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RIESZ LACUNARY ALMOST CONVERGENT DOUBLE SEQUENCE SPACES DEFINED BY SEQUENCE OF ORLICZ FUNCTIONS OVER N-NORMED SPACES

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ABSTRACT. In the present paper we introduce a new concept for strong almost convergence with respect to sequence of Orlicz function, difference sequence, Reisz mean for double sequence, double lacunary sequence and n-normed space. We examine some topological properties, inclusion relation between these newly defined sequence spaces and establish relation with Riesz lacunary almost statistically convergence sequence spaces.

Keywords: double sequence, difference sequence, Pringsheim convergence, Riesz convergence, statistical convergence, Orlicz function, n-normed space.

AMS Subject Classification: 46A40, 40A35, 40G15, 46A45.

1. Introduction and preliminaries

The initial works on double sequences is found in Bromwich [5]. Later on, it was studied by Hardy [14], Moricz [23], Moricz and Rhoades [24], Mursaleen [25, 26], Başarir and Sonalcan [3], Altay and Basar [2], Başar and Sever [4], Mursaleen [30, 35], Shakhmurov [43] and many others. Mursaleen [27, 28] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [29] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [2] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Başar and Sever [4] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known classical sequence space ℓ_q and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [36]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. For more details about sequence spaces see [1, 18, 19, 31-34, 37, 38, 44-46] and references therein.

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An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [20] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [20] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. The notion of difference sequence spaces was introduced by Kızmaz [16], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.

Let n be a non-negative integer, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in Z\},$$

where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_k)$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+v}.$$

Taking $n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Et and Çolak [7].
The concept of 2-normed spaces was initially developed by Gähler [10] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results, see Gunawan [11, 12] and Gunawan and Mashadi [13] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,i\to\infty} ||x_k - x_i, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let $A = (a_{jk}^{mn}), j, k = 0, 1, ...$ be a doubly infinite matrix of real numbers for all $m, n = 0, 1, ...$ Forming the sums

$$y_{mn} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk}$$

called the A- transform of the sequence $x = (x_{j,k})$, which yields a method of summability. More exactly, we say that a sequence x is A-summable to the limit L if the A- transform Ax of x exists for all $m, n = 0, 1, ...$ in the sense of Pringsheim's convergence:

$$\lim_{p,q\to\infty} \sum_{j=0}^{p}\sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = y_{mn}$$

and

$$\lim_{m,n\to\infty} y_{mn} = L.$$

We say that a triangular matrix A is bounded-regular or RH-regular if every bounded and convergent sequence x is A-summable to the same limit and the A-means are also bounded. Necessary and sufficient conditions for A to be bounded-regular are

(1) $\lim_{m,n\to\infty} a_{jk}^{mn} = 0 \ \ (j, k = 0, 1, 2, ..)$

(2) $\lim_{m,n\to\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}^{mn} = 1$

(3) $\lim_{m,n\to\infty} \sum_{j=0}^{\infty} |a_{jk}^{mn}| = 0 \ \ (k = 0, , ...)$

(4) $\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |a_{jk}^{mn}| = 0 \ \ (j = 0, , ...)$

(5) $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |a_{jk}^{mn}| \leq C < \infty \ \ (m, n = 0, 1, ...)$

These conditions were first established by Robison [39]. Actually (1) is a consequence of each (3) and (4). We say that a matrix A is strongly regular if every almost convergent sequence x is A-summable to the same limit, and the A-means are also bounded.

Let $n, m \geq 1$. A double sequence $x = (x_{k,l})$ of real numbers is called almost P-convergent to a limit L if

$$P - \lim_{n,m\to\infty} \sup_{\mu,\eta\geq 0} \Big| \frac{1}{nm} \sum_{k=\mu}^{\mu+n-1} \sum_{l=\eta}^{\eta+m-1} x_{k,l} - L \Big| = 0,$$

i.e., the average value of $(x_{k,l})$ taken over any rectangle

$$\big\{(k,l) : \mu \leq k \leq \mu + n - 1, \eta \leq l \leq \eta + m - 1\big\}$$

tends to L as both n and m tends to ∞, and this convergence is uniform in μ and η.
A double sequence x is called strongly almost P-convergent to a number L if

$$P-\lim_{n,m\to\infty}\sup_{\mu,\eta\geq 0}\frac{1}{nm}\sum_{k=\mu}^{\mu+n-1}\sum_{l}^{\eta+m-1}=\eta|x_{k,l}-L|=0.$$

Let denote the set of sequences with this property as $[\hat{c}^2]$. By $[\hat{c}^2]$, we denote the space of all almost convergent double sequences. It is easy to see that the inclusion $c_2^\infty\subset[\hat{c}^2]\subset\hat{c}^2\subset l_2^\infty$ strictly hold, where l_2^∞ and c_2^∞ denote the spaces of bounded and bounded convergent double sequences, respectively. As in the case of single sequences, every almost convergent double sequences is bounded. But a convergent double sequence need not be bounded. Thus a convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent. We use the following definition which may be called convergence in Pringsheim's sense as follows

$$(x_{k,l}-\lambda)=0(1),\ \ (k,l\to\infty).$$

Let $(p_n),(\bar{p}_m)$ be two sequences of positive numbers and

$$P_n=p_1+p_2+...+p_n,\ \ \bar{P}_m=\bar{p}_1+\bar{p}_2+...+\bar{p}_m.$$

Then the transformation given by

$$T_{n,m}(x)=\frac{1}{P_n\bar{P}_m}\sum_{k=1}^{n}\sum_{l=1}^{m}p_k\bar{p}_lx_{k,l}$$

is called the Riesz mean of double sequence $x=(x_{k,l})$. If $P-\lim_{n,m}T_{n,m}=L$, $L\in\mathbb{R}$, then the sequence $x=(x_{k,l})$ is said to be Riesz convergent to L. If $x=(x_{k,l})$ is Riesz convergent to L, then we write $P_R-\lim x=L$.
The double sequence $\theta_{r,s}=\{(k_r,l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that $k_0=0,h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$ and $l_0=0,\bar{h}_s=l_s-l_{s-1}\to\infty$ as $s\to\infty$.
Let $K_{r,s}=k_rl_s,\ \ h_{r,s}=h_r\bar{h}_s$ and $\theta_{r,s}$ is determined by

$$I_{r,s}=\big\{(k,l):k_{r-1}<k\leq k_r\ \text{ and; }l_{s-1}<l\leq l_{s-1}\big\},\ \ q_r=\frac{k_r}{k_{r-1}},\ \ \bar{q}_s=\frac{l_s}{l_{s-1}}\ \text{ and }\ q_{r,s}=q_r\bar{q}_s.$$

Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence and (p_k), $(\bar{p}_l)$ be sequences of positive real numbers such that $P_{k_r}=\sum_{k\in(0,k_r]}p_k$, $\bar{P}_{l_s}=\sum_{l\in(0,l_s]}\bar{p}_l$ and $H_r=\sum_{k\in(k_{r-1},k_r]}p_k$, $\bar{H}_s=\sum_{l\in(l_{s-1},l_s]}\bar{p}_l$.
Clearly, $H_r=P_{k_r}-P_{k_{r-1}}$, $\bar{H}_s=\bar{P}_{l_s}-\bar{P}_{l_{s-1}}$. If the Riesz transformation of double sequences is RH-regular, and $H_r=P_{k_r}-P_{k_{r-1}}\to\infty$ as $r\to\infty$, $\bar{H}_s=\bar{P}_{l_s}-\bar{P}_{l_{s-1}}\to\infty$, as $s\to\infty$, then $\theta'_{r,s}=\{P_{k_r},\bar{P}_{l_s}\}$ is a double lacunary sequence. Throughout the paper, we assume that $P_n=p_1+p_2+...+p_n\to\infty$ as $n\to\infty$, $\bar{P}_m=\bar{p}_1+\bar{p}_2+...\bar{p}_m\to\infty$, as $m\to\infty$, such that $H_r=P_{k_r}-P_{k_{r-1}}\to\infty$ as $r\to\infty$ and $\bar{H}_s=\bar{P}_{l_s}-\bar{P}_{l_{s-1}}\to\infty$ as $s\to\infty$. Let $P_{k_{r,s}}=P_{k_r}\bar{P}_{l_s}$, $H_{r,s}=H_r\bar{H}_s$, $I'_{r,s}=\{(k,l):P_{k_{r-1}}<k\leq P_{k_r}$ and $\bar{P}_{l_{s-1}}<l\leq\bar{P}_{l_s}\}$, $\theta_r=\frac{P_{k_r}}{P_{k_{r-1}}}$, $\bar{\theta}_s=\frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}}$ and $\theta_{r,s}=\theta_r\bar{\theta}_s$. If we take $p_k=1$, $\bar{p}_l=1$ for all k and l, then $H_{r,s},P_{k_{r,s}},\theta_{r,s}$ and $I'_{r,s}$ reduce to $h_{r,s},k_{r,s},q_{r,s}$ and $I_{r,s}$. Let $P_{k_{r,s}}=P_{k_r}\bar{P}_{l_s}$, $H_{r,s}=H_r\bar{H}_s$, $I'_{r,s}=\{(k,l):P_{k_{r-1}}<k\leq P_{k_r}$ and $\bar{P}_{l_{r-1}}<l\leq\bar{P}_{l_s}\}$, $Q_r=\frac{P_{k_r}}{P_{k_{r-1}}}$, $\bar{Q}_s=\frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}}$ & $Q_{r,s}=Q_r\bar{Q}_s$.

Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz function and $u=(u_{k,l})$ be any factorable double sequence of strictly positive real numbers. In this section we define the following sequence spaces over n-normed spaces:

$$\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big] =$$

$$\Big\{x = (x_{k,l}) : P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}$$

$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0\Big\}$$

$$\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big]_0 =$$

$$\Big\{x = (x_{k,l}) : P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}$$

$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we have

$$\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big] =$$

$$\Big\{x = (x_{k,l}) : P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{u_{k,l}}$$

$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0\Big\}$$

$$\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big]_0 =$$

$$\Big\{x = (x_{k,l}) : P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{u_{k,l}}$$

$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } \rho > 0\Big\}.$$

If we take $u = (u_{k,l}) = 1$, we have

$$\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big] =$$

$$\Big\{x = (x_{k,l}) : P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]$$

$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0\Big\}$$

$$\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]_0 =$$

$$\Big\{x = (x_{k,l}) : P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]$$

$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } \rho > 0\Big\}.$$

If we take $p_k = 1$, $\bar{p}_l = 1$ for all k and l, then we obtain the following sequence spaces $\Big[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]$ and $\Big[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]_0$ which can be seen in [34].

The following inequality will be used throughout the paper. If $0 \le p_{k,l} \le \sup p_{k,l} = H$, $D = \max(1, 2^{H-1})$ then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \le D\{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\} \tag{1}$$

for all k, l and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to introduce the new type of sequence spaces $\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big]$ and $\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big]_0$ in the first section. In the second section of this paper we prove some topological properties and in the third section we establish inclusion relation between above defined sequence spaces and the sequence spaces which we defined in the third section of the paper. In the last section of paper we make an effort to study statistical convergence.

2. Some topological properties

Theorem 2.1 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $u = (u_{k,l})$ be a factorable double sequence of positive real numbers. Then the spaces $\Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big]$ and $\Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big]_0$ are linear spaces over the field $\mathbb{C}$ of complex numbers.*

Proof. Let $x = (x_{k,l}),\ y = (y_{k,l}) \in \Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big]_0$ and $\alpha, \beta\ \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}} = 0, \text{ uniformly in } m \text{ and } n$$

for some $\rho_1 > 0$, and

$$\lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}} = 0, \text{ uniformly in } m \text{ and } n$$

for some $\rho_2 > 0$.

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is a sequence of non-decreasing convex functions, by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v (\alpha x_{k+m,l+n} + \beta y_{k+m,l+n})}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}} \\
&\leq D\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} \frac{1}{2^{u_{k,l}}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v (x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}} \\
&\quad + D\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} \frac{1}{2^{u_{k,l}}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v (y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}} \\
&\leq D\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v (x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}} \\
&\quad + D\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v (y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}} \\
&\longrightarrow 0 \text{ as } r, s \longrightarrow \infty, \text{ uniformly in } m \text{ and } n.
\end{aligned}$$

Thus, we have $\alpha x + \beta y \in \Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big]_0$. Hence $\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big]_0$ is a linear space. Similarly, we can prove that $\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\Big]$ is a linear space. □

Theorem 2.2 *For any sequence of Orlicz functions* $\mathcal{M} = (M_{k,l})$ *and* $u = (u_{k,l})$ *be a factorable double sequence of positive real numbers, the space* $\left[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot||\right]_0$ *is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}} \sum_{k,l \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\},$$

where $K = \max(1, \sup_{k,l} u_{k,l} < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_{k,l}) \in \left[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\right]_0$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{H_{r,s}} \sum_{k,l \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1.$$

