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A PRIORI ESTIMATES FOR SOLUTIONS TO DIRICHLET BOUNDARY VALUE PROBLEMS FOR POLYHARMONIC EQUATIONS IN GENERALIZED MORREY SPACES

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Abstract. A priori estimates are derived for solutions to Dirichlet problem for polyharmonic equations in bounded smooth domains. A problem in generalized Morrey spaces is considered. Based on a priori estimates, the solvability of this problem in generalized Morrey spaces is proved. Similar problem for higher order uniformly elliptic equations is considered. Also, $L_{p,\lambda} - L_{q,\lambda}$ regularity estimates are obtained.

Keywords: generalized Morrey spaces, polyharmonic equations, a priory estimates, uniformly elliptic equations.

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1. Introduction

The classical Morrey spaces $L_{p,\lambda}$ have been originally introduced to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and Holder spaces permits to obtain regularity of the solution to elliptic boundary problems.

For the properties and applications of the classical Morrey spaces we refer the readers to [22, 24].

In [7], Chiarenza and Frasca showed boundness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(R^n)$ that allows to prove continuity of fractional and classical Calderon-Zygmund operators in these spaces and solvability of boundary value problem.

In [21], Mizuhara gave a generalization of these spaces considering a weight function $\omega(x,r) : R^n \times R_+ \to R_+$ instead of r^λ. He studied also a continuity in $L_{p,\omega}$ of some classical integral operators. Later Nakai extended the results of Chiarenza and Frasca in $L_{p,\omega}$ imposing some integral and doubling conditions on ω [23]. Taking a weight $\omega = \varphi^p r^n$, the conditions of Mizuhara-Nakai become

$$\int_r^\infty \varphi^p(x,t)\frac{dt}{t} \le C\varphi^p(x,r), C^{-1} \le \frac{\varphi(x,t)}{\psi(x,t)} \le C, \forall r \le t \le 2,$$

where the constants do not depend on t, r and $x \in R^n$.

Guliyev studied the continuity of sublinear operators in generalized Morrey spaces generated by various integral operators such as the ones of Calderon-Zygmund, Riesz, etc [3, 17, 18]. These results extend the results of Nakai to Morrey-type spaces which are called generalized Morrey spaces. These are new functional spaces, for their applications in the differential equations theory see [19] and the references therein.

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2. Definitions and problem statement

The domain $\Omega \in R^n, n \geq 2$ is supposed to be bounded with $\partial\Omega \in C^{1,1}$.

Definition 2.1. *Let $\varphi : \Omega \times R_+ \to R_+$ be a measurable function and $1 \leq p \leq \infty$. For any domain Ω, the generalized Morrey space $M_{p,\varphi}(\Omega)$ (the weakly generalized Morrey space $WM_{p,\varphi}(\Omega)$) consists of all $f \in L_p^{\text{loc}}(\Omega)$*

$$\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x\in\Omega, 0<r<d} \varphi^{-1}(x,r)\, r^{-n/p}\, \|f\|_{L_p(\Omega(x,r))} < \infty,$$

$$\Big(\|f\|_{WM_{p,\varphi}(\Omega)} = \sup_{x\in\Omega, 0<r<d} \varphi^{-1}(x,r)\, r^{-n/p}\, \|f\|_{WL_p(\Omega(x,r))} < \infty\Big),$$

where $d = \sup_{x,y\in\Omega} |x-y|$, $B(x,r) = \{y \in R^n : |x-y| < r\}$ and $\Omega(x,r) = \Omega \cap B(x,r)$.

In the case of $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$, the generalized Morrey space $M_{p,\varphi}$ (the weakly generalized Morrey space $WM_{p,\varphi}$) is a classical Morrey space $L_{p,\lambda}$ (classical weak Morrey space $WL_{p,\lambda}$).

