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ON $q-$BERNARDI INTEGRAL OPERATOR

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Abstract. In this paper, we introduce the q-Bernardi integral operator for analytic functions using the concept of q-calculus. We also introduce two new classes of analytic functions with respect to $q-$derivative. The convexity and integral preserving properties for these classes in open unit disc are investigated. Special cases of our main results are discussed, which appear to be new ones. The ideas and techniques of this paper may inspire further research in this field.

Keywords: q-calculus, q-Bernardi integral operator.

AMS Subject Classification: 30C45.

1. Introduction

Quantum calculus or q-calculus is an ordinary calculus without notion of limit. In recent years, q-calculus attracted attention of many researchers due to its vast applications in Mathematics and Physics. Jackson [6, 7] introduced and studied the q-derivative and q-integral in systematic way. A firm footing was actually provided and the basic q- hypergeometric functions were first used in geometric function theory by Srivastava [18]. Making use of q-derivative, Ismail [5] introduced and studied a class of q-starlike functions. The q-analogue of close-to-convex functions is defined in [13]. Sahoo and Sharma [15] obtained several interesting results for q-close-to-convex functions. Using the convolution of normalized analytic functions, q-operators are defined with several interesting results, see [2], [3]. Geometric properties of these q-operators in some classes of analytic functions in compact disc are studied in [10]. Recently, Selvakumaran et al. [16] introduced the q-integral operators for certain analytic functions in a unit disc, by using the concept and theory of fractional q-calculus. They also studied some convexity properties of such q-integral operators for some classes of analytic functions which was defined by a linear multiplier fractional q-differintegral operator. For the recent work on the usefulness of q-calculus, see [12, 18] and the references therein. It is an interesting problem to study the applications of q-calculus to derive integral inequalities for relative harmonic preinvex functions [11].

Geometric function theory is a highly developed branch of mathematics which suggests the significance of geometric ideas and problems in complex analysis. These ideas also occur in real analysis, but geometry has had a greater impact in complex analysis. There are numerous applications of geometric function theory in other branches of mathematics and physics.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

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analytic in the open unit disc $E = \{z : |z| < 1\}$. Let S, C, S^*, K be the subclasses of $\mathcal{A}$ of univalent, convex, starlike, close-to-convex functions, respectively.

Let $f \in \mathcal{A}$. Then the operator $I : \mathcal{A} \to \mathcal{A}$ is defined as:

$$I(f)(z) = \frac{(1+c)}{z^c} \int_0^z t^{c-1} f(t)dt, , c = 1, 2, 3, ... \tag{2}$$

is called Bernardi integral operator, which was introduced by Bernardi in [4]. He proved that the subclasses S^*, C, K of S are closed under this operator.

We note that, for $c = 1$ in (2), we have the Libera integral operator, see [9].

Let $f(z)$ be given by (1.1) and $g(z)$ be defined as

$$g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then the Hadamard product (or convolution) is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

We now recall the basic concepts and results of q-calculus. To be more precise, the q-derivative of function $f \in \mathcal{A}$ is defined as, (see [7])

$$d_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0), \tag{3}$$

and $d_q f(0) = f'(0)$, where $q \in (0, 1)$. For a function $g(z) = z^n$, the q-derivative is

$$d_q g(z) = [n]_q z^{n-1}, \tag{4}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

We note that as $q \to 1^-$, $d_q f(z) \to f'(z)$. Here $f'(z)$ is ordinary derivative and $[n]_q \to n$ as $q \to 1^-$. From (1.4) we deduce that

$$d_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^n. \tag{5}$$

Jackson [6] introduced the q-integral of a function f as:

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(q^n z), \tag{6}$$

provided that series converges.

The subclasses of q-convex and q-starlike functions are defined by, (see ([5]),

$$S_q^* = \left\{ f \in A : Re \frac{z d_q f(z)}{f(z)} > 0, \quad z \in E \right\},$$

$$C_q = \left\{ f \in A : Re \frac{d_q (z d_q f(z)}{d_q f(z)} > 0, \quad z \in E \right\}.$$

We note that when $q \longrightarrow 1^-$, the above classes reduce to well known classes of convex and starlike functions.

