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ON eBE-ALGEBRAS

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Abstract. In this paper, we introduce a new algebra, called an eBE-algebra, which is a generalization of a BE-algebra and discuss its basic properties. Also, the notion of filters in this structure is studied. We show that every filter can state as a union of extension of upper sets.

Keywords: eBE-algebra, self distributive, filter.

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1. Introduction and preliminaries

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2]). They have shown that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. It is known that several generalizations of a BCI/BCK-algebra were extensively investigated by many researchers and properties have been considered systematically. H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra ([3]). They defined BE-algebra as an algebra $(X; *, 1)$ of type $(2, 0)$ (i.e. a non-empty set with a binary operation "$*$" and a constant 1) satisfying the following axioms:

(BE1) $x * x = 1$,
(BE2) $x * 1 = 1$,
(BE3) $1 * x = x$,
(BE4) $x * (y * z) = y * (x * z)$,
for all $x, y, z \in X$.

A. Walendziak investigated the relationship between BE-algebras, implication algebras, and J-algebras ([7]). A. Rezaei et al. got some results on BE-algebras and introduced the notion of commutative ideals in BE-algebras and proved several characterizations of such ideals ([4, 5, 6]). For development of many-valued logical system, it is needed to make clear the corresponding algebraic structures. It is motivated us to focus on a new algebraic structure, namely eBE-algebra, as a generalization of BE-algebra and so investigate some properties. Also, we discuss on filters of eBE-algebras.

2. A New extension of BE-algebras

Definition 2.1. *Let X be a non-empty set. By an eBE-algebra we shall mean an algebra $(X; *, A)$ such that "$*$" is a binary operation on X and A is a non-empty subset of X satisfying the following axioms:*

- *(eBE1)* $x * x \in A$,

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- *(eBE2)* $x * A \subseteq A$,
- *(eBE3)* $A * x = \{x\}$,
- *(eBE4)* $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$.

In Text, $A * x = \{a * x : a \in A\}$ *and similarly* $x * A = \{x * a : a \in A\}$.

We note that if $A = X$, then $(X; *, X)$ is an eBE-algebra.

Let $a, b \in A$. By (eBE3), we have $a * b = b \in A$ and $b * a = a \in A$. Hence A is a closed subset of X.

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y \in A$. By (eBE1) the relation "$\leq$" is reflexive.

Theorem 2.1. *Every BE-algebra is an eBE-algebra.*

Proof. Put $A := \{1\}$, we can see that $(X; *, A)$ is an eBE-algebra. □

In the next example we show that every eBE-algebra is not a BE-algebra in general.

Example 2.1. Let $X = \{a, b, c\}$ be a set and $A = \{a, b\}$ with the following table.

$*_1$	a	b	c
a	a	b	c
b	a	b	c
c	b	b	a

Then $(X; *_1, A)$ is an eBE-algebra. Since $b *_1 b = b$ and $c *_1 c = a$, there is not an element $1 \in X$, such that $x *_1 x = 1$, for all $x \in X$. Hence $(X; *_1, A)$ is not a BE-algebra.

In the following example we show that axioms "(eBE1)" to "(eBE4)" are independence.

Example 2.2. Let $X = \{a, b, c\}$ be a set and $A = \{a, b\}$ with the following tables:

(i).

$*_2$	a	b	c
a	a	b	c
b	a	b	c
c	a	a	c

Then $(X; *_2, A)$ satisfies axioms (eBE2), (eBE3) and (eBE4). Since $c *_2 c = c \notin A$, $(X; *_2, A)$ does not satisfy the axiom (eBE1).

(ii).

$*_3$	a	b	c
a	a	b	c
b	a	b	c
c	c	c	b

Then $(X; *_3, A)$ satisfies axioms (eBE1), (eBE3) and (eBE4). Since $c *_3 A = c *_3 \{a, b\} = \{c\} \nsubseteq A$, $(X; *_3, A)$ does not satisfy the axiom (eBE2).

(iii).

$*_4$	a	b	c
a	b	b	b
b	c	b	c
c	b	b	b

Then $(X; *_4, A)$ satisfies axioms $(eBE1)$, $(eBE2)$ and $(eBE4)$. Since $A *_4 a = \{a, b\} *_4 a = \{b, c\} \neq \{a\}$, $(X; *_4, A)$ does not satisfy the axiom $(eBE3)$.

