ON eBE-ALGEBRAS

A. REZAEI¹, A. BORUMAND SAEID², A. RADFAR¹

Abstract. In this paper, we introduce a new algebra, called an eBE-algebra, which is a generalization of a BE-algebra and discuss its basic properties. Also, the notion of filters in this structure is studied. We show that every filter can state as a union of extension of upper sets.

Keywords: eBE-algebra, self distributive, filter.

AMS Subject Classification: 06D20, 06F35, 03G25.

1. Introduction and preliminaries

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2]). They have shown that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. It is known that several generalizations of a BCI/BCK-algebra were extensively investigated by many researchers and properties have been considered systematically. H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra ([3]). They defined BE-algebra as an algebra \((X; *, 1)\) of type \((2, 0)\) (i.e. a non-empty set with a binary operation “*” and a constant 1) satisfying the following axioms:

(BE1) \(x \ast x = 1\),
(BE2) \(x \ast 1 = 1\),
(BE3) \(1 \ast x = x\),
(BE4) \(x \ast (y \ast z) = y \ast (x \ast z)\),

for all \(x, y, z \in X\).

A. Walendziak investigated the relationship between BE-algebras, implication algebras, and J-algebras ([7]). A. Rezaei et al. got some results on BE-algebras and introduced the notion of commutative ideals in BE-algebras and proved several characterizations of such ideals ([4, 5, 6]). For development of many-valued logical system, it is needed to make clear the corresponding algebraic structures. It is motivated us to focus on a new algebraic structure, namely eBE-algebra, as a generalization of BE-algebra and so investigate some properties. Also, we discuss on filters of eBE-algebras.

2. A new extension of BE-algebras

Definition 2.1. Let \(X\) be a non-empty set. By an eBE-algebra we shall mean an algebra \((X; *, A)\) such that “*” is a binary operation on \(X\) and \(A\) is a non-empty subset of \(X\) satisfying the following axioms:

- (eBE1) \(x \ast x \in A\),
(eBE2) $x * A \subseteq A$, 
(eBE3) $A * x = \{x\}$, 
(eBE4) $x * (y * z) = y * (x * z)$,
for all $x, y, z \in X$.

In Text, $A * x = \{a * x : a \in A\}$ and similarly $x * A = \{x * a : a \in A\}$.

We note that if $A = X$, then $(X; *, X)$ is an eBE-algebra.

Let $a, b \in A$. By (eBE3), we have $a * b = b \in A$ and $b * a = a \in A$. Hence $A$ is a closed subset of $X$.

We introduce a relation “$\leq$” on $X$ by $x \leq y$ if and only if $x * y \in A$. By (eBE1) the relation “$\leq$” is reflexive.

Theorem 2.1. Every BE-algebra is an eBE-algebra.

Proof. Put $A := \{1\}$, we can see that $(X; *, A)$ is an eBE-algebra. \hfill \Box

In the next example we show that every eBE-algebra is not a BE-algebra in general.

Example 2.1. Let $X = \{a, b, c\}$ be a set and $A = \{a, b\}$ with the following table.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>b</td>
<td>a</td>
</tr>
</tbody>
</table>

Then $(X; * _1, A)$ is an eBE-algebra. Since $b * _1 b = b$ and $c * _1 c = a$, there is not an element $1 \in X$, such that $x * _1 x = 1$, for all $x \in X$. Hence $(X; * _1, A)$ is not a BE-algebra.

In the following example we show that axioms “(eBE1)” to “(eBE4)” are independence.

Example 2.2. Let $X = \{a, b, c\}$ be a set and $A = \{a, b\}$ with the following tables:

(i).

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>c</td>
</tr>
</tbody>
</table>

Then $(X; * _2, A)$ satisfies axioms (eBE2), (eBE3) and (eBE4). Since $c * _2 c = c \not\in A$, $(X; * _2, A)$ does not satisfy the axiom (eBE1).

(ii).

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>b</td>
</tr>
</tbody>
</table>

Then $(X; * _3, A)$ satisfies axioms (eBE1), (eBE3) and (eBE4). Since $c * _3 A = c * _3 \{a, b\} = \{c\} \not\subseteq A$, $(X; * _3, A)$ does not satisfy the axiom (eBE2).

(iii).

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

Then $(X; * _4, A)$ satisfies axioms (eBE1), (eBE2) and (eBE4). Since $A * _4 a = \{a, b\} * _4 a = \{b, c\} \neq \{a\}$, $(X; * _4, A)$ does not satisfy the axiom (eBE3).
Proposition 2.1. for all \( x, y, z \)

Corollary 2.2. can see that valid.