Thus

$$\begin{aligned}
&\Big(\tfrac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \\
&\leq \Big(\frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \\
&\leq 1,
\end{aligned}$$

for each r, s, m and n. Suppose that $x_{k,l} \neq 0$ for each $k, l \in \mathbb{N}$. This implies that $\Delta^v x_{k+m,l+n} \neq 0$, for each $k, l, m, n \in \mathbb{N}$. Let $\epsilon \to 0$, then $\Big(||\frac{\Delta^v x_{k+m,l+n}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big) \to \infty$. It follows that

$$\Big(\frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \to \infty,$$

which is a contradiction. Therefore, $\Delta^v x_{k+m,l+n} = 0$ for each k, l, m and n and thus $x_{k,l} = 0$ for each $k, l \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1$$

and

$$\Big(\frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v y_{k+m,l+n}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1$$

for each r, s, m and n. Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v(x_{k+m,l+n}+y_{k+m,l+n})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}}$$

$$\begin{aligned}
&\leq \Big(\sum_{(k,l)\in I_{r,s}}\Big[\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)p_k\bar{p}_l\Big(M_{k,l}\Big(||\frac{\Delta^v(x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)p_k\bar{p}_l\Big(M_{k,l}\Big(||\frac{\Delta^v(y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v(x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}}\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v(y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}}\\
&\leq 1.
\end{aligned}$$

Since $\rho's$ are non-negative, so we have
$g(x+y)$

$$\begin{aligned}
&= \inf\Big\{\rho^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v(x_{k+m,l+n}+y_{k+m,l+n})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}}\\
&\qquad \leq 1, r, s \in \mathbb{N}\Big\},\\
&\leq \inf\Big\{\rho_1^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v(x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}}\\
&\qquad \leq 1, r, s \in \mathbb{N}\Big\}\\
&+ \inf\Big\{\rho_2^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v(y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}}\\
&\qquad \leq 1, r, s \in \mathbb{N}\Big\}.
\end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v \lambda x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\}.$$

Then

$$g(\lambda x) = \inf\Big\{(|\lambda|t)^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{u_{r,s}} \leq \max(1, |\lambda|^{\sup u_{r,s}})$, we have
$g(\lambda x) \leq \max(1, |\lambda|^{\sup u_{r,s}})$

$$\inf\Big\{t^{\frac{u_{r,s}}{K}} : \Big(\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{u_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

To prove the next theorem we need the following lemma.

Lemma 2.1. *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $M(x) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Theorem 2.3 *For a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ which satisfies Δ_2-condition, we have $\Big[\tilde{R}^2, \theta_{r,s}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big] \subseteq \Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]$.*

Proof. Let $x = (x_{k,l}) \in \Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]$ so that for each m and n, we have

$$D_{r,s} = \Big\{x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l ||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}|| = 0,$$

$$\text{uniformly in } m \text{ and } n \text{ for some } L\Big\}.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_{k,l}(t) < \epsilon$ for every t with $0 \leq t < \delta$. Now, we have

$$\begin{aligned}
&\lim_{r,s} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l ||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}|| \\
&= \lim_{r,s} \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ ||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}|| < \delta}} p_k \bar{p}_l \Big(M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big) \\
&+ \lim_{r,s} \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ ||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}|| \geq \delta}} p_k \bar{p}_l \Big(M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big) \\
&\leq \frac{1}{H_{r,s}}(H_{r,s}\epsilon) \\
&+ \lim_{r,s} \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ ||\frac{||\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}|| > \delta}} p_k \bar{p}_l \Big(M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big) \\
&< \frac{1}{H_{r,s}}(H_{r,s}\epsilon) + \frac{1}{H_{r,s}} K\delta^{-1} p_k \bar{p}_l \big(M_{k,l}\big)(2) H_{r,s} D_{r,s}.
\end{aligned}$$

Therefore by above Lemma as r and s goes to infinity in the Pringsheim sense, for each m and n, we have $x = (x_{k,l}) \in \Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]_0$. This completes the proof of the theorem. □

Theorem 2.4 *Let $0 < \inf u_{k,l} = h \leq u_{k,l} \leq \sup u_{k,l} = H < \infty$ and $\mathcal{M} = (M_{k,l})$, $\mathcal{M}' = (M'_{k,l})$ be two sequences of Orlicz functions which satisfying Δ_2-condition, we have* *(i)* $\Big[\tilde{R}^2,\theta_{r,s},\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]\subset\Big[\tilde{R}^2,\theta_{r,s},\mathcal{M}\circ\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]$ *and*

(ii) $\Big[\tilde{R}^2,\theta_{r,s},\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]_0\subset\Big[\tilde{R}^2,\theta_{r,s},\mathcal{M}\circ\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]_0$.

Proof. Let $x=(x_{k,l})\in\Big[\tilde{\mathrm{R}}^2,\theta_{r,s},\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]$. Then we have

$$\lim_{r,s}\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}}p_k\bar{p}_l\Big[M'_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{u_{k,l}}=0,$$

uniformly in m and n for some L and $\rho>0$.

Let $\epsilon>0$ and choose δ with $0<\delta<1$ such that $M_{k,l}(t)<\epsilon$ for $0\le t\le\delta$. Let

$$y_{k,l}=p_k\bar{p}_l\Big(M'_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\ \text{ for all }\ k,l\in\mathbb{N}.$$

We can write

$$\begin{aligned}\frac{1}{H_{r,s}}\sum_{(k,l)\in I_{r,s}}p_k\bar{p}_l[M_{k,l}(y_{k,l})]^{u_{k,l}} &= \frac{1}{h_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ y_{k,l}\le\delta}}p_k\bar{p}_l[M_{k,l}(y_{k,l})]^{u_{k,l}}\\ &+ \frac{1}{H_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ y_{k,l}>\delta}}p_k\bar{p}_l[M_{k,l}(y_{k,l})]^{u_{k,l}}.\end{aligned}$$

Since $\mathcal{M}=(M_{k,l})$ satisfies Δ_2-condition, we have

$$\begin{aligned}\frac{1}{H_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ y_{k,l}\le\delta}}p_k\bar{p}_l[M_{k,l}(y_{k,l})]^{u_{k,l}} &\le p_k\bar{p}_l[M_{k,l}(1)]^H\frac{1}{H_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ y_{k,l}\le\delta}}p_k\bar{p}_l[M_{k,l}(y_{k,l})]^{u_{k,l}}\\ &\le p_k\bar{p}_l[M_{k,l}(2)]^H\frac{1}{H_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ y_{k,l}\le\delta}}p_k\bar{p}_l[M_{k,l}(y_{k,l})]^{p_{k,l}} \qquad (2.1)\end{aligned}$$