Definition 2.2. *The generalized Sobolev-Morrey space $W_{2,p,\varphi}(\Omega)$ consists of all Sobolev functions $u \in W_{2,p}(\Omega)$with distributional derivatives $D^s u \in Mp, \varphi(\Omega)$, endowed with the norm*

$$\|u\|_{W_{2,p,\varphi}(\Omega)} = \sum_{0\leq|s|\leq 2} \|D^s f\|_{M_{p,\varphi}(\Omega)}$$

The space $W_{2,p,\varphi}(\Omega) \bigcap \overset{0}{\underset{p,\varphi}{W}}(\Omega)$ consists of all functions $u \in W_{2,p}(\Omega) \bigcap \overset{0}{W}_{1,p}(\Omega)$ with $D^s u \in M_{p,\varphi}(\Omega)$, and is endowed with the same norm. Recall that $\overset{0}{W}_{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W_{1,p}(\Omega)$.

Before the Dirichlet boundary value problem for polyharmonic equation, we consider

$$(-\Delta)^m u = f \,\text{in}\, \Omega \tag{1}$$

$$u = \frac{\partial u}{\partial \vartheta} = ... = \frac{\partial^{m-1} u}{\partial \vartheta} = g \text{ on } \partial\Omega,$$

where $\Omega \subset R^n, n \geq 2$ is a bounded domain with sufficiently smooth boundary.

Note. For simplicity, we will denote $L = (-\Delta)^m$ in (1),but in fact the estimates that we will derive hold for any uniformly elliptic operator L of order $2m$.

Now we give estimates for the Green function and the Poisson kernels for the solutions of the problem (1). Later we will obtain a priori estimates for the solution and the solvability of problem (1) in generalized Morrey spaces.

Let $G_m(x,y)$ be the Green function and $K_j(x,y), j = \overline{o, m-1}$ be the Poisson kernels of problem (1). Then the solution of problem (1) can be written as

$$u(x) = \int_\Omega G_m(x,y) f(y) dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(x,y) g(y) d\sigma_y$$

for corresponding f and g.
For example, when $m = 2$ and $n = 2$ it is known that there is a constant $C(\Omega)$ such that

$$|G_2(x,y)| \leq C(\Omega) d(x) d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}, \tag{2}$$

where d is the distance between x and the boundary $\partial\Omega$:

$$d(x) = \inf_{\tilde{\tilde{x}} \in \partial\Omega} |x - \tilde{x}| . \tag{3}$$

However,we would like to mention that for G_m and K_j the estimates are optimal for deriving regularity results in spaces that involve to behavior at the boundary. Coming back to the $m = n = 2$ it follows from (2) that the solution of problem (1) satisfies for appropriate f at $g = 0$

$$\left\| ud^2 \right\|_{L_\infty(\Omega)} \le C(\Omega) \left\| f \right\|_{L_1(\Omega)} ,$$

$$\left\| u \right\|_{L_\infty(\Omega)} \le C(\Omega) \left\| fd^2 \right\|_{L_1(\Omega)} .$$

For general m and n , $L_{p,\lambda}$- $L_{q,\lambda}$ estimates will be also obtained.

A special case for $m = 1$ has been considered in [13].

We will derive estimates also for the kernels of derivatives. We also will focus on the estimates that contain growth rates near the boundary. These estimates will be optimal. Indeed, when we consider $G_m(x, y)$ for the balls $\Omega = B\,(x,\, r) \subset R^n$, the growth rates near the boundary are sharp [14]. For $m = 1$ or $m \ge 2$ and $\Omega = B(x, r)$ it is known that the Green function is positive and can even be estimated from below by a positive function with the same singular behavior [15]. Let us remind that for $m \ge 2$ the Green function in general is not positive. For general domains the optimal behavior in absolute values is reflected in our estimates. Sharp estimates for K_{m-1} and K_{m-2} in case of a ball can be found in [16]. By integrating pointwise the estimates for parabolic kernel $p(t, x, y)$ with respect to t from 0 to ∞, pointwise estimates for the Green function have been obtained by Barbatis in [4] who considered higher order parabolic problems on domains and derived pointwise estimates for kernels. Classical estimates by Eidelman [12] for higher order parabolic systems do not cover domains with boundary. For a survey of spectral theory of higher order elliptic operators, including some estimates for the corresponding kernels, we refer the readers to [10].