Theorem 2.4. Corollary 2.1. can see that valid.

Theorem 2.3. Proof. BE-algebra.

202 TWMS J. PURE APPL. MATH., V.7, N.2, 2016

Then \( (X;*, A) \) satisfies axioms (eBE1), (eBE2) and (eBE3). Since

\[
c * 5 (d * 5 d) = c * 5 b = b \neq c = d * 5 c = d * 5 (c * 5 d),
\]

\( (X;*, A) \) does not satisfy the axiom (eBE4).

**Theorem 2.2.** Let \( (X;*, A) \) be an eBE-algebra. If \( A \) is a singleton set, then \( (X;*, A) \) is a BE-algebra.

**Proof.** Let \( A = \{a\} \) be a singleton set. If we put \( 1 := a \), then \( (X;*, 1) \) is a BE-algebra. \( \square \)

**Theorem 2.3.** Let \( (X;*, A_1) \) and \( (X;*, A_2) \) be two eBE-algebras. Then \( (X;*, A_1 \cap A_2) \) is, too.

**Proof.** Let \( x \in X \). Since \( x \times x \in A_1 \) and \( x \times x \in A_2 \), we have \( x \times x \in A_1 \cap A_2 \) and so (eBE1) is valid.

For (eBE2), let \( a \in x \times (A_1 \cap A_2) \). Hence there exists \( b \in A_1 \cap A_2 \) such that \( a = x \times b \). Since \( b \in A_1, x \times b \in A_1 \) and \( b \in A_2, x \times b \in A_2 \), we have \( a = x \times b \in A_1 \cap A_2 \) and so \( x \times (A_1 \cap A_2) \subseteq A_1 \cap A_2 \).

Let \( a \in (A_1 \cap A_2) \times x \). Then there exists \( b \in A_1 \cap A_2 \) such that \( a = b \times x \). Since \( b \times x = x \), we can see that \( a = x \) and so \( (A_1 \cap A_2) \times x = \{x\} \). Therefore (eBE3) is valid.

Also, it is obvious that (eBE4) is valid. \( \square \)

**Corollary 2.1.** If \( (X;*, A_i) \), for \( i \in \Lambda \), be a family of eBE-algebras, then \( (X;*, \bigcap_{i \in \Lambda} A_i) \) is, too.

**Theorem 2.4.** Let \( (X;*, A_1) \) and \( (X;*, A_2) \) be two eBE-algebras. Then \( (X;*, A_1 \cup A_2) \) is, too.

**Proof.** Let \( x \in X \). Since \( x \times x \in A_1 \) and \( x \times x \in A_2 \), we have \( x \times x \in A_1 \cup A_2 \) and so (eBE1) is valid.

For (eBE2), let \( a \in x \times (A_1 \cup A_2) \). Hence there exists \( b \in A_1 \cup A_2 \) such that \( a = x \times b \). If \( b \in A_1 \), then \( a \in A_1 \). Also, if \( b \in A_2 \), then \( a \in A_2 \). Thus \( a \in A_1 \cup A_2 \) and so \( x \times (A_1 \cup A_2) \subseteq A_1 \cup A_2 \).

Let \( a \in (A_1 \cup A_2) \times x \). Then there exists \( b \in A_1 \cup A_2 \) such that \( a = b \times x \). Since \( b \times x = x \), we can see that \( a = x \) and so \( (A_1 \cup A_2) \times x = \{x\} \). Therefore (eBE3) is valid.

Also, it is obvious that (eBE4) is valid. \( \square \)

**Corollary 2.2.** If \( (X;*, A_i) \), for \( i \in \Lambda \), is a family of eBE-algebras, then \( (X;*, \bigcup_{i \in \Lambda} A_i) \) is, too.

**Proposition 2.1.** Let \( (X;*, A) \) be an eBE-algebra. Then

(i) \( (X;*, X \setminus A) \) is not an eBE-algebra,
(ii) \( x \times (y \times x) \in A \),
(iii) \( x \leq y \times z \) implies \( y \leq x \times z \),
(iv) \( x \leq (x \times y) \times x \),
(v) \( y \times z \in A \) implies \( x \times (y \times z) \in A \) and \( y \times (x \times z) \in A \),
(vi) \( x \times (y \times z) \not\in A \) implies \( x \times z \not\in A \),

for all \( x, y, z \in X \).
Proof. (i). Since \( x \ast A \subseteq A \), we have \( x \ast A \not\subseteq X \setminus A \) and so (eBE2) is not valid.

