbb{N} \times \mathbb{N}$ such that $A(\varepsilon) \in \mathcal{I}_2$, i.e., $\{X_{jk}\}$ is an $\mathcal{I}_2^E$-double Cauchy sequence. □

Theorem 4.3. *Let $(E, \|.,.\|)$ be a fuzzy 2-normed space and $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. If a sequence $\{X_{jk}\}$ is $\mathcal{I}_2^E$-convergent, then it is an $\mathcal{I}_2^E$-double Cauchy sequence.*

Proof. Suppose that $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. Then for each $\varepsilon >)$ and a non-zero $Z \in E$, we have

$$A(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2.$$

Since $\mathcal{I}_2$ is an admissible ideal, there exists an $(j_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that $(j_0, k_0) \notin A(\varepsilon)$. Let

$$A_1(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_{j_0 k_0}, Z\|_0^+ \geq 2\varepsilon\}.$$

Since $\|.,.\|_0^+$ is a 2-norm in the usual sense, we get

$$\|X_{jk} - X_{j_0 k_0}, Z\|_0^+ \leq \|X_{jk} - X_0, Z\|_0^+ + \|X_{j_0 k_0} - X_0, Z\|_0^+.$$

Observe that if $(j,k) \in A_1(\varepsilon)$, then

$$\|X_{jk} - X_0, Z\|_0^+ + \|X_{j_0 k_0} - X_0, Z\|_0^+ \geq 2\varepsilon.$$

On the other hand, since $(j_0, k_0) \notin A(\varepsilon)$, we have

$$\|X_{j_0 k_0} - X_0, Z\|_0^+ < \varepsilon.$$

So we can conclude that $\|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon$, hence $(j,k) \in A(\varepsilon)$. This implies that $A_1(\varepsilon) \subset A(\varepsilon)$, for each $\varepsilon > 0$ and a non-zero $Z \in E$. This gives $A_1(\varepsilon) \in \mathcal{I}_2$ which shows that $\{X_{jk}\}$ is an $\mathcal{I}_2^E$-double Cauchy sequence. □

Theorem 4.4. *Let $(E, \|.,.\|)$ be a fuzzy 2-normed space, $\mathcal{I}_2$ an admissible ideal in $\mathbb{N} \times \mathbb{N}$, $\{X_{jk}\}$ a double sequence in E, and $A_{nm} = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_{nm}, Z\|_0^+ \geq \varepsilon\}$, where $(n,m) \in \mathbb{N} \times \mathbb{N}$. If $\{X_{jk}\}$ is an I_2^N-double Cauchy sequence, then for every $\varepsilon > 0$, there exists $B \subset \mathbb{N} \times \mathbb{N}$ with $B \in \mathcal{I}_2$ such that $\|X_{jk} - X_{lt}, Z\|_0^+ < \varepsilon$, for all $(j,k), (l,t) \notin B$.*

Proof. Let $\varepsilon > 0$ and a non-zero $Z \in E$ be given. Set $B = A_{nm}(\frac{\varepsilon}{2})$, where $(n,m) \in \mathbb{N} \times \mathbb{N}$. Since $\{X_{jk}\}$ be a double sequence in E, we have $B \in \mathcal{I}_2$ and for all $(j,k), (l,t) \notin B$, we get

$$\|X_{jk} - X_{nm}, Z\|_0^+ < \frac{\varepsilon}{2} \text{ and } \|X_{nm} - X_{lt}, Z\|_0^+ < \frac{\varepsilon}{2}.$$

Then we have $\|X_{jk} - X_{lt}, Z\|_0^+ < \varepsilon$, for all $(j,k), (l,t) \notin B$, by the triangle inequality, because $\|.,.\|$ is a 2-norm in the usual norm. □

Now we will prove that $\mathcal{I}_2^{*E}$-convergent implies $\mathcal{I}_2^E$-Cauchy condition in a fuzzy 2-normed space.

Theorem 4.5. *Let $(E, \|.,.\|)$ be a fuzzy 2-normed space and $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. If a double sequence $\{X_{jk}\}$ is $\mathcal{I}_2^{*E}$-convergent, then it is an $\mathcal{I}_2^E$-double Cauchy sequence.*

