TWMS J. Pure Appl. Math. V.7, N.2, 2016, pp.185-199

SECOND HANKEL DETERMINANT FOR A CERTAIN SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, we consider a subclass of the function class Σ of bi-univalent analytic functions in the open unit disk Δ and we obtain the functional $|a_2a_4 - a_3^2|$ for the function class. Also we give upper bounds for $|a_2a_4 - a_3^2|$. Our result gives corresponding $|a_2a_4 - a_3^2|$ for the subclasses of Σ defined in the literature.

Keywords: Univalent functions, analytic functions, Bi-univalent functions, coefficient bounds, Hankel determinant.

AMS Subject Classification: 30C45

1. Introduction

Let $\mathcal{A}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in Δ. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$ in Δ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ in Δ. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$
$$\text{and} \quad f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

where

$$f^{-1}(w) = g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1). Earlier, Brannan and Taha [4] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order α denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex function of order α denoted by $\mathcal{K}_\Sigma(\alpha)$ corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively.

A function $f \in \mathcal{A}$ is in the class of strongly bi-starlike(and strongly bi-convex) functions $\mathcal{S}^*_\Sigma[\alpha]$(*and* $\mathcal{K}_\Sigma[\alpha]$) [4, 19] of order α $(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \text{ and } \left|arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}$$

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Manuscript received February 2015.

and
$$\left|arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\alpha\pi}{2} \text{ and } \left|arg\left(1+\frac{wg''(w)}{g'(w)}\right)\right|<\frac{\alpha\pi}{2}$$
where g is the extension of f^{-1} to Δ. For each of the function classes $\mathcal{S}^*_\Sigma[\alpha]($ *and* $\mathcal{K}_\Sigma[\alpha])$ non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found [4, 19]. But the coefficient problem for each of the following Taylor-Maclaurin coefficients:
$$|a_n| \qquad (n\in\mathbb{N}\setminus\{1,2\};\ \ \mathbb{N}:=\{1,2,3,\cdots\})$$
is still an open problem (see [3, 4, 13, 16, 19]).

An analytic function f is subordinate to an analytic function g, written $f(z)\prec g(z)$, provided there is an analytic function w defined on Δ with $w(0)=0$ and $|w(z)|<1$ satisfying $f(z)=g(w(z))$. Ma and Minda [14] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{z\,f'(z)}{f(z)}$ or $1+\frac{z\,f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk Δ, $\phi(0)=1$, $\phi'(0)>0$ and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f\in\mathcal{A}$ satisfying the subordination $\frac{z\,f'(z)}{f(z)}\prec\phi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f\in\mathcal{A}$ satisfying the subordination $1+\frac{z\,f''(z)}{f'(z)}\prec\phi(z)$.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$ (see [1]). In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ, satisfying $\phi(0)=1$, $\phi'(0)>0$ and $\phi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form
$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots,\quad (B_1>0). \tag{3}$$

We need the following lemma for our investigation.

Lemma 1.1. *(see [7], p. 41) Let $\mathcal{P}$ be the class of all analytic functions $p(z)$ of the form*
$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n \tag{4}$$
satisfying $\Re(p(z))>0$ $(z\in\Delta)$ and $p(0)=1$. Then
$$|p_n|\le 2\ (n=1,2,3,...).$$
This inequality is sharp for each n. In particular, equality holds for all n for the function
$$p(z)=\frac{1+z}{1-z}=1+\sum_{n=1}^{\infty}2z^n.$$

In 1976, Noonan and Thomas [17] defined the qth Hankel determinant of f for $q\ge1$ by
$$H_q(n)\ =\ \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}.$$

Further, Fekete and Szegö [8] considered the Hankel determinant of $f\in A$ for $q=2$ and $n=1$, $H_2(1)\ =\ \begin{vmatrix} a_1 & a_2\\ a_2 & a_3\end{vmatrix}$. They made an early study for the estimates of $|a_3-\mu a_2^2|$ when

$a_1 = 1$ with μ real. The well known result due to them states that if $f \in \mathcal{A}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3 & \text{if} \quad \mu \geq 1, \\ 1 + 2\, exp(\frac{-2\mu}{1-\mu}) & \text{if} \quad 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if} \quad \mu \leq 0. \end{cases}$$

Furthermore, Hummel [10, 11] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is convex functions and also Keogh and Merkes [12] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is close-to-convex, starlike and convex in Δ.