(ii). Using (eBE4) and (eBE2), we have

\[
x \ast (y \ast x) = y \ast (x \ast x) \in y \ast A \subseteq A.
\]

(iii). Let \( x \leq y \ast z \). Hence \( x \ast (y \ast z) \in A \). Then by using (eBE4), we have

\[
y \ast (x \ast z) = x \ast (y \ast z) \in A.
\]

Therefore \( y \leq x \ast z \).

(iv). From (eBE4), (eBE1) and (eBE3) we have

\[
x \ast ((x \ast y) \ast x) = (x \ast y) \ast (x \ast x) \in (x \ast y) \ast A \subseteq A.
\]

Therefore \( x \leq (x \ast y) \ast x \).

(v). Let \( y \ast z \in A \). Hence \( x \ast (y \ast z) \in x \ast A \subseteq A \). Now, using (eBE4) we have \( y \ast (x \ast z) \in A \).

(vi). The proof is obvious by (v).

\[\Box\]

**Theorem 2.5.** Let \((X; *, A)\) be an eBE-algebra. Consider \(Y := (X \setminus A) \cup \{1\}\) and define the operation \(\ast\) on \(Y\) as follows:

\[
x \ast y = \begin{cases} 
  x \ast y & \text{if } x, y \neq 1 \text{ and } x \ast y \not\in A \\
  1 & \text{if } x, y \neq 1 \text{ and } x \ast y \in A \\
  y & \text{if } x = 1 \\
  1 & \text{if } y = 1
\end{cases}
\]

Then \((Y; \ast, 1)\) is a BE-algebra.

**Proof.** By (eBE1), \( x \ast x \in A \), for all \( x \in X \). Thus \( x \ast x = 1 \), for all \( x \in Y \) and so (BE1) holds.

By definition of “\(\ast\)”, (BE2) and (BE3) are hold. To prove \((Y; \ast, 1)\) is a BE-algebra it is sufficient to prove that \( x \ast (y \ast z) = y \ast (x \ast z) \), for all \( x, y, z \in Y \). If \( x = 1 \) or \( y = 1 \) or \( z = 1 \), then we have \( x \ast (y \ast z) = y \ast (x \ast z) \). Now, let \( x, y, z \neq 1 \).

If \( y \ast z \in A \), then \( y \ast z = 1 \) and so \( x \ast (y \ast z) = 1 \). On the other hand, if \( x \ast z \in A \), then \( x \ast z = 1 \) and \( y \ast (x \ast z) = y \ast 1 = 1 = x \ast (y \ast z) \). If \( x \ast z \not\in A \), then \( x \ast z = x \ast z \). By Proposition 2.1(v), and \( y \ast z \in A \), we have \( y \ast (x \ast z) \in A \). Hence \( y \ast (x \ast z) = 1 = x \ast (y \ast z) \).

If \( y \ast z \not\in A \), then \( y \ast z = y \ast z \). We have two cases: \( x \ast (y \ast z) \in A \) or \( x \ast (y \ast z) \not\in A \). If \( x \ast (y \ast z) = x \ast (y \ast z) \in A \), then \( x \ast (y \ast z) = 1 \). By (eBE4), \( y \ast (x \ast z) \in A \) and so \( y \ast (x \ast z) = y \ast (x \ast z) = 1 \) or \( y \ast (x \ast z) = y \ast 1 = 1 \). Thus \( x \ast (y \ast z) = y \ast (x \ast z) \), in this case.

If \( x \ast (y \ast z) \not\in A \), then \( x \ast (y \ast z) = x \ast (y \ast z) \). By Proposition 2.1(vi), \( x \ast z \not\in A \) and so \( x \ast z = x \ast z \). Also, by (eBE4), \( y \ast (x \ast z) = x \ast (y \ast z) \). Hence

\[
y \ast (x \ast z) = y \ast (x \ast z) = y \ast (x \ast z) = x \ast (y \ast z) = x \ast (y \ast z).
\]

Thus \( x \ast (y \ast z) = y \ast (x \ast z) \). Therefore \((Y; \ast, 1)\) is a BE-algebra.

**Example 2.3.** Let \(X = \{a, b, c, d\}\) and \(A = \{a, b\}\). Consider the following table:

<table>
<thead>
<tr>
<th>*6</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>
Then \((X;*_6,A)\) is an eBE-algebra. By Theorem 2.5, we get \(Y = \{1,c,d\}\) with the following table:

<table>
<thead>
<tr>
<th>(\circ)</th>
<th>1</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Where \((Y;\circ,1)\) is a BE-algebra.