For $y_{k,l}>\delta$

$$y_{k,l}<\frac{y_{k,l}}{\delta}<1+\frac{y_{k,l}}{\delta}.$$

Since $\mathcal{M}=(M_{k,l})$ is non-decreasing and convex, it follows that

$$p_k\bar{p}_l\big(M_{k,l}(y_{k,l})\big)<p_k\bar{p}_l\big(M_{k,l}\big)\Big(1+\frac{y_{k,l}}{\delta}\Big)<\frac{1}{2}p_k\bar{p}_l\big(M_{k,l}(2)\big)+\frac{1}{2}p_k\bar{p}_l\Big(M_{k,l}\Big(\frac{2y_{k,l}}{\delta}\Big)\Big).$$

Also $(M_{k,l})$ satisfies Δ_2-condition, we can write

$$p_k\bar{p}_l\Big(M_{k,l}(y_{k,l})\Big)<\frac{1}{2}T\frac{y_{k,l}}{\delta}p_k\bar{p}_l\big(M_{k,l}(2)\big)+\frac{1}{2}T\frac{y_{k,l}}{\delta}p_k\bar{p}_1\big(M_{k,l}(2)\big)=T\frac{y_{k,l}}{\delta}p_k\bar{p}_l\big(M_{k,l}(2)\big).$$

Hence,

$$\begin{aligned}&\frac{1}{H_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ y_{k,l}>\delta}}p_k\bar{p}_l[M_{k,l}(y_{k,l})]^{u_{k,l}}\\ &\le\max\Big(1,\Big(\frac{Tp_k\bar{p}_l\big(M_{k,l}(2)\big)}{\delta}\Big)^H\Big)\frac{1}{H_{r,s}}\sum_{\substack{(k,l)\in I_{r,s}\\ y_{k,l}>\delta}}[(y_{k,l})]^{u_{k,l}} \qquad (2.2)\end{aligned}$$

by the inequalities (2.1) and (2.2), we have $x=(x_{k,l})\in\Big[\tilde{\mathrm{R}}^2,\theta_{r,s},\mathcal{M}\circ\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]$. This completes the proof of (i). Similarly, we can prove that $\Big[\tilde{\mathrm{R}}^2,\theta_{r,s},\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]_0\subset\Big[\tilde{\mathrm{R}}^2,\theta_{r,s},\mathcal{M}\circ\mathcal{M}',p,\Delta^v,u,||\cdot,\cdots,\cdot||\Big]_0$. □

3. Inclusion relations

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz function, $u = (u_{k,l})$ be any factorable double sequence of strictly positive real numbers and (p_μ) , $(\bar{p}_\eta)$ be sequences of positive numbers and $P_\mu = p_1 + p_2 + ... + p_\mu$, $\bar{P}_\eta = \bar{p}_1 + \bar{p}_2 + ... + \bar{p}_\eta$. In this section we define the following sequence spaces:
$\left[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\right] =$

$$\Big\{x = (x_{k,l}) : P - \lim_{\mu,\eta\to\infty} \frac{1}{P_\mu \bar{P}_\eta} \sum_{k=1}^{\mu} \sum_{l=1}^{\eta} p_k \bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0\Big\}$$

and
$\left[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\right]_0 =$

$$\Big\{x = (x_{k,l}) : P - \lim_{\mu,\eta\to\infty} \frac{1}{P_\mu \bar{P}_\eta} \sum_{k=1}^{\mu} \sum_{l=1}^{\eta} p_k \bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we have
$\left[\tilde{\mathrm{R}}^2, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\right] =$

$$\Big\{x = (x_{k,l}) : P - \lim_{\mu,\eta\to\infty} \frac{1}{P_\mu \bar{P}_\eta} \sum_{k=1}^{\mu} \sum_{l=1}^{\eta} p_k \bar{p}_l \Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)^{u_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0\Big\}$$

and
$\left[\tilde{\mathrm{R}}^2, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\right]_0 =$

$$\Big\{x = (x_{k,l}) : P - \lim_{\mu,\eta\to\infty} \frac{1}{P_\mu \bar{P}_\eta} \sum_{k=1}^{\mu} \sum_{l=1}^{\eta} p_k \bar{p}_l \Big(\|\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)^{u_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0\Big\}.$$

If we take $u_{k,l} = 1$, for all $k, l \in \mathbb{N}$, then we get
$\left[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, \|\cdot, \cdots, \cdot\|\right] =$

$$\Big\{x = (x_{k,l}) : P - \lim_{\mu,\eta\to\infty} \frac{1}{P_\mu \bar{P}_\eta} \sum_{k=1}^{\mu} \sum_{l=1}^{\eta} p_k \bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0\Big\}$$

and
$\left[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, \|\cdot, \cdots, \cdot\|\right]_0 =$

$$\Big\{x = (x_{k,l}) : P - \lim_{\mu,\eta\to\infty} \frac{1}{P_\mu \bar{P}_\eta} \sum_{k=1}^{\mu} \sum_{l=1}^{\eta} p_k \bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0\Big\}.$$

In this section of the paper we study inclusion relations between the spaces $\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big]$, $\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big]_0$, $\Big[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big]$ and $\Big[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big]_0$.

Theorem 3.1 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and (p_k), $(\bar{p}_l)$ be the sequences of positive numbers. If $\lim_r \inf Q_r > 1$ and $\lim_s \inf \bar{Q}_s > 1$, then $\Big[\tilde{R}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big] \subseteq \Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u\|\cdot, \cdots, \cdot\|\Big]$.*

Proof. Assume that $\lim_r \inf Q_r > 1$ and $\lim_s \inf \bar{Q}_s > 1$, then there exists $\delta > 0$ such that $Q_r > 1+\delta$ and $\bar{Q}_s > 1+\delta$. This implies $\frac{H_r}{P_{k_r}} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{H}_s}{\bar{P}_{l_s}} \geq \frac{\delta}{1+\delta}$. Then for $x \in \Big[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big]$, we can write for each m and n

$$
\begin{aligned}
A_{r,s} &= \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&= \frac{1}{H_{r,s}} \sum_{k=1}^{k_r}\sum_{l=1}^{l_s} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&- \frac{1}{H_{r,s}} \sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&- \frac{1}{H_{r,s}} \sum_{k=k_{r-1}+1}^{k_r}\sum_{l=1}^{l_{s-1}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&- \frac{1}{H_{r,s}} \sum_{k=1}^{k_{r-1}}\sum_{l=l_{s-1}+1}^{l_s} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&= \frac{P_{k_r}\bar{P}_{l_s}}{H_{r,s}}\Big(\frac{1}{P_k\bar{P}_{l_s}} \sum_{k=1}^{k_r}\sum_{l=1}^{l_s} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}\Big) \\
&- \frac{P_{k_{r-1}}\bar{P}_{l_{s-1}}}{H_{r,s}}\Big(\Big(\frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}\Big) \\
&- \frac{1}{H_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{\bar{P}_{l_{s-1}}}{\bar{H}_s}\frac{1}{\bar{P}_{l_{s-1}}} \sum_{l=1}^{l_{s-1}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&- \frac{1}{\bar{H}_s} \sum_{l=l_{s-1}+1}^{l_s} \frac{P_{k_{r-1}}}{H_r}\frac{1}{\bar{P}_{k_{r-1}}} \sum_{k=l}^{k_{r-1}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}.
\end{aligned}
$$