Let Ω be the family of functions $w(z)$ analytic in the open unit disc E and satisfy the conditions $w(0) = 0, \quad |w(z)| < 1$ for $z \in E$. For arbitrary fixed numbers $A, B, -1 \leq B < A \leq 1$, denote by $P(A, B)$, the class of functions p analytic in E such that

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{7}$$

for some functions $w(z) \in \Omega$. This class was introduced by Janowski [8].

The aim of this research paper is to introduce and study q-Bernardi integral operator. Using the q-calculus, we introduce and study some new classes of analytic functions, which can be viewed as significant generalizations of the previously known classes. We investigate the properties of q-Bernardi integral operator for these new classes. Several consequences of the main results are mentioned.

2. Main results

We here define and study some new classes of analytic functions using the concept of q-calculus.

Definition 2.1. *Let $f \in \mathcal{A}$. Then $f \in S_q^*(A, B)$, if and only if,*

$$\left\{ \frac{zd_q f(z)}{f(z)} \right\} \in P(A, B), \quad -1 \leq B < A \leq 1, \quad z \in E. \tag{8}$$

We note the following.

$$\begin{aligned} (i). \quad S_q^*(1 - 2\alpha, -1) &= S_q^*(\alpha) \qquad 0 \leq \alpha < 1, \\ &= \left\{ f \in \mathcal{A} : Re \frac{zd_q f(z)}{f(z)} > \alpha, z \in E \right\}, \quad \text{see [1]} \end{aligned}$$

$$(ii). \quad S_q^*(1, -1) = S_q^* = \left\{ f \in \mathcal{A} : Re \frac{zd_q f(z)}{f(z)} > 0, z \in E \right\}, \quad \text{see [5].}$$

$$\begin{aligned} (iii). \quad S_q^*((1 - 2\alpha)\beta, -\beta) &= S_q^*(\alpha, \beta), \quad (0 \leq \alpha < 1), \quad (0 \leq \beta < 1) \\ &= \left\{ f \in \mathcal{A} : \left| \frac{\frac{zd_q f(z)}{f(z)} - 1}{\frac{zd_q f(z)}{f(z)} + 1 - 2\alpha} \right| < \beta, z \in E \right\}, \end{aligned}$$

$$(iv). \quad \lim_{q \to 1^-} S_q^*(A, B) = \left\{ f \in A : \lim_{q \to 1^-} \frac{zd_q f(z)}{f(z)} \in P(A, B) \right\} = S^*(A, B), \quad \text{see[9].}$$

Definition 2.2. *Let $f \in \mathcal{A}$. Then, $f \in K_q(A, B)$, if there exists a function $g \in \bigcap\limits_{0<q<1} S_q^*(A, B)$, $(-1 \leq B < A \leq 1,\ z \in E)$, such that*

$$\left\{ \frac{zd_q f(z)}{g(z)} \right\} \in P(A, B), \tag{9}$$

We note the following.

$$(i).\quad K_q(1-2\alpha,-1) = K_q(\alpha) \quad (0\le\alpha<1)$$
$$= \left\{f\in\mathcal{A}: Re\frac{zd_qf(z)}{g(z)}>\alpha, g\in\bigcap_{0<q<1}S_q^*(\alpha), z\in E\right\},$$

$$(ii).\quad K_q(1,-1)=K_q=\left\{f\in\mathcal{A}: Re\frac{zd_qf(z)}{g(z)}>0, g\in\bigcap_{0<q<1}S_q^*, z\in E\right\},$$

$$(iii).\quad {}_q((1-2\alpha)\beta,-\beta) = K_q(\alpha,\beta), \quad (0\le\alpha<1), \quad (0\le\beta<1)$$
$$= \left\{f\in\mathcal{A}: \left|\frac{\frac{zd_qf(z)}{g(z)}-1}{\frac{zd_qf(z)}{g(z)}+1-2\alpha}\right|<\beta, g\in\bigcap_{0<q<1}S_q^*(\alpha,\beta), z\in E\right\},$$

$$(iv).\quad \lim_{q\to1^-}K_q(A,B) = \left\{f\in\mathcal{A}: \lim_{q\to1^-}\frac{zd_qf(z)}{g(z)}\in P(A,B), g\in\bigcap_{0<q<1}S_q^*(A,B), z\in E\right\}$$
$$= K(A,B), \quad \text{see[9]}.$$

We now define the q-Bernardi integral operator.

