TWMS J. Pure Appl. Math., V.8, N.1, 2017, pp.112-114

SOME REMARKS ON THE PAPER, ENTITLED "FRACTIONAL AND OPERATIONAL CALCULUS WITH GENERALIZED FRACTIONAL DERIVATIVE OPERATORS AND MITTAG-LEFFLER TYPE FUNCTIONS" BY Z. TOMOVSKI, R. HILFER AND H. M. SRIVASTAVA

H.M. SRIVASTAVA[1,2], N.A. ALIEV[3], G.H. MAMMADOVA[3], FIKRET A. ALIEV[3,4]

Abstract. In this note, we would like to bring the readers' attention toward the fact that [9, Section 3, Eq.(3.2)] is not a solution of the fractional differential equation [9, Section 3, Eq.(3.1)].

Keywords: generalized Mittag-Leffer type functions, fractional differential equations, Dirac delta function.

AMS Subject Classification: Primary 26A33, 33C20; Secondary 33E12.

As is well known (see, for example, [1-5], [8], a great deal of attention has recently been paid to the solution of differential equations involving both ordinary and partial derivatives of fractional order. However, we propose to show here that [9, Section 3, Eq.(3.2)] is not a solution of the fractional differential equation [9, Section 3, Eq.(3.1)].

We begin by recalling the following formulas:

$$\left(D_{0+}^{\alpha} f\right)(x) = \lambda f(x) \tag{1}$$

and

$$f(x) = x^{1-\alpha} E_{\alpha,\alpha}(\lambda x^{\alpha}). \tag{2}$$

Here the equations (1) and (2) correspond to Eq. (3.1) and Eq. (3.2) in [9]. The Mittag-Lefler type function occurring in the equation (2) is defined by

$$E_{\mu,\nu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}. \tag{3}$$

Making use of Eq. (2) and the definition (3), we observe that

$$f(x) = x^{1-\alpha} \sum_{n=0}^{\infty} \frac{(\lambda x^{\alpha})^n}{\Gamma(\alpha n + \alpha)} = \sum_{n=0}^{\infty} \frac{\lambda^n x^{\alpha n + 1 - \alpha}}{\Gamma(\alpha n + \alpha)}$$
$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(\alpha n + \alpha)} \cdot \Gamma(\alpha n + 2 - \alpha) \cdot \frac{x^{\alpha n + 1 - \alpha}}{\Gamma(\alpha n + 2 - \alpha)},$$

[1]Department of Mathematics and Statistics, University of Victoria, Victoria, Canada
[2]Department of Medical Research, China Medical University Hospital, China Medical University, Taiwan, Republic of China
[3]Institute of Applied Mathematics, Baku State University, Baku, Republic of Azerbaijan
[4]Institute of Information Technology, Azerbaijan National Academy of Sciences, Baku, Republic of Azerbaijan
e-mail: harimsri@math.uvic.ca, gamar.mamadova@gmail.com, f_aliev@yahoo.com
Manuscript received January 2017.

so that

$$\begin{aligned}\left(D_{0+}^{\alpha} f\right)(x) &= \sum_{n=0}^{\infty} \frac{\lambda^n \, \Gamma(\alpha n + 2 - \alpha)}{\Gamma(\alpha n + \alpha)} \, D_{0+}^{\alpha} \left(\frac{x^{\alpha n + 1 - \alpha}}{\Gamma(\alpha n + 2 - \alpha)} \right) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n \, \Gamma(\alpha n + 2 - \alpha)}{\Gamma(\alpha n + \alpha)} \, \frac{x^{\alpha n + 1 - 2\alpha}}{\Gamma(\alpha n + 2 - 2\alpha)} \neq \lambda f(x) .\end{aligned}$$

Hence, clearly, (2) is *not* a solution of (1).