The paper is organized as follows. We give in Section 1 some information about previous results. In Section 2 we give some definitions. In Section 3 we obtain the estimates for the Green function and the Poisson kernels. In Section 4 we will show applications of our results to regularity estimates in weighted spaces, solvability in generalized Morrey spaces. In Section 5 we prove the solvability of uniformly elliptic boundary problem in a general Morrey space.

Proposition 2.1. *([15]) Let f and g be functions on $\Omega \times \Omega$ with $g \ge 0$. Then we denote $f \sim g$ on $\Omega \times \Omega$ if and only if there are $C_1, C_2 > 0$ such that*

$$C_1 f(x, y) \le g(x, y) \le C_2 f(x, y) \text{ for all } x, y \in \Omega.$$

Let f be a function on $\Omega \times \Omega$ and $\alpha, \beta \in N^n$. Derivatives of f are denoted as

$$D_x^\alpha D_y^\beta f(x, y) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdot \ldots \cdot \partial x_n^{\alpha_n}} \cdot \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1} \partial y_2^{\beta_2} \cdot \ldots \cdot \partial y_n^{\beta_n}} f(x, y),$$

where $|\alpha| = \sum_{k=1}^n \alpha_k, |\beta| = \sum_{k=1}^n \beta_k$. Now we give some auxillary results.

Theorem 2.1. *([9],[24]). Let $G_m(x, y)$ be the Green function of the problem (1). Then for every $x, y \in \Omega$ the following estimates hold:*

1. if $2m - n > 0$, then

$$|G_m(x,y)| \leq d^{m-\frac{1}{2}n}(x) \cdot d^{m-\frac{1}{2}n}(y) \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}n},$$

if $2m-n=0$*, then*

$$|G_m(x,y)| \leq \log\left(1+\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)^m\right),$$

3. if $2m-n<0$*, then*

$$|G_m(x,y)| \leq |x-y|^{2m-n} \min\left(1, \left(1, \frac{d(x)d(y)}{|x-y|^2}\right)^m\right).$$

Theorem 2.2. *([9],[20]) Let* $K_j(x,y)$*,* $j=\overline{0,m-1}$ *be the Poisson kernels of the problem (1). Then for every* $x \in \Omega, y \in \partial\Omega$ *the following estimates holds:*

$$|K_j(x,y)| \leq \frac{d^m(x)}{|x-y|^{n-j+m-1}}. \tag{4}$$

Remark 2.1. If $n-1<j\leq m-1$, then from (4) we get on $\Omega \times \partial\Omega$

$$|K_j(x,y)| \leq d^{1+j-n}(x). \tag{5}$$

Remark 2.2. The estimates in Theorems 2.2 hold for any uniformly elliptic operator of order $2m$.

In [15] the estimates as in Theorem 2.1 are given for the case $\Omega = B(x,r)$ in R^n. The authors in [15] used the explicit formula for the Green function given in [5].

For general domains one cannot expect an explicit formula and instead we will proceed by the estimates for $G_m(x,y)$ and $K_j(x,y)$ given in [9], [20]. For sufficiently regular domains Ω, the authors in [9] and [20] proved that the Green function and Poisson kernels exist and obtained estimates for these functions.

We will prove estimates for $G_m^{(x,y)}$ and $K_j(x,y)$ depending on the distance to the boundary.We will do so by estimating the j-th derivative through an integration of the $(j+1)$-th derivative along a path to the boundary. Distance to the boundary $d(x)$ will depend on the proportionality between the arch which joins internal point with the boundary. The arch will be constructed explicitly in Lemma 2.1.