Here we consider the Hankel determinant of $f \in \mathcal{A}$ for $q = 2$ and $n = 2$,

$$H_2(2) \;=\; \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

To prove our main result, we need the following lemmas.

Lemma 1.2. *If the function $p(z) \in \mathcal{P}$ is given by the series (4), then*

$$2p_2 = p_1^2 + x(4 - p_1^2), \tag{5}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \tag{6}$$

for some x, z with $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

Lemma 1.3. [9] *The power series for $p(z)$ given in (4) converges in Δ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots \tag{7}$$

and $p_{-k} = \overline{p_k}$, are all nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k z}), \;\; \rho_k > 0, \;\; t_k \text{ real}$$

and $t_k \neq t_j$ for $k \neq j$ in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

Many researchers (see [1, 2, 5, 18, 20, 21]) have introduced and investigated several interesting subclasses (by generalization or in associated with certain linear operators see [15]) of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The object of the present paper is to determine the functional $|a_2 a_4 - a_3^2|$ for a subclass of the function class Σ. Also we give upper bounds for $|a_2 a_4 - a_3^2|$. Our result gives corresponding $|a_2 a_4 - a_3^2|$ for the subclasses of Σ defined in the literature.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{F}_\Sigma(\phi, \alpha)$

For $\alpha \geq 0$ we let a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{F}_\Sigma(\phi, \alpha)$, if the following conditions are satisfied:

$$(1 - \alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z) \tag{8}$$

and

$$(1 - \alpha)\left(\frac{wg'(w)}{g(w)}\right) + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \phi(w), \tag{9}$$

where g is the inverse of f given by (2).

Definition 2.1. *Taking $\alpha = 0$, we let $\mathcal{F}_{\Sigma}(\phi,\alpha) \equiv \mathcal{S}^*_{\Sigma}(\phi)$ and if $f \in \mathcal{S}^*_{\Sigma}(\phi)$, then*

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \tag{10}$$

and

$$\frac{wg'(w)}{g(w)} \prec \phi(w), \tag{11}$$

where the function g is the inverse of f given by (2).

Definition 2.2. *Taking $\alpha = 1$, we let $\mathcal{F}_{\Sigma}(\phi,\alpha) \equiv \mathcal{K}_{\Sigma}(\phi)$ and if $f \in \mathcal{K}_{\Sigma}(\phi)$, then*

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \tag{12}$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec \phi(w), \tag{13}$$

where the function g is the inverse of f given by (2).

Theorem 2.1. *Let f given by (1) be in the class $\mathcal{F}_{\Sigma}(\phi,\alpha)$. Then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} H(2), & f(\alpha,B_1,B_2,B_3) \geq 0, C(\alpha,B_1,B_2,B_3) \geq 0 \\ \max\left\{\frac{B_1^2}{4(1+2\alpha)^2}, H(2)\right\}, & f(\alpha,B_1,B_2,B_3) > 0, C(\alpha,B_1,B_2,B_3) < 0 \\ \frac{B_1^2}{4(1+2\alpha)^2}, & f(\alpha,B_1,B_2,B_3) \leq 0, C(\alpha,B_1,B_2,B_3) \leq 0 \\ \max\{H(t_0), H(2)\}, & f(\alpha,B_1,B_2,B_3) < 0, C(\alpha,B_1,B_2,B_3) > 0, \end{cases} \tag{14}$$

where

$$H(2) = \frac{B_1|2B_1 - 2B_2 + B_3|(1+\alpha)^2(1+2\alpha)^2 + {B_1}^4(1+2\alpha)^2 + 2B_1|B_2 - B_1(1+\alpha)^3(1+2\alpha)^2|}{3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}$$