(iv).

$*_5$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	b	b	b	c
d	b	b	c	b

Then $(X;*_5,A)$ satisfies axioms (eBE1), (eBE2) and (eBE3). Since

$$c *_5 (d *_5 d) = c *_5 b = b \neq c = d *_5 c = d *_5 (c *_5 d),$$

$(X;*_5,A)$ does not satisfy the axiom (eBE4).

Theorem 2.2. *Let $(X;*,A)$ be an eBE-algebra. If A is a singleton set, then $(X;*,A)$ is a BE-algebra.*

Proof. Let $A=\{a\}$ be a singleton set. If we put $1:=a$, then $(X;*,1)$ is a BE-algebra. □

Theorem 2.3. *Let $(X;*,A_1)$ and $(X;*,A_2)$ be two eBE-algebras. Then $(X,*,A_1\cap A_2)$ is, too.*

Proof. Let $x\in X$. Since $x*x\in A_1$ and $x*x\in A_2$, we have $x*x\in A_1\cap A_2$ and so (eBE1) is valid.

For (eBE2), let $a\in x*(A_1\cap A_2)$. Hence there exists $b\in A_1\cap A_2$ such that $a=x*b$. Since $b\in A_1$, $x*b\in A_1$ and $b\in A_2$, $x*b\in A_2$, we have $a=x*b\in A_1\cap A_2$ and so $x*(A_1\cap A_2)\subseteq A_1\cap A_2$.

Let $a\in (A_1\cap A_2)*x$. Then there exists $b\in A_1\cap A_2$ such that $a=b*x$. Since $b*x=x$, we can see that $a=x$ and so $(A_1\cap A_2)*x=\{x\}$. Therefore (eBE3) is valid.

Also, it is obvious that (eBE4) is valid. □

Corollary 2.1. *If $(X;*,A_i)$, for $i\in\Lambda$, be a family of eBE-algebras, then $(X;*,\bigcap_{i\in\Lambda} A_i)$ is, too.*

Theorem 2.4. *Let $(X;*,A_1)$ and $(X;*,A_2)$ be two eBE-algebras. Then $(X,*,A_1\cup A_2)$ is, too.*

Proof. Let $x\in X$. Since $x*x\in A_1$ and $x*x\in A_2$, we have $x*x\in A_1\cup A_2$ and so (eBE1) is valid.

For (eBE2), let $a\in x*(A_1\cup A_2)$. Hence there exists $b\in A_1\cup A_2$ such that $a=x*b$. If $b\in A_1$, then $a\in A_1$. Also, if $b\in A_2$, then $a\in A_2$. Thus $a\in A_1\cup A_2$ and so $x*(A_1\cup A_2)\subseteq A_1\cup A_2$.

Let $a\in (A_1\cup A_2)*x$. Then there exists $b\in A_1\cup A_2$ such that $a=b*x$. Since $b*x=x$, we can see that $a=x$ and so $(A_1\cup A_2)*x=\{x\}$. Therefore (eBE3) is valid.

Also, it is obvious that (eBE4) is valid. □

Corollary 2.2. *If $(X;*,A_i)$, for $i\in\Lambda$, is a family of eBE-algebras, then $(X;*,\bigcup_{i\in\Lambda} A_i)$ is, too.*

Proposition 2.1. *Let $(X;*,A)$ be an eBE-algebra. Then*

(i) *$(X;*,X\setminus A)$ is not an eBE-algebra,*
(ii) *$x*(y*x)\in A$,*
(iii) *$x\le y*z$ implies $y\le x*z$,*
(iv) *$x\le (x*y)*x$,*
(v) *$y*z\in A$ implies $x*(y*z)\in A$ and $y*(x*z)\in A$,*
(vi) *$x*(y*z)\notin A$ implies $x*z\notin A$,*

for all $x,y,z\in X$.

Proof. (i). Since $x * A \subseteq A$, we have $x * A \nsubseteq X \setminus A$ and so (eBE2) is not valid.