Let $f\in\mathcal{A}$. Then $L:\mathcal{A}\to\mathcal{A}$ is said to be q-Bernardi integral operator defined by $L(f)=F$, where F is given by

$$F(z) = \frac{[1+c]_q}{z^c}\int_0^z t^{c-1}f(t)d_qt \tag{10}$$
$$= \sum_{n=1}^{\infty}\left(\frac{[1+c]_q}{[n+c]_q}\right)a_nz^n, \quad c=1,2,3,...$$

We note that, for $c=1$ in (10), we have the q-Libera integral operator defined as:

$$F_1(z) = \frac{[2]_q}{z}\int_0^z f(t)d_qt \tag{11}$$
$$= \sum_{n=1}^{\infty}\left(\frac{[2]_q(1-q)}{1-q^{n+1}}\right)a_nz^n, \quad 0<q<1, \quad z\in E, \quad \text{see [11,17]}. \tag{12}$$

Also the that radius of convergence R of $\sum_{n=1}^{\infty}\left(\frac{[2]_q(1-q)}{1-q^{n+1}}\right)a_nz^n$ and $\sum_{n=1}^{\infty}\left(\frac{[1+c]_q}{[n+c]_q}\right)a_nz^n$ is q, for $0<q<1$ and we have

$$\lim_{q\longrightarrow1^-}F_(z)=\sum_{n=1}^{\infty}\frac{(1+c)}{(n+c)}a_nz^n,$$

,

$$\lim_{q\longrightarrow1^-}F_1(z)=z+\sum_{n=1}^{\infty}\frac{2}{(n+1)}a_nz^n,$$

which are defined in [14].

Lemma 2.1. *Let N and T be analytic in E, $N(0)=T(0)=0$, $d_qN(0)/d_qT(0)=1$, $0<q<1$. Then*

$$\frac{d_qN(z)}{d_qT(z)}\in P(A,B),$$

implies that

$$\frac{N(z)}{T(z)} \in P(A,B), \quad T \in \bigcap_{0<q<1} S_q^*(A,B), \quad z \in E.$$

Proof. Consider

$$\frac{d_qN(z)}{d_qT(z)} \in P(A,B). \tag{13}$$

For $p(z) \in P(A,B)$, we have

$$\left|p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| < \frac{|B-A|r}{1-B^2r^2}, \quad \text{see[9]}.$$

On other hand from (13), we can write

$$\frac{d_qN(z)}{d_qT(z)} = p(z). \tag{14}$$

From (13) and (14), we have

$$\left|\frac{d_qN(z)}{d_qT(z)} - a(r)\right| < b(r), \tag{15}$$

where

$$a(r) = \frac{1-ABr^2}{1-B^2r^2}, \quad b(r) = \frac{|B-A|r}{1-B^2r^2}.$$

Choose $A(z)$ so that

$$A(z)d_qT(z) = d_qN(z) - a(r)d_qT(z) \quad \textit{and} \quad |A(z)| < b(r).$$

For fixed z_0, $z_0 \in E$, let L be the line segment from 0 to $T(z_0)$, which maps one sheet of the starlike image of E by the mapping T, since $T \in \bigcap_{0<q<1} S_q^*(A,B) \subset S^*(A,B) \subset S^*$. Let L^{-1} be the pre-image of L under T and let $r = max|z|$, where $z \in L^{-1}$. Then we have

$$\begin{aligned}
|N(z_0) - a(r)T(z_0)| &= \left|\int_0^{z_0} [d_qN(t) - a(t)d_qT(t)]d_qt\right| \\
&= \left|\int_0^{z_0} [A(t)d_qT(t)]d_qt\right| \\
&\leq \int_0^{z_0} |A(t)d_qT(t)|\, d_qt \\
&< b(r)|T(z_0)|.
\end{aligned}$$

It follows that

$$\frac{N(z)}{T(z)} \in P(A,B), \quad z \in E.$$

This completes the proof. □

Lemma 2.2. *Let $f \in S_q^*(A,B)$. Then*

$$\sigma(z) = \int_0^z t^{c-1} f(t) d_qt, \quad -1 \leq B < A \leq 1, \quad 0 < q < 1, \quad c = 1,2,3..., \tag{16}$$

is multivalent starlike in E.