$$\begin{aligned} H\left(t_0 = \sqrt{\tfrac{-2C(\alpha,B_1,B_2,B_3)}{f(\alpha,B_1,B_2,B_3)}}\right) &= \frac{B_1^2}{4(1+2\alpha)^2} - \frac{[C(\alpha,B_1,B_2,B_3)]^2}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)f(\alpha,B_1,B_2,B_3)}, \\ f(\alpha,B_1,B_2,B_3) &= 4B_1|2B_1 - 2B_2 + B_3|(1+\alpha)^2(1+2\alpha)^2 + 4{B_1}^4(1+2\alpha)^2 \\ &\quad -3{B_1}^3(1+\alpha)(1+2\alpha)(1+3\alpha) - 12{B_1}^2(1+\alpha)^2(1+2\alpha)^2 \\ &\quad +3{B_1}^2(1+\alpha)^3(1+3\alpha), \\ C(\alpha,B_1,B_2,B_3) &= 3{B_1}^3(1+\alpha)(1+2\alpha)(1+3\alpha) + 8B_1|B_2 - B_1|(1+\alpha)^3(1+2\alpha)^2 \\ &\quad +12{B_1}^2(1+\alpha)^2(1+2\alpha)^2 - 6{B_1}^2(1+\alpha)^3(1+3\alpha)\big]. \end{aligned}$$

Proof. Let $f \in \mathcal{F}_{\Sigma}(\phi,\alpha)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \Delta \to \Delta$, with $u(0) = v(0) = 0$, satisfying

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \phi(u(z)) \tag{15}$$

and

$$(1-\alpha)\left(\frac{wg'(w)}{g(w)}\right) + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = \phi(v(w)). \tag{16}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots .$$

It follows that,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)z^3 + \cdots\right] \tag{17}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right)z^2 + \left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right)z^3 + \cdots\right]. \tag{18}$$

Then $p(z)$ and $q(z)$ are analytic in Δ with $p(0) = 1 = q(0)$. Using (17) and (18), it is clear that,

$$\begin{aligned}\phi(u(z)) = 1 + \frac{B_1p_1}{2}z + \left[\frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right]z^2 \\ + \left[\frac{B_1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{B_2p_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_3p_1^3}{8}\right]z^3 + \cdots\end{aligned} \tag{19}$$

and

$$\begin{aligned}\phi(v(w)) = 1 + \frac{B_1q_1}{2}w + \left[\frac{B_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2q_1^2\right]w^2 \\ + \left[\frac{B_1}{2}\left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right) + \frac{B_2q_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{B_3q_1^3}{8}\right]w^3 + \cdots .\end{aligned} \tag{20}$$

Equating the coefficients in (15) and (16), we get,

$$(1+\alpha)a_2 = \frac{B_1p_1}{2}, \tag{21}$$

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = \frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2, \tag{22}$$

$$\begin{aligned}&3(1+3\alpha)a_4 - 3(1+5\alpha)a_2a_3 + (1+7\alpha)a_2^3 \\ = &\frac{B_1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{B_2p_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_3p_1^3}{8}\end{aligned} \tag{23}$$

and

$$-(1+\alpha)a_2 = \frac{B_1q_1}{2}, \tag{24}$$

$$-2(1+2\alpha)a_3 + (3+5\alpha)a_2^2 = \frac{B_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2q_1^2, \tag{25}$$

$$\begin{aligned}&-3(1+3\alpha)a_4 + 6(2+5\alpha)a_2a_3 - 2(5+11\alpha)a_2^3 \\ = &\frac{B_1}{2}\left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right) + \frac{B_2q_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{B_3q_1^3}{8}.\end{aligned} \tag{26}$$

From (21) and (24) gives

$$a_2 = \frac{B_1p_1}{2(1+\alpha)} = -\frac{B_1q_1}{2(1+\alpha)}, \tag{27}$$

which implies

$$p_1 = -q_1. \tag{28}$$

Now from (22), (25) and by using (27), we obtain

$$a_3 = \frac{B_1^2 p_1^2}{4(1+\alpha)^2} + \frac{B_1(p_2-q_2)}{8(1+2\alpha)}. \tag{29}$$