Since $x \in \Big[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big]$ the last two terms tend to zero uniformly in m and n in the Pringsheim sense, thus for each m and n, we have

$$
\begin{aligned}
A_{r,s} &= \frac{P_{k_r}\bar{P}_{l_s}}{H_{r,s}}\Big(\frac{1}{p_k\bar{p}_{l_s}} \sum_{k=1}^{k_r}\sum_{l=1}^{l_s} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}\Big) \\
&- \frac{P_{k_{r-1}}\bar{P}_{l_{s-1}}}{H_{r,s}}\Big(\Big(\frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}\Big) + 0(1).
\end{aligned}
$$

Since $H_{r,s} = P_{k_r}P_{l_s} - P_{k_{r-1}}P_{l_{s-1}}$, for each m and n we have the following:

$$\frac{P_{k_r}P_{l_s}}{H_{r,s}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{P_{k_{r-1}}P_{l_{s-1}}}{H_{r,s}} \leq \frac{1}{\delta}.$$

The terms

$$\frac{1}{P_{k_r}\bar{P}_{l_s}} \sum_{k=1}^{k_r}\sum_{l=1}^{l_s} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}$$

and

$$\frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}$$

are both Pringsheim null sequences for all m and n. Thus $A_{r,s}$ is a Pringsheim null sequence for each m and n. Therefore $x \in \Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big]$. This completes the proof of the theorem. □

Theorem 3.2 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and (p_k), $(\bar{p}_l)$ be sequences of positive numbers. If $\limsup_r Q_r < \infty$ and $\limsup_s \bar{Q}_s < \infty$, then $\Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big] \subseteq \Big[\tilde{R}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\|\Big]$.*

Proof. Since $\limsup_r Q_r < \infty$ and $\limsup_r \bar{Q}_s < \infty$, there exists $H > 0$ such that $Q_r < H$ and $\bar{Q}_s < H$ for all r and s. Let $x \in \Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big]$ and $\epsilon > 0$. Then there exist $r_0 > 0$ and $s_0 > 0$ such that for every $i \geq r_0$ and $j \geq s_0$ and for all m and n,

$$A'_{i,j} = \frac{1}{H_{i,j}} \sum_{(k,l)\in I_{i,j}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} < \epsilon.$$

Let $\mathcal{M}' = \max\{A'_{i,j} : 1 \leq i \leq r_0 \text{ and } 1 \leq j \leq s_0\}$ and n and m be such that $k_{r-1} < n \leq k_r$ and $l_{s-1} < m \leq l_s$. Thus we obtain the following:

$$\frac{1}{P_n\bar{P}_m} \sum_{k=1}^{n}\sum_{l=1}^{m} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}$$

$$\begin{aligned}
&\leq \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{k=1}^{k_r}\sum_{l=1}^{l_s} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&= \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{t,u=1,1}^{r,s} \Big(\sum_{(k,l)\in I_{t,u}} p_k\bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}\Big) \\
&= \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{t,u=1,1}^{r_0,s_0} H_{t,u}A'_{t,u} + \frac{1}{P_{k_{r-1}}\bar{P}_{l_{s-1}}} \sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} H_{t,u}A'_{t,u}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mathcal{M}' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \Big(\sup_{t \geq r_0 \cup u \geq s_0} A'_{t,u} \Big) \frac{1}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} H_{t,u} \\
&\leq \frac{\mathcal{M}' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \frac{\epsilon}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} \sum_{(r_0 < t \leq r) \cup (s_0 < u \leq s)} H_{t,u} \\
&\leq \frac{\mathcal{M}' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \frac{P_{k_r}}{P_{k_{r-1}}} \frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}} \epsilon = \frac{\mathcal{M}' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + Q_r \bar{Q}_s \epsilon \\
&\leq \frac{\mathcal{M}' P_{k_{r_0}} \bar{P}_{l_{s_0}}}{P_{k_{r-1}} \bar{P}_{l_{s-1}}} + \epsilon H^2.
\end{aligned}
$$

Since $P_{k,l} \to \infty$ and $\bar{P}_{l_{s-1}} \to \infty$ as $r, s \to \infty$, it follows that

$$
\frac{1}{P_n \bar{P}_m} \sum_{k=1}^{n} \sum_{l=1}^{m} p_k \bar{p}_l \Big[M_{k,l} \Big(\| \frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1} \| \Big) \Big]^{u_{k,l}} \to 0,
$$

uniformly in m and n. Therefore $x \in \Big[\tilde{\mathrm{R}}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\| \Big]$. □

Theorem 3.3 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and (p_k), $(\bar{p}_l)$ be the sequences of positive numbers. If $\limsup_r Q_r < \infty$ and $\limsup_s \bar{Q}_s < \infty$, then $\Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot|| \Big] = \Big[\tilde{R}^2, \mathcal{M}, p, \Delta^v, u, \|\cdot, \cdots, \cdot\| \Big]$.*

Proof. It is easy to prove by using Theorem 3.1 and Theorem 3.2. □

Theorem 3.4 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions. Then the following statements are true:*
(a) If $p_k < 1$ and $\bar{p}_l < 1$ for all $k, l \in \mathbb{N}$, then
$\Big[AC_{\theta r,s}, \mathcal{M}, \Delta^v, u, \|\cdot, \cdots, \cdot\| \Big] \subset \Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot|| \Big]$ with $\Big[AC_{\theta r,s}, \mathcal{M}, \Delta^v, u, \|\cdot, \cdots, \cdot\| \Big] - P - \lim x = \Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot|| \Big] - P - \lim x = L$.
(b) If $p_k > 1$ and $\bar{p}_l > 1$ for all $k, l \in \mathbb{N}$, then
$\Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot|| \Big] \subset \Big[AC_{\theta r,s}, \mathcal{M}, \Delta^v, u, \|\cdot, \cdots, \cdot\| \Big]$ with
$\Big[\tilde{R}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u, ||\cdot, \cdots, \cdot|| \Big] - P - \lim x = \Big[AC_{\theta r,s}, \mathcal{M}, \Delta^v, u, \|\cdot, \cdots, \cdot\| \Big] - P - \lim x = L$.