(ii). Using (eBE4) and (eBE2), we have

$$x * (y * x) = y * (x * x) \in y * A \subseteq A.$$

(iii). Let $x \leq y * z$. Hence $x * (y * z) \in A$. Then by using (eBE4), we have

$$y * (x * z) = x * (y * z) \in A.$$

Therefore $y \leq x * z$.

(iv). From (eBE4), (eBE1) and (eBE3) we have

$$x * ((x * y) * x) = (x * y) * (x * x) \in (x * y) * A \subseteq A.$$

Therefore $x \leq (x * y) * x$.

(v). Let $y * z \in A$. Hence $x * (y * z) \in x * A \subseteq A$. Now, using (eBE4) we have $y * (x * z) \in A$.

(vi). The proof is obvious by (v).

□

Theorem 2.5. *Let $(X; *, A)$ be an eBE-algebra. Consider $Y := (X \setminus A) \cup \{1\}$ and define the operation $\diamond$ on Y as follows:*

$$x \diamond y = \begin{cases} x * y & \text{if } x, y \neq 1 \text{ and } x * y \notin A \\ 1 & \text{if } x, y \neq 1 \text{ and } x * y \in A \\ y & \text{if } x = 1 \\ 1 & \text{if } y = 1 \end{cases}$$

Then $(Y; \diamond, 1)$ is a BE-algebra.

Proof. By (eBE1), $x * x \in A$, for all $x \in X$. Thus $x \diamond x = 1$, for all $x \in Y$ and so (BE1) holds.

By definition of "$\diamond$", (BE2) and (BE3) are hold. To prove $(Y; \diamond, 1)$ is a BE-algebra it is sufficient to prove that $x \diamond (y \diamond z) = y \diamond (x \diamond z)$, for all $x, y, z \in Y$. If $x = 1$ or $y = 1$ or $z = 1$, then we have $x \diamond (y \diamond z) = y \diamond (x \diamond z)$. Now, let $x, y, z \neq 1$.

If $y * z \in A$, then $y \diamond z = 1$ and so $x \diamond (y \diamond z) = 1$. On the other hand, if $x * z \in A$, then $x \diamond z = 1$ and $y \diamond (x \diamond z) = y \diamond 1 = 1 = x \diamond (y \diamond z)$. If $x * z \notin A$, then $x \diamond z = x * z$. By Proposition 2.1(v), and $y * z \in A$, we have $y * (x * z) \in A$. Hence $y \diamond (x \diamond z) = 1 = x \diamond (y \diamond z)$.

If $y * z \notin A$, then $y \diamond z = y * z$. We have two cases: $x * (y * z) \in A$ or $x * (y * z) \notin A$. If $x * (y \diamond z) = x * (y * z) \in A$, then $x \diamond (y \diamond z) = 1$. By (eBE4), $y * (x * z) \in A$ and so $y \diamond (x \diamond z) = y \diamond (x * z) = 1$ or $y \diamond (x \diamond z) = y \diamond 1 = 1$. Thus $x \diamond (y \diamond z) = y \diamond (x \diamond z)$, in this case.

If $x * (y * z) \notin A$, then $x \diamond (y \diamond z) = x * (y * z)$. By Proposition 2.1($vi$), $x * z \notin A$ and so $x \diamond z = x * z$. Also, by (eBE4), $y * (x * z) = x * (y * z) \notin A$. Hence

$$y \diamond (x \diamond z) = y \diamond (x * z) = y * (x * z) = x * (y * z) = x \diamond (y \diamond z).$$

Thus $x \diamond (y \diamond z) = y \diamond (x \diamond z)$. Therefore $(Y; \diamond, 1)$ is a BE-algebra. □

Example 2.3. Let $X = \{a, b, c, d\}$ and $A = \{a, b\}$. Consider the following table:

$*_6$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	a	a	a	d
d	a	a	a	a

Then $(X;*_6,A)$ is an eBE-algebra. By Theorem 2.5, we get $Y=\{1,c,d\}$ with the following table:

$\diamond$	1	c	d
1	1	c	d
c	1	1	d
d	1	1	1

Where $(Y;\diamond,1)$ is a BE-algebra.