On the otherhand, subtracting (26) from (23) and by using (27), (29), we get

$$\begin{aligned} a_4 = {} & \frac{5B_1^3 p_1^3}{16(1+\alpha)^3} + \frac{5B_1^2 p_1(p_2-q_2)}{32(1+\alpha)(1+2\alpha)} + \frac{B_1(p_3-q_3)}{12(1+3\alpha)} \\ & + \frac{p_1^3(B_1-2B_2+B_3)}{24(1+3\alpha)} + \frac{(B_2-B_1)p_1(p_2+q_2)}{12(1+3\alpha)} - \frac{p_1^3 B_1^3(11+29\alpha)}{48(1+\alpha)^3}. \end{aligned} \tag{30}$$

Thus we establish that

$$\begin{aligned} a_2a_4 - a_3^2 = {} & \frac{2B_1(B_1-2B_2+B_3)(1+\alpha)^3 - 2(1+\alpha)B_1^4}{96(1+\alpha)^4(1+3\alpha)} p_1^4 - \frac{B_1^2(p_2-q_2)^2}{64(1+2\alpha)^2} \\ & + \frac{B_1^3 p_1^2(p_2-q_2)}{64(1+\alpha)^2(1+2\alpha)} + \frac{B_1(B_2-B_1)p_1^2(p_2+q_2)}{24(1+3\alpha)} + \frac{B_1^2 p_1(p_3-q_3)}{24(1+\alpha)(1+3\alpha)} \end{aligned} \tag{31}$$

According to Lemma 1.2, we have

$$2p_2 = p_1^2 + x(4-p_1^2) \quad \text{and} \quad 2q_2 = q_1^2 + y(4-q_1^2), \tag{32}$$

hence by (28), we have

$$p_2 - q_2 = \frac{4-p_1^2}{2}(x-y) \tag{33}$$

$$p_2 + q_2 = p_1^2 + \frac{4-p_1^2}{2}(x+y) \tag{34}$$

and further

$$\begin{aligned} 4p_3 &= p_1^3 + 2(4-p_1^2)p_1 x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z, \\ 4q_3 &= q_1^3 + 2(4-q_1^2)q_1 y - q_1(4-q_1^2)y^2 + 2(4-q_1^2)(1-|y|^2)w, \end{aligned}$$

for some x, y, z, w with $|x| \le 1, |y| \le 1, |z| \le 1, |w| \le 1$ and $p_1, q_1 \in [0,2]$.

Thus,

$$p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1(4-p_1^2)}{2}(x+y) - \frac{p_1(4-p_1^2)}{4}(x^2+y^2) + \frac{4-p_1^2}{2}\left[(1-|x|^2)z - (1-|y|^2)w\right]. \tag{35}$$

Using (33) - (35) in (31), we get,

$$\begin{aligned} a_2a_4 - a_3^2 = {} & \left(\frac{B_1(2B_1-2B_2+B_3)(1+\alpha)^2 - B_1^4 + 2B_1(B_2-B_1)(1+\alpha)^3}{48(1+\alpha)^3(1+3\alpha)}\right) p_1^4 \\ & + \frac{B_1^3 p_1^2(4-p_1^2)(x-y)}{128(1+\alpha)^2(1+2\alpha)} + \left(\frac{B_1(B_2-B_1)(1+\alpha)+B_1^2}{48(1+\alpha)(1+3\alpha)}\right)(4-p_1^2)p_1^2(x+y) \\ & - \frac{B_1^2(4-p_1^2)p_1^2}{96(1+\alpha)(1+3\alpha)}(x^2+y^2) - \frac{B_1^2(4-p_1^2)^2}{256(1+2\alpha)^2}(x-y)^2 \\ & + \frac{B_1^2 p_1(4-p_1^2)}{48(1+\alpha)(1+3\alpha)}[(1-|x|^2)z - (1-|y|^2)w]. \end{aligned} \tag{36}$$