Proof. (a) If $p_k < 1$ and $\bar{p}_l < 1$ for all $k, l \in \mathbb{N}$, then $H_r < h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s < \bar{h}_s$ for all $s \in \mathbb{N}$, respectively. Then there exist two constants M and N such that $0 < M \leq \frac{H_r}{h_r} < 1$ for all $r \in \mathbb{N}$ and $0 < N \leq \frac{\bar{H}_s}{\bar{h}_s} < 1$ for all $s \in \mathbb{N}$. Let $x = (x_{k,l})$ be a double sequence with $P - \lim x = L$ in $\Big[AC_{\theta r,s}, \mathcal{M}, \Delta^v, u, \|\cdot, \cdots, \cdot\| \Big]$, then for each m and n

$$
\begin{aligned}
&\frac{1}{H_{r,s}} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l} \Big(\| \frac{\Delta^v x_{k+m,l+n} - L}{\rho} \| \Big) \Big]^{u_{k,l}} \\
&= \frac{1}{H_r \bar{H}_s} \sum_{(k,l) \in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l} \Big(\| \frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1} \| \Big) \Big]^{u_{k,l}} \\
&< \frac{1}{M h_r} \frac{1}{N \bar{h}_s} \sum_{(k,l) \in I_{r,s}} \Big[M_{k,l} \Big(\| \frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1} \| \Big) \Big]^{u_{k,l}} \\
&= \frac{1}{MN} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \Big[M_{k,l} \Big(\| \frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1 \cdots, z_{n-1} \| \Big) \Big]^{u_{k,l}}.
\end{aligned}
$$

Hence, we obtain the result by taking the P-limit as $r, s \to \infty$.
(b) Let $\frac{H_r}{h_r}$ and $\frac{\bar{H}_s}{\bar{h}_s}$ be upper bounded and $p_k > 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l > 1$ for all $l \in \mathbb{N}$. Then $H_r > h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s > \bar{h}_s$ for all $s \in \mathbb{N}$. Let there exists two constants M and N such that $1 < \frac{H_r}{h_r} \leq M < \infty$ for all $r \in \mathbb{N}$ and $1 < \frac{\bar{H}_s}{\bar{h}_s} \leq N < \infty$ for all $s \in \mathbb{N}$. Suppose that the double sequence $x = (x_{k,l})$ converges to the $P - \lim L \in \Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big]$, with $\Big[\tilde{\mathrm{R}}^2, \theta_{r,s}, \mathcal{M}, p, \Delta^v, u||\cdot, \cdots, \cdot||\Big] - P - \lim x = L$, then for each m and n we have
$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}\|\Big)\Big]^{u_{k,l}}$

$$\begin{aligned}
&= \frac{1}{h_r \bar{h}_s} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&< \frac{M}{H_r}\frac{N}{\bar{H}_s} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}} \\
&= \frac{1}{MN}\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k \bar{p}_l \Big[M_{k,l}\Big(\|\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1 \cdots, z_{n-1}\|\Big)\Big]^{u_{k,l}}.
\end{aligned}$$

Hence, the result is obtained by taking the $P - \lim$ as $r, s \to \infty$. □

4. Statistical convergence

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [42] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [9], Connor [6], Salat [40], Mursaleen and Edely [29], Mursaleen and Mohiuddine [30], Isık [15], Savas [41], Kolk [17], Maddox [21] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.
A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists: $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.
A sequence $x = (x_{k,l})$ is said to be lacunary Δ^v-statistically convergent to L, if for every $\epsilon > 0$

$$\lim_{r,s} \frac{1}{H_{r,s}} \Big|\Big\{(k,l) \in I_{r,s} : ||(\Delta^v x_{k,l} - L), z_1, \cdots, z_{n-1}|| \geq \epsilon\Big\}\Big| = 0.$$

In this case we write $x_{k,l} \to L\Big(S_{\theta_{r,s}}(\Delta^v)\Big)$. The set of all lacunary Δ^v-statistically convergent sequences is denoted by $S_{\theta_{r,s}}(\Delta^v)$.

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. The double number sequence x is said to be $S_{R^2,\theta_{r,s},\Delta^v} - P$-convergent to L provided that for every $\epsilon > 0$,

$$P - \lim_{r,s} \frac{1}{H_{r,s}} \sup_{m,n} \Big|\big\{(k,l) \in I'_{r,s} : p_k \bar{p}_l \big|\Delta^v x_{k+m,l+n} - L\big| \geq \epsilon\big\}\Big| = 0.$$

In this case we write $S_{R^2,\theta_{r,s},\Delta^v} - P - \lim x = L$.

A double sequence $x = (x_{k,l})$ is said to be Riesz lacunary almost P-convergent to L if $P - \lim\limits_{r,s} \omega_{rs}^{mn}(x) = L$, uniformly in m and n, where $\omega_{rs}^{mn} = \omega_{rs}^{mn}(x) = \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l(\Delta^v x_{k+m,l+n})$.

A double sequence $x = (x_{k,l})$ is said to be Riesz lacunary almost statistically summable to L if for every $\epsilon > 0$ the set

$$K_\epsilon = \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : |\omega_{rs}^{mn} - L| \geq \epsilon\right\}$$

has double natural density zero, i.e., $\delta_2(K_\epsilon) = 0$. In this case, we write $(\tilde{\mathrm{R}}, \theta, \Delta^v)_{st_2} - P - \lim x = L$. That is, for every $\epsilon > 0$, $P - \lim_{m,n} \left|\{r \leq m, s \leq n : |\omega_{rs}^{mn} - L| \geq \epsilon\}\right| = 0$, uniformly in m and n. Hence, a double sequence $x = (x_{k,l})$ is Riesz lacunary almost statistically summable to L if and only if the double sequence $(\omega_{rs}^{mn}(x))$ is almost statistically P-convergent to L. Note that since a convergent double sequence is also statistically convergent to the same value, a Riesz lacunary almost convergent double sequence is also Riesz lacunary almost statistically summable with the same P-limit.

A double sequence $x = (x_{k,l})$ is said to be strongly $\left[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v\right]_q$-almost convergent $(0 < q < \infty)$ to the number L if $P - \lim\limits_{r,s} \omega_{rs}^{mn}(|\Delta^v x - L|^q) = 0$, uniformly in m and n. In this case, we write $x_{k,l} \to L\left(\left[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v\right]_q\right)$ and L is called $\left[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v\right]_q - P - \lim$ of x. Also, we denote the set of all strongly $\left[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v\right]_q$-almost P-convergent double sequences by $\left[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v\right]_q$.