In the following lemma we state the existence of such an arch.

Lemma 2.1. *Let* $x \in \Omega$ *and* $y \in \overline{\Omega}$*. There exists a curve* $\gamma_x^y : [0,1] \to \Omega$ *with* $\gamma_x^y(0) = x$*,* $\gamma_x^y(1) \in \partial\Omega$ *such that*

$$1.\ |\gamma_x^y(t) - y| \geq \frac{1}{2}|x-y|,\ \textit{for every } t \in [0,1], \tag{6}$$

$$2.\ l \leq (1+\pi)d(x),\ \textit{where } l \textit{ is the length of } \gamma_x^y. \tag{7}$$

Moreover, let $\widetilde{\gamma}_x^y : [0,l] \to \overline{\Omega}$ *be the parametrization by arch length of* γ_x^y*. The following estimate holds:*

$$3.\ \frac{1}{5}S \leq |x - \widetilde{\gamma}_x^y(s)| \leq S \textit{ for } s \in [0,l]. \tag{8}$$

We proceed with the proof of Theorem 2.1 and start from the estimates in [20] of the m-th derivative of $G_m(x,y)$.

Integrating this function along the arch γ_x^y of Lemma 2.1, we find the estimates of the $(m-1)$-th derivative of $G_m(x,y)$ in terms of the distance to the boundary. Iterating the procedure m times we find the results as stated in Theorem 2.1.

Later we use some auxiliary results which can be easily obtained from [20]. By these results we prove the following theorem.

Theorem 2.3. *([9],[20]) Let $G_m(x,y)$ be the Green function for the problem (1), $k \in N^n$. Then for every $x,y \in \Omega$ the following estimates hold:*

1. For $|k| \geq m$: if $2m-n-|k|<0$, then

$$\left|D_x^k G_m(x,y)\right| \leq C\,|x-y|^{2m-n-|k|}\min\left(1,\frac{d(y)}{|x-y|}\right)^m;$$

if $2m-n-|k|=0$, then

$$\left|D_x^k G_m(x,y)\right| \leq C\log\left(1+\frac{d^m(y)}{|x-y|^m}\right)\sim$$

$$\sim\log\left(2+\frac{d(y)}{|x-y|}\right)\min\left(1,\frac{d(y)}{|x-y|}\right)^m; \tag{9}$$

if $2m-n-|k|>0$, then

$$\left|D_x^k G_m(x,y)\right| \leq Cd^{2m-n-|k|}(y)\min\left(1,\frac{d(y)}{|x-y|}\right)^{n+|k|-m}.$$

2. For $|k|<m$: if $2m-n-|k|<0$, then

$$\left|D_x^k G_m(x,y)\right| \leq C\,|x-y|^{2m-n-|k|}\min\left(1,\frac{d(x)}{|x-y|}\right)^{m-|k|}\min\left(1,\frac{d(y)}{|x-y|}\right)^m.$$

if $2m-n-|k|=0$, then

$$\left|D_x^k G_m(x,y)\right| \leq C\log\left(1+\frac{d^m(y)d^{m-|k|}(x)}{|x-y|^{2m-|k|}}\right)\sim\log\left(2+\frac{d(y)}{|x-y|}\right)\times$$

$$\times\min\left(1,\frac{d(y)}{|x-y|}\right)^m\min\left(1,\frac{d(x)}{|x-y|}\right)^{m-|k|}; \tag{10}$$

If $2m-n-|k|>0$, and moreover

a) $m-\frac{n}{2}\leq|k|$, then

$$\left|D_x^k G_m(x,y)\right| \leq Cd^{2m-n-|k|}(y)\min\left(1,\frac{d(x)}{|x-y|}\right)^{m-|k|}\min\left(1,\frac{d(y)}{|x-y|}\right)^{n-m+|k|}$$

b) $|k|<m-\frac{n}{2}$, then

$$|D_x^k G_m(x,y)| \leq C\cdot d(y)^{m-\frac{n}{2}}d^{2m-\frac{n}{2}-|k|}(x)\min\left(1,\frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{n}{2}}.$$