Since $p(z) \in P$, so $|p_1| \le 2$. Thus, letting $p_1 = t$ and applying triangle inequality on (36), with $\lambda = |x| \le 1$ and $\mu = |y| \le 1$, we obtain

$$|a_2a_4 - a_3^2| \le C_1 + C_2(\lambda+\mu) + C_3(\lambda^2+\mu^2) + C_4(\lambda+\mu)^2 = F(\lambda,\mu), \tag{37}$$

where

$$\begin{aligned} C_1 &= C_1(t) = \frac{t}{48(1+\alpha)^3(1+3\alpha)}\left\{B_1|2B_1-2B_2+B_3|(1+\alpha)^2\ t^3+B_1{}^4\ t^3\right. \\ &\quad \left. +2B_1|B_2-B_1|(1+\alpha)^3\ t^3+2B_1^2(1+\alpha)^2(4-t^2)\right\} \geq 0, \\ C_2 &= C_2(t) = \frac{B_1(4-t^2)t^2}{384(1+\alpha)^2(1+2\alpha)(1+3\alpha)}\left\{3B_1^2(1+3\alpha)\right. \\ &\quad \left. +8|B_2-B_1|(1+\alpha)^2(1+2\alpha)+B_1(1+\alpha)(1+2\alpha)\right\} \geq 0, \\ C_3 &= C_3(t) = \frac{B_1^2(4-t^2)t(t-2)}{96(1+\alpha)(1+3\alpha)} \leq 0, \\ C_4 &= C_4(t) = \frac{B_1^2(4-t^2)^2}{256(1+2\alpha)^2} \geq 0. \end{aligned}$$

Now, we need to maximize function $F(\lambda,\mu)$ in the closed square, $S=\{(\lambda,\mu):0\leq\lambda\leq1,0\leq\mu\leq1\}$. Since, coefficients of the function $F(\lambda,\mu)$ has dependent variable t, we need to maximize $F(\lambda,\mu)$ in the cases $t=0, t=2$ and $t\in(0,2)$.

1. Firstly, let $t=0$. Therefore, from (37), we write

$$F(\lambda,\mu)=\frac{B_1^2}{16(1+2\alpha)^2}(\lambda+\mu)^2.$$

We can see easily the maximum of function $F(\lambda,\mu)$ occurs at $\lambda=\mu=1$ and

$$\max\{F(\lambda,\mu):0\leq\lambda\leq1,0\leq\mu\leq1\}=\frac{B_1^2}{4(1+2\alpha)^2} \tag{38}$$

2. Secondly, let $t=2$. In this case, $F(\lambda,\mu)$ is a constant function as follows

$$F(\lambda,\mu)=\frac{|2B_1-2B_2+B_3|B_1(1+\alpha)^2+B_1{}^4+2B_1|B_2-B_1|}{3(1+\alpha)^3(1+3\alpha)}. \tag{39}$$

3. Thirdly, let $t\in(0,2)$. In this case, if we change $\lambda+\mu=\xi$ and $\lambda.\mu=\eta$, then

$$F(\lambda,\mu)=C_1(t)+C_2(t)\xi+[C_3(t)+C_4(t)]\xi^2-2C_3(t)\eta=G(\xi,\eta),\quad 0\leq\xi\leq2,0\leq\eta\leq1. \tag{40}$$

Now, we investigate maximum of $G(\xi,\eta)$ in $D=\{(\xi,\eta):0\leq\xi\leq2,0\leq\eta\leq1\}$.

From definition of function $G(\xi,\eta)$, we have

$$\begin{aligned} G'_\xi(\xi,\eta) &= C_2(t)+2[C_3(t)+C_4(t)]\xi=0, \\ G'_\eta(\xi,\eta) &= -2C_3(t)=0. \end{aligned}$$

From this, it is clear that, the function has no critical point in D. Thus, $F(\lambda,\mu)$ has no critical point in square S. Then, the function can not take maximum value in square S.

Now, we investigate maximum of $F(\lambda,\mu)$ on the boundary of the square S.

3.1. Firstly, let $\lambda=0, 0\leq\mu\leq1$ (similarly, $\mu=0,0\leq\lambda\leq1$). In this case, we write

$$F(0,\mu)=C_1(t)+C_2(t)\mu+[C_3(t)+C_4(t)]\mu^2=\varphi_1(\mu).$$

Then,

$$\varphi_1'(\mu)=C_2(t)+2[C_3(t)+C_4(t)]\mu.$$

Case (i) If $C_3(t)+C_4(t)\geq0$, then $\varphi_1'(\mu)>0$ and the function is increasing and the maximum occurs at $\mu=1$.