**Theorem 2.6.** Let \((X;*,1)\) be a BE-algebra and \(A_0\) be a set such that \(A_0 \cap X = \emptyset\). If we define \(Y = X \cup A_0\), \(A = A_0 \cup \{1\}\) and define the operation “\(\circ\)” on \(Y\) as follows:

\[
x \circ y = \begin{cases} x*y & \text{if } x,y \notin A_0 \\ y & \text{otherwise} \end{cases}
\]

Then \((Y;\circ,A)\) is an eBE-algebra.

**Proof.** Let \(x \in Y\). If \(x \in X\), then \(x \circ x = x*x = 1 \in A\). If \(x \in A_0\), then \(x \circ x = x \in A_0 \subseteq A\). Thus \(x \circ x \in A\), for all \(x \in Y\) and (eBE1) holds.

By definition of “\(\circ\)”, we have \(x \circ A = A \subseteq A\) and \(A \circ x = \{x \}\). Hence (eBE2) and (eBE3) holds.

To prove \(Y\) satisfies axiom (eBE4) we must investigate eight cases. By easy calculation we get if \(x,y,z \notin A_0\), then

\[
x \circ (y \circ z) = x*(y*z) = y*(x*z) = y \circ (x \circ z).
\]

On the seven rest following cases, \(x \circ (y \circ z) = z = y \circ (x \circ z)\). Therefore \((Y;\circ,A)\) is an eBE-algebra.

**Definition 2.2.** An eBE-algebra \(X\) is said to be self distributive if \(x*(y*z) = (x*y)*(x*z)\), for all \(x,y,z \in X\).

**Example 2.4.** (i). According Example 2.3, \((X;*_6,A)\) is a self distributive eBE-algebra.

(ii). In Example 2.1, since

\[
c*_1(c*_1c) = c*_1a = b \neq a = a*_1a = (c*_1c)*_1(c*_1c).
\]

Then \((X;*_1,A)\) is not a self distributive eBE-algebra.

**Proposition 2.2.** Let \((X;*,A)\) be a self distributive eBE-algebra. Then

(i) if \(x*y \in A\), then \(z*x \leq z*y\),

(ii) \(y*z \leq (x*y)*(x*z)\),

for all \(x,y,z \in X\).

**Proof.** (i). Let \(x*y \in A\) and \(z \in X\). Since \(X\) is a self distributive, we have

\[
(z*x)*(z*y) = z*(x*y) \in z*A \subseteq A.
\]

Therefore \(z*x \leq z*y\).

(ii). Using self distributivity and Proposition 2.1(ii), we have

\[
(y*z)*((x*y)*(x*z)) = (y*z)*((x*y*z)) \in A.
\]

Therefore \(y*z \leq (x*y)*(x*z)\).  

\[\square\]
3. Filters on eBE-algebras

From now on, $X$ is an eBE-algebra, otherwise it is stated.

**Definition 3.1.** A subset $F$ of $X$ is called a filter of $X$ if it satisfies:

(F1) $A \subseteq F$;

(F2) $x \in F$ and $x \ast y \in F$ imply $y \in F$.

We will denote by $F(X)$ the set of all filters in $X$. We have $F(X) \neq \emptyset$, because $X$ and $A$ are filters of $X$.

**Example 3.1.** In Example 2.4(ii), $F = \{a, b, c\}$ is a filter of $X$.

**Proposition 3.1.** Let $F \subseteq F(X)$. If $x \in F$ and $x \leq y$, then $y \in F$.

*Proof.* Let $x \in F$ and $x \leq y$. Then $x \ast y \in A \subseteq F$. Hence $x \ast y \in F$. Since $x \in F$ and $F$ is a filter, we have $y \in F$. □

**Theorem 3.1.** Let $(X; \ast, A)$ be an eBE-algebra. Then $A$ is a filter of $X$.

*Proof.* Since $A \subseteq A$, it is sufficient to show (F2). Let $x, x \ast y \in A$. Since for all $y \in X$, $\{y\} = A \ast y$, we have $y = x \ast y \in A$. Therefore $A$ is a filter of $X$. □

**Proposition 3.2.** Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of filters. Then $\bigcap_{\alpha \in \Lambda} F_\alpha$ is too.