If we take

$$f(x) = x^{\alpha - 1} \, E_{\alpha,\alpha}(\lambda \, x^{\alpha}), \tag{4}$$

then

$$f(x) = \sum_{n=0}^{\infty} \frac{\lambda^n \, x^{\alpha n + \alpha - 1}}{\Gamma(\alpha n + \alpha)}.$$

In this case, we readily see that

$$\begin{aligned}\left(D_{0+}^{\alpha} \, f\right)(x) &= D_{0+}^{\alpha} \left(\frac{x^{\alpha n + \alpha - 1}}{\Gamma(\alpha n)} \right) \Bigg|_{n=0} + \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(\alpha n + \alpha)} \, D_{0+}^{\alpha} \left(\frac{x^{\alpha n + \alpha - 1}}{\Gamma(\alpha n + \alpha)} \right) \\ &= \frac{x^{-1}}{\Gamma(0)} + \sum_{n=1}^{\infty} \frac{\lambda^n \, x^{\alpha n - 1}}{\Gamma(\alpha n)},\end{aligned} \tag{5}$$

where, obviously, [7]

$$\frac{x^{-1}}{\Gamma(0)} = \frac{x^{-1}}{(-1)!} = 0 \qquad \text{whenever} \qquad x \neq 0. \tag{6}$$

Since the Riemann-Liouville fractional calculus is based upon a definite integral which is taken over a *non-empty* interval $(0, x)$, we can tacitly assume that $x > 0$. Thus, for $x > 0$, we find from (5) and (6) that

$$\begin{aligned}\left(D_{0+}^{\alpha} \, f\right)(x) &= \sum_{n=1}^{\infty} \frac{\lambda^n \, x^{\alpha n - 1}}{\Gamma(\alpha n - 1)} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^{n+1} \, x^{\alpha(n+1) - 1}}{\Gamma\big(\alpha(n+1)\big)} \\ &= \lambda \, x^{\alpha - 1} \, E_{\alpha,\alpha}(\lambda \, x^{\alpha}) = \lambda \, f(x) \qquad (x > 0).\end{aligned}$$

It follows that the *correct* solution of (1) is given by (4) under the *explicitly-stated* condition that $x > 0$.

The above-mentioned error was reproduced in a relatively more recent survey-cum-expository article [8] on the theory and applications of the Mittag-Leffler type functions which are associated with various operators of fractional calculus.

Finally, we turn to the familiar fact that the impulse or distributional (or generalized) function $\delta(x)$, which is popularly known as the *Dirac delta function*, is traditionally defined, for any suitably-constrained continuous function $\varphi(x)$, by (see, for details, [2] and [6]; see also [5] and [7]):

$$\delta(x) = 0 \qquad (x \neq 0) \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\varphi(x)\,\mathrm{d}x = \varphi(0), \tag{7}$$

so that, in particular, we have

$$\delta(x) = 0 \qquad (x \neq 0) \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1. \tag{8}$$

In light of Eq. (6), it is possible to set [7]

$$\delta(x) = \frac{x^{-1}}{\Gamma(0)} = \frac{x^{-1}}{(-1)!} = 0 \qquad \text{whenever} \qquad x \neq 0. \tag{9}$$

However, for obvious reasons, Eq. (9) cannot be construed to define the Dirac delta function $\delta(x)$, simply because it does not satisfy the necessary *second* requirement in the definition (8). Consequently, instead of the Dirac delta function $\delta(x)$, use should be (and has been) made in Eq. (5) of the quotient involved in Eq. (6) and Eq. (9).

References

[1] Ashyralyev, A., (2015), A survey of results in the theory of fractional spaces generated by positive operators, TWMS J. Pure Appl. Math., 6(2), pp.129–157.

[2] Dirac, P.A.M., (1958), The Principles of Quantum Mechanics, Fourth edition, Clarendon (Oxford University) Press, Oxford, London and New York.

[3] Hilfer, R., (2000), Fractional time evolution, in Applications of Fractional Calculus in Physics (R. Hilfer, Editor), pp.87–130, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.

[4] Ibarhim, R.W., (2015), Stability of sequential fractional differential equation, Appl. Comput. Math., 14(2), pp.141–149.

[5] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., (2006), Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York.

[6] Lighthill, M.J., (1958), Introduction to Fourier Analysis and Generalised Functions, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, Cambridge, London and New York.

[7] Samko, S.G., Kilbas, A.A., Marichev, O.I., (1993), Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science Publishers, Yverdon, Switzerland.

[8] Srivastava, H.M., (2016) Some families of Mittag-Leffler type functions and associated operators of fractional calculus (survey), TWMS J. Pure Appl. Math., 7(2), pp.123–145.

[9] Tomovski, Ž., Hilfer, R., Srivastava, H. M., (2010), Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, Integral Transforms Spec. Funct., 21, pp.797–814.