Proof. Consider

$$d_q\sigma(z) = z^{c-1}f(z),$$

or

$$zd_q\sigma(z) = z^c f(z).$$

q-logarithmic differentiation and simple calculation leads us to

$$\left\{\frac{d_q(zd_q\sigma(z))}{d_q\sigma(z)}\right\} = \left\{[c]_q + \frac{zd_qf(z)}{f(z)}\right\} \in P(A,B), \tag{17}$$

since $f(z) \in S_q^*(A,B)$, $z \in E$. Let $T(z) = zd_q\sigma(z)$, $N(z) = \sigma(z)$ with $N(0) = T(0) = 0$. From (17), we have $T(z) \in S_q^*(A,B)$ for all $q \in (0,1)$, that is $T(z) \in \bigcap\limits_{0<q<1} S_q^*(A,B)$, $z \in E$.

Now an application of Lemma 2.3 and using the similar technique with $P(A,B) \subset P$given in [9], we have

$$\left\{\frac{N(z)}{T(z)}\right\} \in P(A,B), \quad or \quad \left\{\frac{zd_q\sigma(z)}{\sigma(z)}\right\} \in P(A,B).$$

That is, $\sigma(z) \in S_q^*(A,B)$ for all $q \in (0,1)$, which implies that $\sigma(z) \in \bigcap\limits_{0<q<1} S_q^*(A,B) \subset S^*(A,B) \subset S^*$, which shows that σ is $p-$valently starlike in E. □

Lemma 2.3. *For $0 < q < 1$, $c = 1,2,3,..,$ the function $\phi(z)$ defined by*

$$\phi(z) = \sum_{n=1}^{\infty} \frac{[1+c]_q}{[c+n]_q} z^n, \tag{18}$$

belongs to the class C_q in E.

Proof. From ratio test, one can see that radius of convergence for $\sum_{n=1}^{\infty} \frac{[1+c]_q}{[c+n]_q} z^n$ is q, $0 < q < 1$, we have

$$zd_q\phi(z) = \sum_{n=1}^{\infty} \frac{[1+c]_q(1-q^n)}{(1-q^{n+c})} z^n.$$

Now q-logarithmic differentiation and simple computation gives us

$$Re\left\{\frac{d_q(zd_q\phi(z)}{d_q\phi(z)}\right\} = \sum_{n=1}^{\infty}\left(\frac{1-q^n}{1-q}\right) = \sum_{n=1}^{\infty}(1+q+q^2+...q^{n-1}) > 0,$$

for $0 < q < 1$, $z \in E$. This shows that $\phi \in C_q$, which is the required result. □

Theorem 2.1. *If $f \in S_q^*(A,B)$, then the function $F(z)$ defined by*

$$F(z) = \frac{[1+c]_q}{z^c}\int_0^z t^{c-1}f(t)d_qt, \tag{19}$$

belongs to the class $\bigcap\limits_{0<q<1} S_q^*(A,B)$, $(0 \le B < A \le 1,\ z \in E)$.

Proof. Consider

$$\begin{aligned}\frac{zd_qF(z)}{F(z)} &= \frac{zd_qF(z)}{F(z)}\frac{z^c}{z^c}\\ &= \frac{[1+c]_qf(z) - [c]_qF(z)}{F(z)}\\ &= \frac{z^cf(z) - [c]_q\sigma(z)}{\sigma(z)} = \frac{N(z)}{T(z)},\end{aligned} \tag{20}$$

where $N(z) = z^c f(z) - [c]_q\sigma(z)$ and $T(z) = \sigma(z)$, $\sigma(z)$ is given by (16) is p-valently starlike by Lemma 2.4. $d_qN(0)/d_qT(0) = zd_qf(z)/f(z)|_{z=0} = 1$,

$$\frac{d_qN(z)}{d_qT(z)} = \frac{zd_qf(z)}{f(z)} \in P(A,B),$$

since $f(z) \in S_q^*(A,B)$. Thus, Lemma 2.3 leads us

$$\frac{N(z)}{T(z)} \in P(A,B),$$

It follows that $F(z) \in S_q^*(A,B)$ for all $q \in (0,1)$, that is, $F(z) \in \bigcap\limits_{0<q<1} S_q^*(A,B)$ in E. This completes the proof. □

For $c = 1$ in Theorem 2.6, we have the following result, called q-Libera Integral operator, see [12].