*Proof.* Since $A \subseteq F_\alpha$, for all $\alpha \in \Lambda$, we have $A \subseteq \bigcap_{\alpha \in \Lambda} F_\alpha$. Let $x, x \ast y \in \bigcap_{\alpha \in \Lambda} F_\alpha$. Then $x, x \ast y \in F_\alpha$, for all $\alpha \in \Lambda$. Since $F_\alpha$ is a filter, we have $y \in F_\alpha$, for all $\alpha \in \Lambda$. Therefore $y \in \bigcap_{\alpha \in \Lambda} F_\alpha$. □

**Theorem 3.2.** Let $(X; \ast, A)$ be an eBE-algebra and $F$ be a filter. Then $F_1 = (F \setminus A) \cup \{1\}$ is a filter of $(Y; \circ, 1)$, which is defined in Theorem 2.5.

*Proof.* It is obvious that $1 \in F_1$. Let $x \in F_1$ and $x \circ y \in F_1$. If $x = 1$, then $y = 1 \circ y \in F_1$.

Now let $x \neq 1$. If $y = 1$, then $y = 1 \in F_1$. Now let $x, y \neq 1$. So $x \in F \setminus A$ and $y \in X \setminus A$. If $x \circ y = 1$ by definition of “$\circ$” we imply that $x \ast y \in A$. By definition of filter $x \ast y \in A \subseteq F$ and $x \in F$, we imply that $y \in F$. Beside $y \notin A$, $y \in F \setminus A \subseteq F_1$.

If $x \circ y \neq 1$, then by definition of “$\circ$”, $x \ast y \notin A$ and $x \circ y = x \ast y \in F_1$. Thus $x \ast y \in F_1 \subseteq F$. Beside $F$ is a filter and $x \in F$ we get $y \in F$. Since $y \notin A$, $y \in F \setminus A \subseteq F_1$. Therefore $F_1$ is a filter of $Y$. □

**Example 3.2.** From Theorem 2.5 and Example 2.4 (i), we get $Y = \{1, c, d\}$ with the following table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>1</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

which is a BE-algebra. We can see that $F = \{a, b, c\}$ is a filter of $(X; \ast_6, A)$ and $F_1 = (F \setminus A) \cup \{1\} = \{1, c\}$ is a filter of $(Y; \circ, 1)$.

**Theorem 3.3.** Let $(X; \ast, 1)$ be a BE-algebra, $F$ be a filter of $X$ and $A_0$ be a set such that $X \cap A_0 = \emptyset$. Then $F_0 = F \cup A_0$ is a filter of an eBE-algebra $(Y; \circ, A)$, which is defined in Theorem 2.6.
Proof. Since $F$ is a filter, $1 \in F$, and so $A \subseteq F_0$. Now, let $x \in F_0$ and $x \circ y \in F_0$. If $x \circ y \in A_0$, then by definition of “$\circ$”, $x \circ y = y$ and so $y \in A_0 \subseteq F_0$.

If $x \circ y \in F$, by definition of “$\circ$”, we have $x, y \notin A$ and $x \circ y = x \ast y$. Besides $x \in F_0 = F \cup A_0$, consequently $x \in F$. Since $F$ is a filter, $x \in F$ and $x \ast y \in F$, we get that $y \in F \subseteq F_0$. Hence $F_0$ is a filter.

\[\square\]

**Example 3.3.** Let $X = \{1, c, d\}$ and $A_0 = \{a, b\}$. According to Example 3.2, $(X; \circ, 1)$ is a $BE$-algebra. We can see that $F = \{1, c\}$ is a filter of $X$. By Theorem 3.3, we get $Y = \{1, a, b, c, d\}$, $A = \{1, a, b\}$ and $(Y; \circ, A)$ is an $eBE$-algebra with the following table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We can see that $F_0 = F \cup A_0 = \{1, a, b, c\}$ is a filter of $Y$.

Let $(X; \ast, A)$ be an $eBE$-algebra and $a \in X$. Put $F_a := \{x \in X : a \ast x \in A\}$. Since $a \ast a \in A$ and $a \ast A \subseteq A$, we have $a \in F_a$ and $A \subseteq F_a$. Hence $F_a$ is not an empty set.

Also, if put $F^a := \{x \in X : x \ast a \in A\}$, then $a \in F^a$ but $A \not\subseteq F^a$ in general.