Theorem 2.6. *Let $(X;*,1)$ be a BE-algebra and A_0 be a set such that $A_0\cap X=\emptyset$. If we define $Y=X\cup A_0$, $A=A_0\cup\{1\}$ and define the operation "$\circ$" on Y as follows:*

$$x\circ y=\begin{cases} x*y & \text{if } x,y\notin A_0\\ y & \text{otherwise}\end{cases}$$

Then $(Y;\circ,A)$ is an eBE-algebra.

Proof. Let $x\in Y$. If $x\in X$, then $x\circ x=x*x=1\in A$. If $x\in A_0$, then $x\circ x=x\in A_0\subseteq A$. Thus $x\circ x\in A$, for all $x\in Y$ and (eBE1) holds.

By definition of "$\circ$", we have $x\circ A=A\subseteq A$ and $A\circ x=\{x\}$. Hence (eBE2) and (eBE3) holds.

To prove Y satisfies axiom (eBE4) we must investigate eight cases. By easy calculation we get if $x,y,z\notin A_0$, then

$$x\circ(y\circ z)=x*(y*z)=y*(x*z)=y\circ(x\circ z).$$

On the seven rest following cases, $x\circ(y\circ z)=z=y\circ(x\circ z)$. Therefore $(Y;\circ,A)$ is an eBE-algebra. □

Definition 2.2. *An eBE-algebra X is said to be self distributive if $x*(y*z)=(x*y)*(x*z)$, for all $x,y,z\in X$.*

Example 2.4. (i). According Example 2.3, $(X;*_6,A)$ is a self distributive eBE-algebra.

(ii). In Example 2.1, since

$$c*_1(c*_1c)=c*_1a=b\neq a=a*_1a=(c*_1c)*_1(c*_1c).$$

Then $(X;*_1,A)$ is not a self distributive eBE-algebra.

Proposition 2.2. *Let $(X;*,A)$ be a self distributive eBE-algebra. Then*

(i) *if $x*y\in A$, then $z*x\leq z*y$,*

(ii) *$y*z\leq(x*y)*(x*z)$,*

for all $x,y,z\in X$.

Proof. (i). Let $x*y\in A$ and $z\in X$. Since X is a self distributive, we have

$$(z*x)*(z*y)=z*(x*y)\in z*A\subseteq A.$$

Therefore $z*x\leq z*y$.

(ii). Using self distributivity and Proposition 2.1(ii), we have

$$(y*z)*((x*y)*(x*z))=(y*z)*(x*(y*z))\in A.$$

Therefore $y*z\leq(x*y)*(x*z)$. □

3. Filters on eBE-algebras

From now on, X is an eBE-algebra, otherwise it is stated.

Definition 3.1. *A subset F of X is called a filter of X if it satisfies:*

(F1) $A \subseteq F$;

(F2) $x \in F$ *and* $x * y \in F$ *imply* $y \in F$.

We will denote by $F(X)$ the set of all filters in X. We have $F(X) \neq \emptyset$, because X and A are filters of X.

Example 3.1. In Example 2.4(ii), $F = \{a, b, c\}$ is a filter of X.

Proposition 3.1. *Let $F \in F(X)$. If $x \in F$ and $x \leq y$, then $y \in F$.*

Proof. Let $x \in F$ and $x \leq y$. Then $x * y \in A \subseteq F$. Hence $x * y \in F$. Since $x \in F$ and F is a filter, we have $y \in F$. □

Theorem 3.1. *Let $(X; *, A)$ be an eBE-algebra. Then A is a filter of X.*

Proof. Since $A \subseteq A$, it is sufficient to show $(F2)$. Let $x, x * y \in A$. Since for all $y \in X$, $\{y\} = A * y$, we have $y = x * y \in A$. Therefore A is a filter of X. □

Proposition 3.2. *Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of filters. Then $\bigcap_{\alpha \in \Lambda} F_\alpha$, is too.*

Proof. Since, $A \subseteq F_\alpha$, for all $\alpha \in \Lambda$, we have $A \subseteq \bigcap_{\alpha \in \Lambda} F_\alpha$. Let $x, x * y \in \bigcap_{\alpha \in \Lambda} F_\alpha$. Then $x, x * y \in F_\alpha$, for all $\alpha \in \Lambda$. Since F_α is a filter, we have $y \in F_\alpha$, for all $\alpha \in \Lambda$. Therefore $y \in \bigcap_{\alpha \in \Lambda} F_\alpha$. □

Theorem 3.2. *Let $(X; *, A)$ be an eBE-algebra and F be a filter. Then $F_1 = (F \setminus A) \cup \{1\}$ is a filter of $(Y; \diamond, 1)$, which is defined in Theorem 2.5.*

Proof. It is obvious that $1 \in F_1$. Let $x \in F_1$ and $x \diamond y \in F_1$. If $x = 1$, then $y = 1 \diamond y \in F_1$.