Proof. Let $x, y \in \Omega$. We use the estimates for derivatives of $G_m(x,y)$ from [9]. The estimates for the lower order derivatives of $G_m(x,y)$ will be obtained by integrating the higher order derivatives along the arch γ_x^y by means of Lemma 2.1. For example, with $\alpha, \beta \in N^n$ and $\widetilde{x} \in \partial\Omega$, for the end point of γ_x^y, we find

$$D_x^\alpha D_y^\beta G_m(x,y) = D_x^\alpha D_y^\beta G_m(\widetilde{x},y) + \int\limits_{\gamma_x^y} \nabla_z D_z^\alpha D_y^\beta G_m(z,y)dz. \tag{11}$$

If $|\alpha| \leq m-1$, then the first term on the right-hand side of (11) equals to zero and we get

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \leq \int\limits_0^l \left|\nabla_x D_x^\alpha D_y^\beta G_m(\overline{\gamma}_x^y(s), y)\right| ds. \tag{12}$$

If $|\beta| \leq m-1$, then similarly by integrating with respect to y we find

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \leq \int\limits_0^l \left|\nabla_y D_y^\beta D_x^\alpha G_m(x, \overline{\gamma}_y^x(s)\right| ds \tag{13}$$

We take $H(x,y) = D_x^\alpha D_y^\beta G_m(x,y)$ and depending on $|k| = r$ we choose α and β.

We distinguish the cases as in the statement of the theorem. For example,

Case 1. $r \geq m$. Let $\beta \in N^n$ with $|\beta| = m-1$. Then from (13) with $k = \alpha$ and using the estimates from [20], we get

$$|D_x^\alpha D_y^\beta G_m(x,y)| \leq |x-y|^{m-n-r}.$$

Case 2. $r < m$. Also, using the estimates from [20] for $\left|D_y^\beta D_x^\alpha D_x^k G_m(x,y)\right|$ and then integrating m times with respect to y and $m-r$ times with respect to x, we get the desired result.

Theorem is proved. □

Remark 2.3. Theorem 2.1 holds in case where there is no symmetry between x and y.

Lemma 2.2. *Let $v_1,\ k \in N$ with $k \geq 2$. If*

$$|\nabla_x H(x,y)| \leq C_1 \left|x-y\right|^{-k} d^{v_1}(x)$$

for $x \in \Omega,\ y \in \partial\Omega$ and $H(\widetilde{x}, y) = 0$ for every $\widetilde{x} \in \partial\Omega$ with $\widetilde{x} \neq y$, then the following inequality holds:

$$|H(x,y)| \leq C_1 \left|x-y\right|^{-k} d^{v_1+1}(x)$$

for $x \in \Omega,\ y \in \partial\Omega$.

Using auxiliary results stated above, we can easily prove this lemma.

The lemma above allows us to prove the following theorem which is a special case of Theorem 2.2.

Theorem 2.4. *([9],[20]) Let $K_j(x,y),\ j = \overline{0, m-1}$ be the Poisson kernels of the problem (1). Let $\alpha \in N^n$ with $|\alpha| \leq m-1$. The following estimate holds for $x \in \Omega,\ y \in \partial\Omega$:*

$$|D_x^\alpha K_j(x,y)| \leq C_2 \frac{d^{m-|\alpha|}(x)}{|x-y|^{n-j+m-1}}.$$

Remark 2.4. The estimates of $D_x^\alpha K_j(x,y)$ for $|\alpha| \geq m$ can be found in [9]. The following estimate holds:

$$|D_x^\alpha K_j(x,y)| \leq C_3 \, |x-y|^{-n+j-|\alpha|+1} .$$

3. Estimates for the solution

Now we will derive regularity estimates for solution of problem (1) when $g = 0$:

$$\begin{cases} (-\Delta)^m u = f \quad \text{in } \Omega \,, \\ \left(\frac{\partial}{\partial n}\right)^k u = 0 \quad \text{on } \Omega \,, \end{cases} \tag{15}$$

where $0 \leq k \leq m-1$, $\Omega \subset R^n$ is bounded. First we recall an estimate involving the Riesz potential. Define $K_j(x) = |x|^{-\gamma}$ and

$$(K_\gamma * f)(x) = \int\limits_\Omega |x-y|^{-\gamma} f(y)dy.$$

The following lemma holds.

Lemma 3.1. *Let $\Omega \subset R^n$ be bounded, $\gamma < n$ and $1 < p < \dfrac{n-\lambda}{\gamma}$. Then condition $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{\gamma}{n-\lambda}$ is necessary and sufficient for the boundedness of $K_\gamma * f$ from $L_{p,\lambda}(\Omega)$ to $L_{q,\lambda}(\Omega)$ and there is $C_4 > 0$ such that for all $f \in L_p$*

$$\|K_\gamma * f\|_{L_{q,\lambda}(\Omega)} \leq C_4 \, \|f\|_{L_{p,\lambda}(\Omega)} \, . \tag{16}$$

As a consequence of the pointwize estimates and using Lemma 3.1, we next state the optimal $L_{p,\lambda} \to L_{q,\lambda}$-regularity results mentioned before.

Let us recall that $d(\cdot)$ is the distance function defined in (3).

Proposition 3.1. *Let $u \in C^{2m}(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ satisfy the conditions of the problem (15).*

1. If $2m > n$, then there exists $k_1 > 0$ such that for all $\theta \in [0,1]$

$$\left\| d(\cdot)^{-m+\theta n} u \right\|_{L_\infty(\Omega)} \leq K_1 \left\| d(\cdot)^{m-(1-\theta)n} f \right\|_{L_1(\Omega)} . \tag{17}$$

2. Let $1 < p < \dfrac{n-\lambda}{\gamma}$. If $\left(\dfrac{1}{p} - \dfrac{1}{q}\right) = \dfrac{\gamma}{n-\lambda} < \min\left\{\dfrac{2m}{n}, 1\right\}$,

$$\alpha \in \left\{ \left(\frac{1}{p} - \frac{1}{q}\right) = \frac{\gamma}{n-\lambda}, \; \min\left\{1, \frac{2m}{n}\right\}\right\}.$$

Then there exists $k_2 > 0$ such that for all $\theta \in [0,1]$

$$\left\| d(\cdot)^{-m+\theta \cdot n \cdot \alpha} u \right\|_{L_{q,\lambda}} \leq K_2 \left\| d(\cdot)^{m-(1-\theta)n\cdot\alpha} f \right\|_{L_{p,\lambda}} . \tag{18}$$

Remark 3.1. Note that the shift in the exponent of $d(\cdot)$ between the right and left hand side of (18) is $2m - n\alpha$, $u \in C^{2m}(\overline{\Omega})$. Hence the shift increases when α tends to $\dfrac{1}{p} - \dfrac{1}{q}$.

Remark 3.2. The conditions $u \in C^{2m}(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ may be considerably weakened for each of the estimates by using a density argument.

Remark 3.3. The estimate in (17) is sharp and does not seem to follow through embedding results. Such estimates will also from the regularity results in case of the problem for L.

Remark 3.4. In a similar way one may also derive estimates for boundary behavior of derivatives. For example, if $n = m = 2$, one finds with $\theta \in [0, 1]$

$$\left\| d^{-1+2\theta}(\cdot) D_x u \right\|_{L_\infty(\Omega)} \leq K_3 \left\| d^{-2\theta}(\cdot) f \right\|_{L_1(\Omega)}.$$

4. Estimates for solutions in generalized Morrey spaces

First, we recall a definition of generalized Sobolev-Morrey space.