**Example 3.4.** In Example 2.4(i), $F^c = \{c, d\}$, but $A \not\subseteq F^c$.

**Theorem 3.4.** Let $(X; \ast, A)$ be a self distributive $eBE$-algebra and $a \in X$. Then $F_a \in F(X)$.

**Proof.** Let $x$, $x \ast y \in F_a$. Then $a \ast x \in A$ and $a \ast (x \ast y) \in A$. Since $X$ is self distributive, we have $(a \ast x) \ast (a \ast y) \in A$. Now, using Theorem 3.1, we have $a \ast y \in A$. Therefore $y \in F_a$. \[\square\]

In the next example we show that in the Theorem 3.4, if $X$ is not self distributive, then $F_a$ is not a filter.

**Example 3.5.** Let $X = \{a, b, c, d\}$ and $A = \{a, b\}$. Consider the following table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
<td>$d$</td>
</tr>
<tr>
<td>$b$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
<td>$d$</td>
</tr>
<tr>
<td>$c$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td>$d$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
</tr>
</tbody>
</table>

Then $(X; \ast, A)$ is an $eBE$-algebra which is not self distributive. We can see that $F_c = \{a, b, c\}$. Because $c \in F_c$ and $c \ast d \in F_c$ but $d \notin F_c$, $F_c$ is not a filter.

**Definition 3.2.** For every $x, y \in X$ define the set

$eA(x, y) = \{z \in X : x \ast (y \ast z) \in A\}$.

We call $eA(x, y)$ an extension upper set of $x, y$. It is easy to see that $A \subseteq eA(x, y)$, $x, y \in eA(x, y)$ and $eA(x, y) = eA(y, x)$.

**Example 3.6.** In Example 3.5, $eA(c, d) = X$ and $eA(a, c) = \{a, b, c\}$. Since $a \ast (c \ast d) = a \ast c = c \notin A$, we conclude that $d \not\in eA(a, c)$.

**Proposition 3.3.** Let $(X; \ast, A)$ be an $eBE$-algebra and $x \in X$. Then

(i) $F_x \subseteq eA(x, y)$, for all $y \in X$.
(ii) If \( x \leq y \), then \( y \in eA(z, x) \), for all \( z \in X \).

(iii) If \( y \in A \), then \( eA(x, y) \subseteq F_x \), for all \( x \in X \).

Proof. (i). Let \( z \in F_x \) and \( y \in X \). Then \( x \ast z \in A \). Using (eBE4) and (eBE2) we have
\[
x \ast (y \ast z) = y \ast (x \ast z) \in y \ast A \subseteq A.
\]
Thus \( x \ast (y \ast z) \in A \) and so \( z \in eA(x, y) \). Therefore \( F_x \subseteq eA(x, y) \).

(ii). Let \( x \leq y \) and \( z \in X \). Hence \( x \ast y \in A \). Using (eBE2) we have
\[
z \ast (x \ast y) \in z \ast A \subseteq A.
\]
Therefore \( y \in eA(z, x) \).

(iii). Let \( x \in X \), \( y \in A \) and \( z \in eA(x, y) \). Using (eBE3) we have
\[
x \ast z = x \ast (y \ast z) \in A.
\]
Thus \( z \in F_x \). Therefore \( eA(x, y) \subseteq F_x \).

\[\square\]

Corollary 3.1. If \( y \in A \), then \( eA(x, y) = F_x \).

Theorem 3.5. Let \((X; \ast, A)\) be an eBE-algebra and \( x \in X \). Then
\[
F_x = \bigcap_{y \in X} eA(x, y).
\]

Proof. From Proposition 3.3(i), we have \( F_x \subseteq \bigcap_{y \in X} eA(x, y) \).

Now, let \( z \in \bigcap_{y \in X} eA(x, y) \). Then \( x \ast (y \ast z) \in A \), for all \( y \in X \). Since \( \emptyset \neq A \subseteq X \), then there exists \( a \in A \) and by (eBE3) we have \( x \ast z = x \ast (a \ast z) \in A \). Hence \( z \in F_x \). Therefore
\[
F_x = \bigcap_{y \in X} eA(x, y).
\]

\[\square\]

Theorem 3.6. Let \((X; \ast, A)\) be a self distributive eBE-algebra. Then the extension upper set \( eA(x, y) \) is a filter of \( X \), where \( x, y \in X \).

Proof. Let \( a \ast b \in eA(x, y) \) and \( a \in eA(x, y) \). Then \( x \ast (y \ast (a \ast b)) \in A \) and \( x \ast (y \ast a) \in A \). It follows from the self distributivity law that \( x \ast ((y \ast a) \ast (y \ast b)) \in A \) and so \( (x \ast (y \ast a)) \ast (x \ast (y \ast b)) \in A \). Now, by Theorem 3.1, since \( A \) is a filter and \( x \ast (y \ast a) \in A \), we have \( x \ast (y \ast b) \in A \). Therefore \( b \in eA(x, y) \).