Now let $x \neq 1$. If $y = 1$, then $y = 1 \in F_1$. Now let $x, y \neq 1$. So $x \in F \setminus A$ and $y \in X \setminus A$. If $x \diamond y = 1$ by definition of "$\diamond$" we imply that $x * y \in A$. By definition of filter $x * y \in A \subseteq F$ and $x \in F$, we imply that $y \in F$. Beside $y \notin A$, $y \in F \setminus A \subseteq F_1$.

If $x \diamond y \neq 1$, then by definition of "$\diamond$", $x * y \notin A$ and $x \diamond y = x * y \in F_1$. Thus $x * y \in F_1 \subseteq F$. Beside F is a filter and $x \in F$ we get $y \in F$. Since $y \notin A$, $y \in F \setminus A \subseteq F_1$. Therefore F_1 is a filter of Y. □

Example 3.2. From Theorem 2.5 and Example 2.4 (i), we get $Y = \{1, c, d\}$ with the following table:

$\diamond$	1	c	d
1	1	c	d
c	1	1	d
d	1	1	1

which is a BE-algebra. We can see that $F = \{a, b, c\}$ is a filter of $(X; *_6, A)$ and $F_1 = (F \setminus A) \cup \{1\} = \{1, c\}$ is a filter of $(Y; \diamond, 1)$.

Theorem 3.3. *Let $(X; *, 1)$ be a BE-algebra, F be a filter of X and A_0 be a set such that $X \cap A_0 = \emptyset$. Then $F_0 = F \cup A_0$ is a filter of an eBE-algebra $(Y; \circ, A)$, which is defined in Theorem 2.6.*

Proof. Since F is a filter, $1 \in F$, and so $A \subseteq F_0$. Now, let $x \in F_0$ and $x \circ y \in F_0$. If $x \circ y \in A_0$, then by definition of "$\circ$", $x \circ y = y$ and so $y \in A_0 \subseteq F_0$.

If $x \circ y \in F$, by definition of "$\circ$", we have $x, y \notin A$ and $x \circ y = x * y$. Besides $x \in F_0 = F \cup A_0$, consequently $x \in F$. Since F is a filter, $x \in F$ and $x * y \in F$, we get that $y \in F \subseteq F_0$. Hence F_0 is a filter. $\square$

Example 3.3. Let $X = \{1, c, d\}$ and $A_0 = \{a, b\}$. According to Example 3.2, $(X; \diamond, 1)$ is a BE-algebra. We can see that $F = \{1, c\}$ is a filter of X. By Theorem 3.3, we get $Y = \{1, a, b, c, d\}$, $A = \{1, a, b\}$ and $(Y; \circ, A)$ is an eBE-algebra with the following table:

$\circ$	1	a	b	c	d
1	1	a	b	c	d
a	1	a	b	c	d
b	1	a	b	c	d
c	1	a	b	1	d
d	1	a	b	1	1

We can see that $F_0 = F \cup A_0 = \{1, a, b, c\}$ is a filter of Y.

Let $(X; *, A)$ be an eBE-algebra and $a \in X$. Put $F_a := \{x \in X : a * x \in A\}$. Since $a * a \in A$ and $a * A \subseteq A$, we have $a \in F_a$ and $A \subseteq F_a$. Hence F_a is not an empty set.

Also, if put $F^a := \{x \in X : x * a \in A\}$, then $a \in F^a$ but $A \nsubseteq F^a$ in general.

Example 3.4. In Example 2.4(i), $F^c = \{c, d\}$, but $A \nsubseteq F^c$.