Definition 4.1. *The generalized Sobolev-Morrey space $W^{2m}_{p,\varphi}(\Omega)$ consists of all Sobolev functions $u \in W^{2m}_p(\Omega)$ with distributional derivatives $D^s_u \in M_{p,\varphi}(\Omega)$, endowed with the norm*

$$\|u\|_{W^{2m}_{p,\varphi}(\Omega)} = \sum_{0 \leq |s| \leq 2m} \|D^s u\|_{M_{p,\varphi}(\Omega)}.$$

The space $W^{2m}_{p,\varphi}(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$ consists of all functions $u \in W^{2m}_p(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$ with $D^s_u \in M_{p,\varphi}(\Omega)$ endowed with the same norm. Recall that $\overset{\circ}{W}{}^1_p(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm in $W^1_p(\Omega)$.

Now we get the estimates for the solution of problem (1) in generalized Morrey spaces with $g = 0$:

$$\|u\|_{W^{2m}_{p,\varphi_1}(\Omega)} \leq C \|f\|_{L_{p,\varphi_2(\Omega)}}.$$

Note that

$$Kf(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \sum_{|\alpha|=2m} D^\alpha_{x_i} G_m(x-y) f(y) dy$$

is a singular Calderon-Zygmund operator. Here and later we assume that the function f is defined in R^n, also this function is continuously extended to the exterior of domain Ω by zero. The function $D^m_{x_i} G_m(x, y) \in C^\infty(R^n \setminus \{0\})$ and the function f is homogeneous of order $m - n$. Hence $D^{2m}_{x_i} G_m(x, y)$ is homogeneous of order $2m - n$ and tends to zero on unit sphere (see [11]). Then from general theory given in [12] it follows that K is a bounded operator on $L_p(R^n)$ for $1 < p < \infty$. Moreover, maximal singular operator

$$\widetilde{K} f(x) = \sup_{\varepsilon > 0} \left| \int\limits_{|x-y|>\varepsilon} \sum_{|\alpha|=2m} D^\alpha G_m(x, y) f(y) dy \right|$$

is also bounded on $L_p(R^n)$ for $1 < p < \infty$.

Theorem 4.1. *Let $\Omega \subset R^n$ be a bounded domain with $\partial\Omega \subset C^2$. Let $1 < p < \infty$ and the pair of functions (φ_1, φ_2) satisfy the condition*

$$\int\limits_r^d \frac{\underset{t<s<\infty}{ess \ \inf} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C_5 \varphi_2(x, r), \qquad (19)$$

where $d = \sup\limits_{x,y\in\Omega} |x-y|$ *and constant* C_5 *is independent of* $x \in \Omega$ *and* $r > 0$. *We assume that* $f \in M_{p,\varphi}$ *and function* u *is a solution of problem (15). There exists a constant* C_6 *which depends only on* n, φ *and* Ω *such that*

$$\|u\|_{W^{2m}_{p,\varphi_2}(\Omega)} \le C_6 \|f\|_{M_{p,\varphi_1}(\Omega)}. \tag{20}$$

Proof. The proof follows from above estimates for the Green function and the inequalities in [11]. There exists constant C_7 which depends only on Ω such that for any $x \in \Omega$ the following inequalities hold:

$$|u(x)| + |D_{x_i}u(x)| \le C_7 Mf(x), \tag{21}$$

$$\left|D_{x_ix_j}u(x)\right| \le C_7(Kf(x) + Mf(x) + |f(x)|). \tag{22}$$

Similarly we can prove the following estimates:

$$|u(x)| + \left|\sum_{|\alpha|\le m} D^{\alpha}_{x_i}u(x)\right| \le C_8 Mf(x), \tag{23}$$

$$\left|\sum_{|\alpha|\le 2m} D^{\alpha}u(x)\right| \le C_8(\widetilde{K}f(x) + Mf(x) + |f(x)|). \tag{24}$$