Proof. By assumption there exists a set

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $E - \lim_m \|X_{j_m k_m} - X_0, Z\|_0^+ = 0$ for each nonzero $Z \in E$, i.e., there exists $n_0 \in \mathbb{N}$ such that $\|X_{j_m k_m} - X_0, Z\|_0^+ < \varepsilon$ for every $\varepsilon > 0$, each non-zero $Z \in E$ and $m > n_0$. Since

$$\|X_{j_m k_m} - X_{j_p k_p}, Z\|_0^+ \le \|X_{j_m k_m} - X_0, Z\|_0^+ + \|X_{j_p k_p} - X_0, Z\|_0^+$$

for every $\varepsilon > 0$, each non-zero $Z \in E$ and $m, p > n_0$, we have

$$\|X_{j_m k_m} - X_{j_p k_p}, Z\|_0^+ \ge \varepsilon$$

i.e., $\{X_{jk}\}$ is an $\mathcal{I}_2^{*E}$-double Cauchy sequence in E. Then by Theorem 4.2, $\{X_{jk}\}$ is an $\mathcal{I}_2^{E}$-double Cauchy sequence. □

Theorem 4.6. *Let $(E, \|.,.\|)$ be a fuzzy 2-normed space and $\mathcal{I}_2$ be an admissible ideal in $\mathbb{N} \times \mathbb{N}$ with property* (AP)*. Then the concepts $\mathcal{I}_2^{E}$-double Cauchy sequence and $\mathcal{I}_2^{*E}$-double Cauchy sequence coincide.*

Proof. If $\{X_{jk}\}$ is an $\mathcal{I}_2^{*E}$-double Cauchy sequence, then it is an $\mathcal{I}_2^{E}$-double Cauchy sequence by Theorem 4.2 (even if $\mathcal{I}_2$ does not have the (AP) property).

So, we have to prove the converse. Let $\{X_{jk}\}$ be an $\mathcal{I}_2^{E}$-double Cauchy sequence. Then by definition, there exists an $j_0 = j_0(\varepsilon), k_0 = k_0(\varepsilon)$ such that

$$A(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_{j_0 k_0}, Z\|_0^+ \ge \varepsilon\} \in \mathcal{I}_2$$

for every $\varepsilon > 0$ and non-zero $Z \in E$.

Let $P_i = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_{s_i t_i}, Z\|_0^+ < \frac{1}{i}\}$, $i = 1, 2, \cdots$, where $s_i = j_0(\frac{1}{i}), t_i = k_0(\frac{1}{i})$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_2)$ for $i = 1, 2, \cdots$. Since $\mathcal{I}_2$ has the property (AP), then by Lemma 3.6 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$, and $P \setminus P_i$ is finite for all i. Now we prove that

$$\lim_{\substack{j,k,s,t \to \infty \\ (j,k),(s,t) \in P}} \|X_{jk} - X_{st}, Z\|_0^+ = 0.$$

To prove this, let $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $m > \frac{2}{\varepsilon}$. If $(j,k), (s,t) \in P$ then $P \setminus P_m$ is finite set, so there exists $r = r(m)$ such that $(j,k), (s,t) \in P$ for all $j,k,s,t > r(m)$. Therefore, $\|X_{jk} - X_{s_m t_m}, Z\|_0^+ < \frac{1}{m}$ and $\|X_{st} - X_{s_m t_m}, Z\|_0^+ < \frac{1}{m}$ for all $j,k,s,t > r(m)$. Hence it follows that

$$\begin{aligned} \|X_{jk} - X_{st}, Z\|_0^+ &< \|X_{jk} - X_{s_m t_m}, Z\|_0^+ + \|X_{st} - X_{s_m t_m}, Z\|_0^+ \\ &< \varepsilon \text{ for all } j,k,s,t > r(m). \end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that $j,k,s,t > r(\varepsilon)$ and $j,k,s,t \in P \in \mathcal{F}(\mathcal{I}_2)$

$$\|X_{jk} - X_{st}, Z\|_0^+ < \varepsilon$$

for every non-zero $Z \in E$. This shows that $\{X_{jk}\}$ is an $\mathcal{I}_2^{*E}$-double Cauchy sequence in E. □

5. Conclusion

In recent years the study of fuzzy numbers and fuzzy normed spaces related to convergence has attracted a big number of works and a wide variety of approaches was developed. In this paper we have focused on convergence of double sequences in fuzzy 2-normed spaces with respect to an ideal on $\mathbb{N} \times \mathbb{N}$ and proved several results. We think that it may be interesting to make a similar investigation for convergence of double sequences in 2-fuzzy 2-normed spaces and related structures, as well as in (non-Archimedean) fuzzy anti-2-normed spaces.

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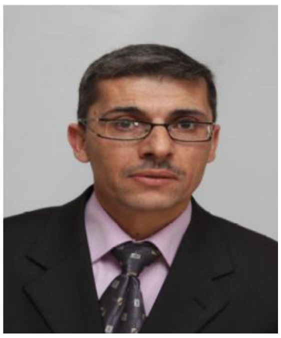

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