Theorem 3.4. *Let $(X; *, A)$ be a self distributive eBE-algebra and $a \in X$. Then $F_a \in F(X)$.*

Proof. Let $x,\ x * y \in F_a$. Then $a * x \in A$ and $a * (x * y) \in A$. Since X is self distributive, we have $(a * x) * (a * y) \in A$. Now, using Theorem 3.1, we have $a * y \in A$. Therefore $y \in F_a$. $\square$

In the next example we show that in the Theorem 3.4, if X is not self distributive, then F_a is not a filter.

Example 3.5. Let $X = \{a, b, c, d\}$ and $A = \{a, b\}$. Consider the following table:

$*_7$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	b	b	b	c
d	b	b	b	b

Then $(X; *_7, A)$ is an eBE-algebra which is not self distributive. We can see that $F_c = \{a, b, c\}$. Because $c \in F_c$ and $c *_7 d \in F_c$ but $d \notin F_c$, F_c is not a filter.

Definition 3.2. For every $x, y \in X$ define the set

$$eA(x, y) = \{z \in X : x * (y * z) \in A\}.$$

We call $eA(x, y)$ an extension upper set of x, y. It is easy to see that $A \subseteq eA(x, y)$, $x, y \in eA(x, y)$ and $eA(x, y) = eA(y, x)$.

Example 3.6. In Example 3.5, $eA(c, d) = X$ and $eA(a, c) = \{a, b, c\}$. Since $a * (c * d) = a * c = c \notin A$, we conclude that $d \notin eA(a, c)$.

Proposition 3.3. *Let $(X; *, A)$ be an eBE-algebra and $x \in X$. Then*

(i) $F_x \subseteq eA(x, y)$, *for all* $y \in X$.

(ii) *If* $x \leq y$*, then* $y \in eA(z, x)$*, for all* $z \in X$*.*
(iii) *If* $y \in A$*, then* $eA(x, y) \subseteq F_x$*, for all* $x \in X$*.*

Proof. (i). Let $z \in F_x$ and $y \in X$. Then $x * z \in A$. Using (eBE4) and (eBE2) we have

$$x * (y * z) = y * (x * z) \in y * A \subseteq A.$$

Thus $x * (y * z) \in A$ and so $z \in eA(x, y)$. Therefore $F_x \subseteq eA(x, y)$.

(ii). Let $x \leq y$ and $z \in X$. Hence $x * y \in A$. Using (eBE2) we have

$$z * (x * y) \in z * A \subseteq A.$$

Therefore $y \in eA(z, x)$.

(iii). Let $x \in X$, $y \in A$ and $z \in eA(x, y)$. Using (eBE3) we have

$$x * z = x * (y * z) \in A.$$

Thus $z \in F_x$. Therefore $eA(x, y) \subseteq F_x$. □

Corollary 3.1. *If* $y \in A$*, then* $eA(x, y) = F_x$*.*

Theorem 3.5. *Let* $(X; *, A)$ *be an eBE-algebra and* $x \in X$*. Then*

$$F_x = \bigcap_{y \in X} eA(x, y).$$

Proof. From Proposition 3.3(i), we have $F_x \subseteq \bigcap_{y \in X} eA(x, y)$.

Now, let $z \in \bigcap_{y \in X} eA(x, y)$. Then $x * (y * z) \in A$, for all $y \in X$. Since $\emptyset \neq A \subseteq X$, then there exists $a \in A$ and by (eBE3) we have $x * z = x * (a * z) \in A$. Hence $z \in F_x$. Therefore $F_x = \bigcap_{y \in X} eA(x, y)$. □

Theorem 3.6. *Let* $(X; *, A)$ *be a self distributive eBE-algebra. Then the extension upper set* $eA(x, y)$ *is a filter of* X*, where* $x, y \in X$*.*

Proof. Let $a * b \in eA(x, y)$ and $a \in eA(x, y)$. Then $x * (y * (a * b)) \in A$ and $x * (y * a) \in A$. It follows from the self distributivity law that $x * ((y * a) * (y * b)) \in A$ and so $(x * (y * a)) * (x * (y * b)) \in A$. Now, by Theorem 3.1, since A is a filter and $x * (y * a) \in A$, we have $x * (y * b) \in A$. Therefore $b \in eA(x, y)$. □

In the following theorem, we give an equivalent condition for the filter in eBE-algebras.