In the following theorem, we give an equivalent condition for the filter in eBE-algebras.

Theorem 3.7. Let \( F \) be a non-empty subset of an eBE-algebra \( X \). Then \( F \) is a filter of \( X \) if and only if \( eA(x, y) \subseteq F \), for all \( x, y \in F \).

Proof. Let \( F \) be a filter and \( x, y \in F \). If \( z \in eA(x, y) \), then \( x \ast (y \ast z) \in A \subseteq F \). Since \( x, y \in F \) and \( F \) is a filter, we have \( z \in F \). Hence \( eA(x, y) \subseteq F \).

Conversely, suppose that \( eA(x, y) \subseteq F \), for all \( x, y \in F \). Since \( A \subseteq eA(x, y) \subseteq F \), we have \( A \subseteq F \). Let \( a \ast b, a \in F \). Using (eBE1) we have \( (a \ast b) \ast (a \ast b) \in A \) and so \( b \in eA(a \ast b, a) \subseteq F \). Therefore \( b \in F \).

\[\square\]

Theorem 3.8. If \( F \) is a filter of \( X \), then
\[
F = \bigcup_{x, y \in F} eA(x, y).
\]
Proof. Let $F$ be a filter of $X$ and $z \in F$. Since $z * (a * z) \in A$, for all $a \in A$, we have $z \in eA(z, a)$. Hence
\[
F \subseteq \bigcup_{z \in F, a \in A} eA(z, a) \subseteq \bigcup_{x, y \in F} eA(x, y), \quad [A \subseteq F]
\]
If $z \in \bigcup_{x, y \in F} eA(x, y)$, then there exists $a, b \in F$ such that $z \in eA(a, b)$. It follows from Theorem 3.7, that $z \in F$, i.e. $\bigcup_{x, y \in F} eA(x, y) \subseteq F$. Therefore $F = \bigcup_{x, y \in F} eA(x, y)$. \hfill \Box

For a non-empty subset $I$ of $X$ we define the binary relation $\sim_I$ in the following way:
\[
x \sim_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I.
\]

The set $\{b : a \sim_I b\}$ will be denoted by $[a]_I$.

Lemma 3.1. In the above relation $\sim_I$, if $A \subseteq I$ and $a \in A$, then $[a]_I = I$.

Proof. Let $x \in I$ and $a \in A$. By using (eBE3) we have $a * x \in A * x = \{x\} \subseteq I$ and so $a * x \in I$. On the other hand from (eBE2) we have $x * a \in x * A \subseteq A \subseteq I$, then $x * a \in I$. Hence $a \sim_I x$. Therefore $I \subseteq [a]_I$.

Conversely, let $a \in A$ and $x \in [a]_I$. Then $x \sim_I a$ and so $x * a \in I$ and $a * x = x \in I$. Hence $[a]_I \subseteq I$. Therefore $[a]_I = I$. \hfill \Box

Theorem 3.9. Let $(X; *, A)$ be a self distributive eBE-algebra and $F \in F(X)$. Then $\sim_F$ is a congruence relation on $X$.

Proof. Since $x * x \in A \subseteq F$, we have $x * x \in F$ and so $x \sim_F x$.

If $x \sim_F y$, then by definition of $\sim_F$, it is obvious that $y \sim_F x$.

Now, let $x \sim_F y$ and $y \sim_F z$. Then $x * y, y * x \in F$ and $y * z, z * y \in F$. By Proposition 2.2(ii), we have $y * z \in (x * y) * (x * z)$ and so by Proposition 3.1, we have $(x * y) * (x * z) \in F$. Since $F$ is a filter and $x * y \in F$, we can see that $x * z \in F$. By a similar way $z * x \in F$. Thus $x \sim_F z$. Therefore $\sim_F$ is an equivalent relation on $X$.

If $x \sim_F y$ and $u \sim_F v$, then $x * y, y * x \in F$ and $u * v, v * u \in F$. By Proposition 2.2(ii), we have $u * v \leq (x * u) * (x * v)$ and $v * u \leq (x * v) * (x * u)$ and so by Proposition 3.1, we have $(x * u) * (x * v) \in F$ and $(x * v) * (x * u) \in F$. Thus $x * u \sim_F x * v$. By the same argument one can prove that $x * v \sim_F y * v$. Since the relation $\sim_F$ is transitive, we have $x * u \sim_F y * v$ which prove that the relation $\sim_F$ is a congruence relation on $X$. \hfill \Box

Proposition 3.4. Let $\sim_I$ be a congruence relation on $X$, $A \subseteq I$ and $a \in A$. Then $[a]_I$ is a filter of $X$.