Theorem 4.1 *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. If $I'_{r,s} \subseteq I_{r,s}$, then $\left[\tilde{R}^2, \theta_{r,s}, p, \Delta^v\right] \subset S_{(R^2,\theta_{r,s},\Delta^v)}$.*

Proof.

$$K_{P_{r,s}}(\epsilon) = \left\{(k,l) \in I'_{r,s} : p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| \geq \epsilon\right\}. \tag{2}$$

Suppose that $x \in \left[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v\right]$. Then for each m and n.

$$P - \lim_{r,s} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| = 0.$$

Since

$$\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| \geq \frac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L|$$

$$= \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s}\\(k,l)\in K_{P_{r,s}(\epsilon)}}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| + \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s}\\(k,l)\notin K_{P_{r,s}(\epsilon)}}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L|$$

$$\geq \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s}\\(k,l)\in K_{P_{r,s}(\epsilon)}}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| = \frac{|K_{P_{r,s}}(\epsilon)|}{H_{r,s}} \ \ \forall\ m \text{ and } n$$

we get $P - \lim\limits_{r,s} \dfrac{|K_{P_{r,s}}(\epsilon)|}{H_{r,s}} = 0$ for each m and n. This implies that $x \in S_{(R^2,\theta_{r,s},\Delta^v)}$. □

Theorem 4.2 *Let M be a constant such that $p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| \leq M$, $\forall k, l \in \mathbb{N}$ and for all m and n. If $I'_{r,s} \subseteq I_{r,s}$, then $S_{(R^2,\theta_{r,s},\Delta^v)} \subset \left[\tilde{R}^2, \theta_{r,s}, p, \Delta^v\right]$ with $\left[\tilde{R}^2, \theta_{r,s}, p, \Delta^v\right] - P - \lim x = S_{(R^2,\theta_{r,s},\Delta^v)} - P - \lim x = L$.*

Proof. Suppose that $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence $p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| \leq M$, $\forall k, l \in \mathbb{N}$ and for all m and n. Let $I'_{r,s} \subseteq I_{r,s}$ and $K_{P_{r,s}}$ be defined in eqn. (4.1). Since $x \in S_{(R^2,\theta_{r,s},\Delta^v)}$ with $S_{(R^2,\theta_{r,s},\Delta^v)} - P - \lim x = L$, then $P - \lim_{r,s} \dfrac{|K_{P_{r,s}}|}{H_{r,s}} = 0$. For a given $\epsilon > 0$ and for all m and n we have

$$\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| \leq \frac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L|$$

$$= \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}(\epsilon)}}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| + \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}(\epsilon)}}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L|$$

$$\leq M\frac{|K_{P_{r,s}}|}{H_{r,s}} + \epsilon.$$

Since ϵ is arbitrary, we get $x \in \left[\tilde{\mathrm{R}}^2, \theta_{r,s}, p, \Delta^v\right]$ with the same $P - \lim$. □

Theorem 4.3 *If $p_k \leq 1$, for all $k \in \mathbb{N}$ and $\bar{p}_l \leq 1$ for all $l \in \mathbb{N}$, then $S_{(\theta_{r,s},\Delta^v)} \subset S_{(R^2,\theta_{r,s},\Delta^v)}$ with $S_{(\theta_{r,s},\Delta^v)} - P - \lim x = S_{(R^2,\theta_{r,s},\Delta^v)} - P - \lim x = L$.*

Proof. If $p_k \leq 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l \leq 1$ for all $l \in \mathbb{N}$, then $H_r \leq h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s \leq \bar{h}_s$ for all $s \in \mathbb{N}$. So, there exist constants M and N such that $0 < M \leq \frac{H_r}{h_r} \leq 1$ for all $r \in \mathbb{N}$ and $0 < N < \frac{\bar{H}_s}{\bar{h}_s} \leq 1$ for all $s \in \mathbb{N}$. Let $x = (x_{k,l})$ be a double sequence which converges to the P-limit L in $S_{(\theta_{r,s},\Delta^v)}$, then for an arbitrary $\epsilon > 0$, and for all m and n, we have

$$\frac{1}{H_{r,s}} \left|\left\{(k,l) \in I'_{r,s} : p_k\bar{p}_l \left|\Delta^v x_{x+m,l+n} - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{H_r H_s} \left|\left\{P_{k_{r-1}} < k \leq P_{k_r} \;\; and \;\; \bar{P}_{l_{s-1}} < l \leq \bar{P}_{l_s} : p_k\bar{p}_l \left|\Delta^v x_{k+m,l+n} - L\right| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{MN}\frac{1}{h_r h_s} \left|\left\{P_{k_{r-1}} \leq k_{r-1} < k \leq P_{k_r} \leq k_r \;\; and\right.\right.$$

$$\left.\left.\bar{P}_{l_{s-1}} < l_{s-1} < l \leq \bar{P}_{l_s} \leq l_s : \left|\Delta^v x_{k+m,l+n} - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{MN}\frac{1}{h_{r.s}} \left|\left\{k_{r-1} < k \leq k_r \;\; and \;\; l_{s-1} < l \leq l_s : \left|\Delta^v x_{x+m,l+n} - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{MN}\frac{1}{h_{r.s}} \left|\left\{(k,l) \in I_{r,s} : \left|\Delta^v x_{x+m,l+n} - L\right| \geq \varepsilon\right\}\right|.$$

Hence, we obtain the result by taking the P-limit as $r, s \to \infty$. □

Theorem 4.4 *If $p_k \geq 1$, for all $k \in \mathbb{N}$ and $\bar{p}_l \geq 1$ for all $l \in \mathbb{N}$ and $\frac{H_r}{h_r}, \frac{\bar{H}_s}{\bar{h}_s}$ are upper bounded, then $S_{(R^2,\theta_{r,s},\Delta^v)} \subset S_{(\theta_{r,s},\Delta^v)}$ with $S_{(R^2,\theta_{r,s},\Delta^v)} - P - \lim x = S_{(\theta_{r,s},\Delta^v)} - P - \lim x = L$.*