Now we need the following auxillary results proved in [3], [17], [18]. □

Lemma 4.1. *Let* $1 \le p < \infty$ *and let there exist a constant* $C_9 > 0$ *such that for any* $x \in R^n$ *and any* $t > 0$

$$\int\limits_r^\infty \frac{\underset{t<s<\infty}{ess} \inf \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dr \le C_9 \varphi_2(x,r). \tag{25}$$

Let also T *be a sublinear bounded operator in* $L_p(R^n)$ *for* $p \in (1,\infty)$ *which satisfies*

$$|Tf(x)| \le C_{10} \int\limits_{R^n} \frac{f(y)}{|x-y|^n} dy, \quad x \notin \operatorname{sup} pf. \tag{26}$$

Then for $p > 1$ *the operator* T *is bounded from* $M_{p,\varphi_1}(R^n)$ *to* $M_{p,\varphi_2}(R^n)$. *Moreover, for* $p > 1$

$$\|Tf\|_{M_{p,\varphi_2}(R^n)} \le \|f\|_{M_{p,\varphi_1}(R^n)}.$$

Lemma 4.2. *Let* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfy the condition (25). Then the operators* M *and* K *are bounded from* $M_{p,\varphi_1}(R^n)$ *to* $M_{p,\varphi_2}(R^n)$ *for* $p > 1$.

Now we prove Theorem 4.1. From Lemmas 4.1 and 4.2 it follows that the operators M and $\widetilde{K}$ are bounded in $M_{p,\varphi}(R^n)$. Therefore, the statement of Theorem 4.1 and estimate (20) are the immediate consequences of the inequalities (23), (24).

5. Estimates for the solutions of any higher order uniformly elliptic equation with smooth coefficients in generalized Morrey spaces

1. Consider the boundary value problem

$$\begin{aligned} Lu = f \quad &\text{in } \ \Omega\,, \\ B_j u = \psi_j \quad &\text{on } \ \partial\Omega, \end{aligned} \tag{27}$$

for $j = 0, ..., m-1$. The following assumptions hold. The operator

$$Lu = \sum_{|\alpha|\le 2m} a_{\alpha,j}(x) D^\alpha u$$

is uniformly elliptic: there exists a constant $\gamma > 0$ such that

$$\gamma^{-1} |\xi|^2 \le \sum_{\alpha,j} a_{\alpha,j}(x)\xi_\alpha \xi_j \le \gamma |\xi|^2\,, \text{ a.e. } x \in \Omega,\ \forall \xi \in R^n\,,$$

$$a_{\alpha,j}(x) = a_{j,\alpha}(x).$$

2. The boundary operators

$$B_j = \sum_{|\beta|\le m_j} b_{j\beta} D^\beta, \quad for\ j = 0,\ m-1$$

satisfy the complementing condition relative to L (see the complementing condition on page 663 of [2].

3. Let $l_1 > \max_j (2m - m_j)$ and $l_0 = \max_j (2m - m_j)$. The coefficients $a_{\alpha j}$ belong to $C^{l_1+1}(\overline{\Omega})$ and the coefficients $b_{j\beta}$ belong to $C^{l_1+1}(\partial\Omega)$.

4. The boundary $\partial\Omega$ belongs to C^{l_1+2m+1}.

5. $f \in M_{p,\varphi}(\Omega)$ with $1 < p < \infty$ and $\varphi : \Omega \times R_+ \to R_+$ is measurable.

Theorem 5.1. *Let us consider the boundary value problem (27) with the conditions (1)-(5) and also the conditions of Theorem 4.1. Then there exists a constant C_7 which depends only on n, φ and Ω such that*

$$\|u\|_{W^{2m}_{p,\varphi_2(\Omega)}} \le C_7 \|f\|_{M_{p,\varphi_1(\Omega)}}\,. \tag{28}$$

The proof of Theorem 5.1 is similar to the one of Theorem 4.1.

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