Proof. From Lemma 3.1, we have $[a]_I = I$. Let $x, x * y \in [a]_I$. Thus $x \sim_I a$ and $x * y \sim_I a$. Since $y \sim_I a$ and $\sim_I$ is a congruence relation, one can see that $a \sim_I x * y \sim_I a * y = y$ (by (eBE3)). Thus $y \in [a]_I$. Therefore $[a]_I$ is a filter of $X$. \hfill \Box

Denote $\frac{X}{\sim_I} = \{[x]_I : x \in X\}$. We define a binary operation “$*$” on $\frac{X}{\sim_I}$ by $[x]_I * [y]_I := [x * y]_I$, in which is well defined by Theorem 3.9. We can define a binary relation “$\leq$” on the quotient set $\frac{X}{\sim_I}$ as follows
\[
[a] \leq [b] \iff (\forall x \in [a])(\exists y \in [b])(x \leq y),
\]

where $[a]$ and $[b]$ are equivalence classes with respect to $\sim_I$.\hfill \Box
Theorem 3.10. Let \((X; \ast, A)\) be a self distributive eBE-algebra, \(F \in F(X)\) and \(a \in A\). Then \((X_{[a]_F}; \ast, [a]_F)\) is a BE-algebra.

Proof. Since \(A \subseteq F\), we can see that \(A \subseteq [a]_F\), for all \(a \in A\). Hence \([a]_F\) is a filter by Proposition 3.4 and so \(\sim_{[a]_F}\) is a congruence relation on \(X\) by Theorem 3.9. Now we have

\begin{align*}
(BE1) \quad [x]_F \ast [y]_F &= [x \ast y]_F = [a]_F, & \text{since } x \ast y \in A \subseteq [a]_F, \\
(BE2) \quad [x]_F \ast [a]_F &= [x \ast a]_F = [a]_F, & \text{since } x \ast a \in x \ast A \subseteq A \subseteq [a]_F, \\
(BE3) \quad [a]_F \ast [x]_F &= [a \ast x]_F = [x]_F, & \text{since } A \ast x = \{x\} \text{ and so } a \ast x = x, \\
(BE4) \quad [x]_F \ast ([y]_F \ast [z]_F) &= [x \ast (y \ast z)]_F = [y \ast (x \ast z)]_F = [y]_F \ast ([x]_F \ast [z]_F).
\end{align*}

Corollary 3.2. Let \((X; \ast, A)\) be a self distributive eBE-algebra, \(F \in F(X)\) and \(|A| = n\). Then there exists at least \(n\) related quotient BE-algebras.

4. Conclusion and future works

Researchers proposed several kinds of algebraic structures related to some axioms in many-valued logic used in many-valued mathematics. Different algebraic structures are important for mathematics and for logic, in particular, non-classical logics and so related algebraic structures are suitable for many-valued reasoning under uncertainty and vagueness.

The goal of this paper is to generalize the notion of BE-algebra by considering the non-empty set substitution with constant 1.

As future works, we shall define the commutative eBE-algebras and we shall study the notion of fuzzy structures on this algebra.

5. Acknowledgment

The authors would like to express their thanks to anonymous referees for their careful reading and valuable suggestions on the earlier version of this paper which improved the presentation and readability.

References

Akbar Rezaei received his Ph.D. in Mathematics, Iran in 2014. He is currently an assistant professor and academic member at the Department of Mathematics in Payame Noor University, Tehran, Iran. His current research interests include algebraic logic, algebraic structures, fuzzy sets and applications, fuzzy logic.

Arsham Borumand Saeid received his Ph.D. in Mathematics, Iran in 2003. He is currently an associate professor at the Department of Pure Mathematics in Shahid Bahonar University of Kerman, Iran. His current research interests include algebraic logic, algebraic structures, fuzzy sets and applications, fuzzy logic.

Akefe Radfar received her Ph.D. in Mathematics, Iran in 2012. She is currently an assistant professor and academic member at the Department of Mathematics in Payame Noor University, Tehran, Iran. Her current research interests include algebraic logic, hyper structures, fuzzy sets and applications, fuzzy logic.