Theorem 3.7. *Let* F *be a non-empty subset of an eBE-algebra* X*. Then* F *is a filter of* X *if and only if* $eA(x, y) \subseteq F$*, for all* $x, y \in F$*.*

Proof. Let F be a filter and $x, y \in F$. If $z \in eA(x, y)$, then $x * (y * z) \in A \subseteq F$. Since $x, y \in F$ and F is a filter, we have $z \in F$. Hence $eA(x, y) \subseteq F$.

Conversely, suppose that $eA(x, y) \subseteq F$, for all $x, y \in F$. Since $A \subseteq eA(x, y) \subseteq F$, we have $A \subseteq F$. Let $a * b, a \in F$. Using $(eBE1)$ we have $(a * b) * (a * b) \in A$ and so $b \in eA(a * b, a) \subseteq F$. Therefore $b \in F$. □

Theorem 3.8. *If* F *is a filter of* X*, then*

$$F = \bigcup_{x, y \in F} eA(x, y).$$

Proof. Let F be a filter of X and $z \in F$. Since $z*(a*z) \in A$, for all $a \in A$, we have $z \in eA(z,a)$. Hence

$$F \subseteq \bigcup_{z\in F, a\in A} eA(z,a) \subseteq \bigcup_{x,y\in F} eA(x,y), \quad [A \subseteq F]$$

If $z \in \bigcup\limits_{x,y\in F} eA(x,y)$, then there exists $a, b \in F$ such that $z \in eA(a,b)$. It follows from Theorem 3.7, that $z \in F$, i.e. $\bigcup\limits_{x,y\in F} eA(x,y) \subseteq F$. Therefore $F = \bigcup\limits_{x,y\in F} eA(x,y)$. □

For a non-empty subset I of X we define the binary relation $\sim_I$ in the following way:

$$x \sim_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

The set $\{b : a \sim_I b\}$ will be denoted by $[a]_I$.

Lemma 3.1. *In the above relation $\sim_I$, if $A \subseteq I$ and $a \in A$, then $[a]_I = I$.*

Proof. Let $x \in I$ and $a \in A$. By using (eBE3) we have $a * x \in A * x = \{x\} \subseteq I$ and so $a * x \in I$. On the other hand from (eBE2) we have $x * a \in x * A \subseteq A \subseteq I$, then $x * a \in I$. Hence $a \sim_I x$. Therefore $I \subseteq [a]_I$.

Conversely, let $a \in A$ and $x \in [a]_I$. Then $x \sim_I a$ and so $x * a \in I$ and $a * x = x \in I$. Hence $[a]_I \subseteq I$. Therefore $[a]_I = I$. □

Theorem 3.9. *Let $(X; *, A)$ be a self distributive eBE-algebra and $F \in F(X)$. Then $\sim_F$ is a congruence relation on X.*

Proof. Since $x * x \in A \subseteq F$, we have $x * x \in F$ and so $x \sim_F x$.

If $x \sim_F y$, then by definition of $\sim_F$, it is obvious that $y \sim_F x$.

Now, let $x \sim_F y$ and $y \sim_F z$. Then $x * y,\ y * x \in F$ and $y * z,\ z * y \in F$. By Proposition 2.2(ii), we have $y * z \leq (x * y) * (x * z)$ and so by Proposition 3.1, we have $(x * y) * (x * z) \in F$. Since F is a filter and $x * y \in F$, we can see that $x * z \in F$. By a similar way $z * x \in F$. Thus $x \sim_F z$. Therefore $\sim_I$ is an equivalent relation on X.

If $x \sim_I y$ and $u \sim_I v$, then $x * y, y * x \in F$ and $u * v, v * u \in F$. By Proposition 2.2(ii), we have $u * v \leq (x * u) * (x * v)$ and $v * u \leq (x * v) * (x * u)$ and so by Proposition 3.1, we have $(x * u) * (x * v) \in F$ and $(x * v) * (x * u) \in F$. Thus $x * u \sim_F x * v$. By the same argument one can prove that $x * v \sim_F y * v$. Since the relation $\sim_I$ is transitive, we have $x * u \sim_F y * v$ which prove that the relation $\sim_I$ is a congruence relation on X. □