Corollary 2.1. If $f \in S_q^*(A,B)$, then the function $F_1(z)$ defined by

$$F_1(z) = \frac{[2]_q}{z}\int_0^z f(t)d_qt, \tag{21}$$

belongs to the class $\bigcap\limits_{0<q<1} S_q^*(A,B)$, $(0 \leq B < A \leq 1,\ z \in E)$.

An application of Theorem 2.6 leads us to the following improved convolution result investigated by Bernardi [4].

Corollary 2.2. Let $f \in \mathcal{A}$, and let $f \in S_q^*(A,B)$, $\phi \in C_q$. Then

$$(f * \phi)(z) \in \bigcap_{0<q<1} S^*(A,B),$$

where ϕ is given by (18).

Proof. We note that $q-$Bernardi operator $L(f) = F$ given by (2.3) can be written in convolution form as:

$$F(z) = \phi(z) * f(z), \tag{22}$$

where f is given by (1.1) and ϕ is defined by (18), ϕ belongs to the class C_q, by Lemma 2.5. Therefore, an application of Theorem 2.6 gives us the desired result. □

Theorem 2.2. *Let $f \in \mathcal{A}$ and let $f \in K_q(A,B)$ be q-close-to-convex functions with respect to g, $g \in \bigcap\limits_{0<q<1} S_q^*(A,B)$, $-1 \leq B < A \leq 1$,*

$$F(z) = \frac{[1+c]_q}{z^c}\int_0^z t^{c-1}f(t)d_qt, \quad G(z) = \frac{[1+c]_q}{z^c}\int_0^z t^{c-1}g(t)d_qt.$$

Then F belongs to the class of q-close-to-convex functions with respect to G in E.

Proof. Let $d_qF(z) = [1+c]_q z^{-1}f(z) - [c]_q z^{-1}F(z)$. Then

$$\begin{aligned}\frac{zd_qF(z)}{G_q(z)} &= \frac{[1+c]_qf(z) - [c]_qF(z)}{G(z)} \\ &= \frac{z^cf(z) - [c]_q\sigma(z)}{\eta(z)} \end{aligned} \tag{23}$$

where σ is given by (16), and

$$\eta(z) = \int_0^z t^{c-1} g(t) d_q t,$$

$\eta(z)$, $G(z) \in \bigcap_{0<q<1} S_q^*(A,B)$, by Lemma 2.4 and Theorem 2.6.

By q-differentiation of nominator and denominator of expression given by (23), we have

$$\frac{d_q(z^c f(z) - [c]_q \sigma(z))}{d_q(\eta(z))} = \frac{z d_q f(z)}{g(z)} \in P(A,B), \tag{24}$$

since $f \in K_q(A,B)$ with respect to g, for $0 < q < 1$ and $z \in E$. Thus, by an application of Lemma 2.3, we have the required result. □

If we take $c = 1$ in Theorem 2.7, then we have the result for $q-$Libera integral operator [12], which improves the result of Libera [9].

Corollary 2.3. Let $f \in \mathcal{A}$ and let $f \in K_q(A,B)$ be $q-$close-to-convex functions with respect to g, $g \in \bigcap_{0<q<1} S^*(A,B)$,

$$F_1(z) = \frac{[2]_q}{z} \int_0^z f(t) d_q t, \quad G_1(z) = \frac{[2]_q}{z} \int_0^z g(t) d_q t.$$

Then F_1 belongs to the class of $q-$close-to-convex functions with respect to G_1 in E.

As an application of Theorem 2.7, we have following:

Corollary 2.4. Let $f \in \mathcal{A}$, and let $f \in K_q(A,B)$, $\phi \in C_q$. Then

$$(f * \phi)(z) \in \bigcap_{0<q<1} K_q(A,B),$$

where ϕ is given by (18) in E.

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