Case (ii) Let $C_3(t)+C_4(t)<0$. Since $C_2(t)+2[C_3(t)+C_4(t)]>0$, we have,

$$C_2(t)+2[C_3(t)+C_4(t)]\mu\geq C_2(t)+2[C_3(t)+C_4(t)]$$

is true for all $\mu \in [0,1]$. So, $\varphi_1'(\mu) > 0$. Therefore, $\varphi_1(\mu)$ is an increasing function. Thus, maximum occurs at $\mu = 1$,

$$\max\{F(0,\mu) : 0 \le \mu \le 1\} = C_1(t) + C_2(t) + C_3(t) + C_4(t). \tag{41}$$

3.2. Secondly, let $\lambda = 1, 0 \le \mu \le 1$ (similarly, $\mu = 1, 0 \le \lambda \le 1$). Then

$$F(1,\mu) = C_1(t) + C_2(t) + C_3(t) + C_4(t) + [C_2(t) + 2C_4(t)]\mu + [C_3(t) + C_4(t)]\mu^2 = \varphi_2(\mu).$$

We can show that $\varphi_2(\mu)$ is an increasing function as similar to previous case.
Therefore,

$$\max\{F(1,\mu) : 0 \le \mu \le 1\} = C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t). \tag{42}$$

Also, for every $t \in (0,2)$, we can see easily that

$$C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t) > C_1(t) + C_2(t) + C_3(t) + C_4(t).$$

Therefore we obtain,

$$\max\{F(\lambda,\mu) : 0 \le \lambda \le 1, 0 \le \mu \le 1\} = C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t).$$

Since $\varphi_1(1) \le \varphi_2(1)$ for $t \in [0,2], \max F(\lambda,\mu) = F(1,1)$ on the boundary of the square S. Thus the maximum of F occurs at $\lambda = 1$ and $\mu = 1$ in the closed square S.
Let us define $H : (0,2) \to \mathbb{R}$ as

$$H(t) = \max F(\lambda,\mu) = F(1,1) = C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t). \tag{43}$$

On substituting the value of $C_1(t), C_2(t), C_3(t)$ and $C_4(t)$ in the above function, we obtain

$$H(t) = \frac{B_1^2}{4(1+2\alpha)^2} + \frac{f(\alpha,B_1,B_2,B_3)\ t^4 + 2\ C(\alpha,B_1,B_2,B_3)\ t^2}{192(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)},$$

where

$$\begin{aligned} f(\alpha,B_1,B_2,B_3) \ &= \ 4B_1|2B_1 - 2B_2 + B_3|(1+\alpha)^2(1+2\alpha)^2 \\ &\quad +4B_1^4(1+2\alpha)^2 - 3B_1^3(1+\alpha)(1+2\alpha)(1+3\alpha) \\ &\quad -12B_1^2(1+\alpha)^2(1+2\alpha)^2 + 3B_1^2(1+\alpha)^3(1+3\alpha), \end{aligned}$$

$$\begin{aligned} C(\alpha,B_1,B_2,B_3) \ &= \ 3B_1^3(1+\alpha)(1+2\alpha)(1+3\alpha) + 8B_1|B_2 - B_1|(1+\alpha)^3(1+2\alpha)^2 \\ &\quad +12B_1^2(1+\alpha)^2(1+2\alpha)^2 - 6B_1^2(1+\alpha)^3(1+3\alpha). \end{aligned}$$

Now, we investigate the maximum value of $H(t)$ in the interval $(0,2)$.
By simple calculation, we obtain

$$H'(t) = \frac{[f(\alpha,B_1,B_2,B_3)t^3 + C(\alpha,B_1,B_2,B_3)]t}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}.$$