Proposition 3.4. *Let $\sim_I$ be a congruence relation on X, $A \subseteq I$ and $a \in A$. Then $[a]_I$ is a filter of X.*

Proof. From Lemma 3.1, we have $[a]_I = I$. Let $x, x*y \in [a]_I$. Thus $x \sim_I a$ and $x*y \sim_I a$. Since $y \sim_I y$ and $\sim_I$ is a congruence relation, one can see that $a \sim_I x * y \sim_I a * y = y$ (by (eBE3)). Thus $y \in [a]_I$. Therefore $[a]_I$ is a filter of X. □

Denote $\frac{X}{\sim_I} = \{[x]_I : x \in X\}$. We define a binary operation "$\star$" on $\frac{X}{\sim_I}$ by $[x]_I \star [y]_I := [x * y]_I$, in which is well defined by Theorem 3.9. We can define a binary relation "$\preceq$" on the quotient set $\frac{X}{\sim_I} = \{[x]_I : x \in X\}$ as follows

$$[a] \preceq [b] \iff (\forall x \in [a])(\exists y \in [b])(x \leq y),$$

where $[a]$ and $[b]$ are equivalence classes with respect to $\sim_I$.

Theorem 3.10. *Let $(X;*,A)$ be a self distributive eBE-algebra, $F \in F(X)$ and $a \in A$. Then $(\frac{X}{\sim_{[a]_F}};\star,[a]_F)$ is a BE-algebra.*

Proof. Since $A \subseteq F$, we can see that $A \subseteq [a]_F$, for all $a \in A$. Hence $[a]_F$ is a filter by Proposition 3.4 and so $\sim_{[a]_F}$ is a congruence relation on X by Theorem 3.9. Now we have

(BE1) $[x]_F \star [x]_F = [x*x]_F = [a]_F$, since $x*x \in A \subseteq [a]_F$,

(BE2) $[x]_F \star [a]_F = [x*a]_F = [a]_F$, since $x*a \in x*A \subseteq A \subseteq [a]_F$,

(BE3) $[a]_F \star [x]_F = [a*x]_F = [x]_F$, since $A*x = \{x\}$ and so $a*x = x$,

(BE4) $[x]_F \star ([y]_F \star [z]_F) = [x*(y*z)]_F = [y*(x*z)]_F = [y]_F \star ([x]_F \star [z]_F)$.

$\square$

Corollary 3.2. *Let $(X;*,A)$ be a self distributive eBE-algebra, $F \in F(X)$ and $|A| = n$. Then there exists at least n related quotient BE-algebras.*

4. Conclusion and future works

Researchers proposed several kinds of algebraic structures related to some axioms in many-valued logic used in many-valued mathematics. Different algebraic structures are important for mathematics and for logic, in particular, non-classical logics and so related algebraic structures are suitable for many-valued reasoning under uncertainty and vagueness.

The goal of this paper is to generalize the notion of BE-algebra by considering the non-empty set substitution with constant 1.

As future works, we shall define the commutative eBE-algebras and we shall study the notion of fuzzy structures on this algebra.

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References

[1] Ceballos, M., Nunez, J., Tenorio, A. F., (2014), Triangular configurations and lie algebras of strictly upper-triangular matrices , Appl. Comput. Math., 13(1), pp.62-70.

[2] Iséki, K., (1980), On BCI-algebras, Math. Sem. Notes, Kobe Univ. 8, pp.125-130.

[3] Kim, H. S., Kim. Y. H., (2007), On BE-algebras, Sci, Math, Jpn., 66(1), pp.113–116.

[4] Rezaei, A., Borumand Saeid, A., (2012), Some results in BE-algebras, An. Univ. Oradea, Fasc. Mat., XIX, pp.33–44.

[5] Rezaei A., Borumand Saeid, A., (2012), Commutative ideals in BE-algebras, Kyungpook Math. J., 52, pp.483–494.

[6] Rezaei, A., Borumand Saeid, A., Borzooei, R. A., (2013), Relation between Hilbert algebras and BE-algebras, Appli. Appli. Math., 8(2), pp.573–584.

[7] Walendziak, A., (2008), On commutative BE-algebras, Sci. Math. Jpn., 69(2), pp.585–588.

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