Proof. If $p_k \geq 1$ for all $k \in \mathbb{N}$ and $\bar{p}_l \geq 1$ for all $l \in \mathbb{N}$ and $\frac{H_r}{h_r}, \frac{\bar{H}_s}{\bar{h}_s}$ are upper bounded, then $H_r \geq h_r$ for all $r \in \mathbb{N}$ and $\bar{H}_s \geq \bar{h}_s$ for all $s \in \mathbb{N}$. So, there exist constants M and N such that $1 \leq \frac{H_r}{h_r} \leq M$ for all $r \in \mathbb{N}$ and $1 \leq \frac{\bar{H}_s}{\bar{h}_s} \leq N$ for all $s \in \mathbb{N}$. Let $x = (x_{k,l})$ be a double sequence which converges to the P-limit L in $S_{(R^2,\theta_{r,s},\Delta^v)}$ with $S_{(R^2,\theta_{r,s},\Delta^v)} - P - \lim x = L$, then for an arbitrary $\epsilon > 0$, and for all m and n, we have

$$\frac{1}{h_r\bar{h}_s}\left|\{(k,l) \in I_{r,s} : |\Delta^v x_{k+m,l+n} - L| \geq \varepsilon\}\right|$$

$$= \frac{1}{H_r\bar{H}_s}\left|\left\{P_{k_r-1} < k \leq P_{k_r} \ \ and \ \ \bar{P}_{l_{s-1}} < l \leq \bar{P}_{l_s} : p_k\bar{p}_l \left|\Delta^v x_{k+m,l+n} - L\right| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{MN}\frac{1}{h_r\bar{h}_s}\left|\left\{P_{k_{r-1}} \leq k_{r-1} < k \leq P_{k_r} \leq k_r \ \ and\right.\right.$$

$$\left.\left.\bar{P}_{l_{s-1}} \leq l_{s-1} < l \leq \bar{P}_{l_s} \leq l_s : |\Delta^v x_{k+m,l+n} - L| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{MN}\frac{1}{h_{r,s}}\left|\{k_{r-1} < k \leq k_r \ \ and \ \ l_{s-1} < l \leq l_s : |\Delta^v x_{k+m,l+n} - L| \geq \varepsilon\}\right|$$

$$= \frac{1}{MN}\frac{1}{h_{r,s}}\left|\{(k,l) \in I_{r,s} : |\Delta^v x_{k+m,l+n} - L| \geq \varepsilon\}\right|.$$

Hence, we obtain the result by taking the P-limit as $r, s \to \infty$. □

Theorem 4.5 *Let $I_{r,s} \subseteq I'_{r,s}$ and $p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| \leq M$ for all $k, l \in \mathbb{N}$ and for all m and n. If the following hold*
(a) $0 < q < 1$ and $1 \leq |\Delta^v x_{k+m,l+n} - L| < \infty$
(b) $1 \leq q < \infty$ and $0 \leq |\Delta^v x_{k+m,l+n} - L| < 1$,
then $S_{(R^2,\theta_{r,s},\Delta^v)} \subset [\tilde{R}^2, \theta_{r,s}, \Delta^v, p]_q$ and $S_{(R^2,\theta_{r,s},\Delta^v)} - P - \lim x = [\tilde{R}^2, \theta_{r,s}, \Delta^v, p]_q - P - \lim x = L$.

Proof. Let $x = (x_{k,l}) \in S_{(R^2,\theta_{r,s},\Delta^v)}$ with $P - \lim_{r,s\to\infty} \frac{1}{H_{r,s}}|K_{P_{r,s}}| = 0$, where $K_{P_{r,s}}(\epsilon)$ was defined in eqn. (4.1). Since $p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L| \leq M$ for all $k, l \in \mathbb{N}$ and for all m and n. If $I_{r,s} \subseteq I'_{r,s}$, then for a given $\epsilon > 0$ and for all m and n, we have

$$\frac{1}{h_r\bar{h}_s}\left|\{(k,l) \in I_{r,s} : |\Delta^v x_{k+m,l+n} - L| \geq \varepsilon\}\right| = \frac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k\bar{p}_l \left|\Delta^v x_{k+m,l+n} - L\right|^q$$

$$\leq \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k\bar{p}_l \left|\Delta^v x_{k+m,l+n} - L\right|^q +$$

$$+\frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\varepsilon)}} p_k\bar{p}_l \left|\Delta^v x_{k+m,l+n} - L\right|^q = A_{r,s} + B_{r,s}$$

where

$$A_{r,s} = \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\epsilon)}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L|^q$$

and

$$B_{r,s} = \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\epsilon)}} p_k\bar{p}_l|\Delta^v x_{k+m,l+n} - L|^q.$$

For $(k,l) \notin K_{P_{r,s}}(\epsilon)$, we have

$$A_{r,s} = \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\epsilon)}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|^q \leq \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\epsilon)}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L| \leq \epsilon.$$

If $(k,l) \in P_{P_{r,s}}(\epsilon)$, then

$$B_{r,s} = \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\epsilon)}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|^q$$

$$\leq \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ (k,l)\notin K_{P_{r,s}}(\epsilon)}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L| \leq \frac{M}{H_{r,s}} |K_{P_{r,s}}(\epsilon)|.$$

Hence

$$\lim_{r,s\to\infty} \frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|^q = 0, \text{ uniformly in } m \text{ and } n.$$

This completes the proof. □

Theorem 4.6 *Let $I_{r,s} \subseteq I'_{r,s}$ and. If the following hold*
(a) $0 < q < 1$ and $0 \leq |\Delta^v x_{k+m,l+n} - L| < 1$
(b) $1 \leq q < \infty$ and $1 \leq |\Delta^v x_{k+m,l+n} - L| < \infty$,
then $[\tilde{R}^2, \theta_{r,s}, \Delta^v, p]_q \subset S_{(R^2,\theta_{r,s},\Delta^v)}$ and $[\tilde{R}^2, \theta_{r,s}, \Delta^v, p]_q - P - \lim x = S_{(R^2,\theta_{r,s},\Delta^v)} - P - \lim x = L$.

Proof. Let $x = (x_{k,l}) \in [\tilde{\mathrm{R}}^2, \theta_{r,s}, \Delta^v, p]_q$-almost P-convergent to the limit L. Since $p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|^q \geq p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|$ for case (a) and (b), then for all m and n, we have

$$\frac{1}{H_{r,s}} \sum_{(k,l)\in I_{r,s}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|^q \geq \frac{1}{H_{r,s}} \sum_{(k,l)\in I'_{r,s}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|$$

$$\geq \frac{1}{H_{r,s}} \sum_{\substack{(k,l)\in I'_{r,s} \\ (k,l)\in K_{P_{r,s}}(\epsilon)}} p_k\bar{p}_l |\Delta^v x_{k+m,l+n} - L|^q \geq \epsilon \frac{1}{H_{r,s}} |K_{P_{r,s}}(\epsilon)|$$

where $K_{P_{r,s}}$ was defined in eqn.(4.1). Taking limit as $r,s \to \infty$ in both sides of the above inequality, we conclude that $S_{R^2,\theta_{r,s},\Delta^v} - P - \lim x = L$. This completes the proof of the theorem. □

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