Let us examine the different cases of $f(\alpha,B_1,B_2,B_3)$ and $C(\alpha,B_1,B_2,B_3)$ as follows:
Case 1: Let $f(\alpha,B_1,B_2,B_3) \ge 0$ and $C(\alpha,B_1,B_2,B_3) \ge 0$, then $H'(t) \ge 0$, so the function is increasing. Thus, maximum point must be on the boundary of $t \in [0,2]$, that is, $t = 2$.
Thus,

$$\begin{aligned} &\max\{F(\lambda,\mu) : 0 \le \lambda \le 1, 0 \le \mu \le 1\} = H(2) \\ &= \frac{B_1|2B_1 - 2B_2 + B_3|(1+\alpha)^2(1+2\alpha)^2 + B_1^4(1+2\alpha)^2 + 2B_1|B_2 - B_1|(1+\alpha)^3(1+2\alpha)^2}{3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}. \end{aligned} \tag{44}$$

Case 2: If $f(\alpha, B_1, B_2, B_3) > 0$ and $C(\alpha, B_1, B_2, B_3) < 0, t_0 = \sqrt{\frac{-2C(\alpha,B_1,B_2,B_3)}{f(\alpha,B_1,B_2,B_3)}}$ is critical point of $H(t)$. Since $H''(t_0) < 0$, the maximum value of function $H(t)$ occurs at $t = t_0$ and

$$H(t_0) = \frac{B_1^2}{4(1+2\alpha)^2} - \frac{[C(\alpha, B_1, B_2, B_3)]^2}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)f(\alpha, B_1, B_2, B_3)}. \tag{45}$$

In this case,

$$H(t_0) < \frac{B_1^2}{4(1+2\alpha)^2}.$$

Therefore,

$$\max\{F(\lambda,\mu) : 0 \le \lambda \le 1, 0 \le \mu \le 1\} = \max\left\{\frac{B_1^2}{4(1+2\alpha)^2}, \frac{B_1|2B_1 - 2B_2 + B_3|(1+\alpha)^2(1+2\alpha)^2 + B_1^4(1+2\alpha)^2 + 2B_1|B_2 - B_1|(1+\alpha)^3(1+2\alpha)^2}{3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\right\}. \tag{46}$$

Case 3: If $f(\alpha, B_1, B_2, B_3) \le 0$ and $C(\alpha, B_1, B_2, B_3) \le 0, H(t)$ is a decreasing function on the interval $(0, 2)$. Thus,

$$\max\{F(\lambda,\mu) : 0 \le \lambda \le 1, 0 \le \mu \le 1\} = \frac{B_1^2}{4(1+2\alpha)^2}. \tag{47}$$

Case 4: If $f(\alpha, B_1, B_2, B_3) < 0$ and $C(\alpha, B_1, B_2, B_3) > 0, t_0$ is a critical point of $H(t)$. Since $H''(t_0) < 0$, the maximum value of $H(t)$ occurs at $t = t_0$. In this case,

$$\frac{B_1^2}{4(1+2\alpha)^2} < H(t_0).$$

Therefore,

$$\max\{F(\lambda,\mu) : 0 \le \lambda \le 1, 0 \le \mu \le 1\} = \max\left\{H(t_0), \frac{B_1|2B_1 - 2B_2 + B_3|(1+\alpha)^2(1+2\alpha)^2 + B_1^4(1+2\alpha)^2 + 2B_1|B_2 - B_1|(1+\alpha)^3(1+2\alpha)^2}{3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\right\}. \tag{48}$$

Thus, from (44),(46),(47) and (48), the proof is completed. □

Corollary 2.1. *Let f given by (1) be in the class $\mathcal{F}_\Sigma(\phi, \alpha)$ and $B_1 < 1, B_1 = 2|B_2|$. Then*

$$|a_2a_4 - a_3^2| \le \frac{B_1|2B_1 - 2B_2 + B_3|(1+\alpha)^2(1+2\alpha)^2 + B_1^4(1+2\alpha)^2 + 2B_1|B_2 - B_1|(1+\alpha)^3(1+2\alpha)^2}{3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}. \tag{49}$$

In particular, if $B_1 = 1/2, B_2 = 1/4$ and $B_3 = 1$, then

$$|a_2a_4 - a_3^2| \le \frac{1}{4(1+\alpha)(1+3\alpha)} + \frac{1}{48(1+\alpha)^3(1+3\alpha)} + \frac{1}{12(1+3\alpha)}. \tag{50}$$

Corollary 2.2. *Let f given by (1) be in the class $\mathcal{F}_{\Sigma}(\phi,\alpha)$ and $B_1<1, B_1\neq 2|B_2|$. Then*

$$|a_2a_4-a_3^2|\leq \max\left\{\frac{B_1^2}{4(1+2\alpha)^2}\,,\right.$$
$$\left.\frac{B_1|2B_1-2B_2+B_3|(1+\alpha)^2(1+2\alpha)^2+B_1^4(1+2\alpha)^2+2B_1|B_2-B_1|(1+\alpha)^3(1+2\alpha)^2}{3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\right\}. \tag{51}$$

In particular, if $B_1=1/2, B_2=1/2$ and $B_3=2$, then

$$|a_2a_4-a_3^2|\leq \frac{1}{3(1+\alpha)(1+3\alpha)}+\frac{1}{48(1+\alpha)^3(1+3\alpha)}.$$

Corollary 2.3. *Let f given by (1) be in the class $\mathcal{F}_{\Sigma}(\phi,\alpha)$ and $B_1\geq 1$. Then*

$$|a_2a_4-a_3^2|\leq \frac{B_1^2}{4(1+2\alpha)^2}.$$

In particular, if $B_1=1$, then

$$|a_2a_4-a_3^2|\leq \frac{1}{4(1+2\alpha)^2}.$$

Corollary 2.4. *Let f given by (1) be in the class $\mathcal{F}_{\Sigma}(\phi,\alpha)$ and $B_1\geq 1, B_1=2|B_2|$. Then*

$$|a_2a_4-a_3^2|\leq \max\left\{H(t_0),\right.$$
$$\left.\frac{B_1|2B_1-2B_2+B_3|(1+\alpha)^2(1+2\alpha)^2+B_1^4(1+2\alpha)^2+2B_1|B_2-B_1|(1+\alpha)^3(1+2\alpha)^2}{3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}\right\}. \tag{52}$$

In particular, if $B_1=1, B_2=1/2$ and $B_3=4$, then

$$|a_2a_4-a_3^2|\leq \frac{5}{3(1+\alpha)(1+3\alpha)}+\frac{1}{3(1+\alpha)^3(1+3\alpha)}+\frac{1}{3(1+3\alpha)}. \tag{53}$$

The following theorems are results of Theorem 2.1.

Theorem 2.2. *Let f given by (1) be in the class $\mathcal{S}^*_{\Sigma}(\phi)$.*
1. If B_1, B_2 and B_3 satisfy the conditions

$$4B_1^3-3B_1^2-B_1+4|2B_1-4B_2+B_3|-8|B_2|\geq 0, 3B_1^2-2B_1+8|B_2|\geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\leq \frac{B_1(B_1^3+|2B_1-4B_2+B_3|)}{3}.$$

2. If B_1, B_2 and B_3 satisfy the conditions

$$4B_1^3-3B_1^2-B_1+4|2B_1-4B_2+B_3|-8|B_2|>0, 3B_1^2-2B_1+8|B_2|<0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\leq \max\left\{\frac{B_1^2}{4},\frac{B_1(B_1^3+|2B_1-4B_2+B_3|)}{3}\right\}.$$

3. If B_1, B_2 and B_3 satisfy the conditions

$$4B_1^3-3B_1^2-B_1+4|2B_1-4B_2+B_3|-8|B_2|\leq 0, 3B_1^2-2B_1+8|B_2|\leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4}.$$

4. If B_1, B_2 and B_3 satisfy the conditions

$$4B_1^3 - 3B_1^2 - B_1 + 4|2B_1 - 4B_2 + B_3| - 8|B_2| < 0, 3B_1^2 - 2B_1 + 8|B_2| > 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \max\left\{\frac{B_1(B_1^3 + |2B_1 - 4B_2 + B_3|)}{3},\right.$$
$$\left.\frac{B_1^2}{4} - \frac{B_1(3B_1^2 - 2B_1 + 8|B_2|)^2}{48[4B_1^3 - 3B_1^2 - B_1 + 4|2B_1 - 4B_2 + B_3| - 8|B_2|]}\right\}.$$

Theorem 2.3. *Let f given by (1) be in the class $\mathcal{K}_\Sigma(\phi)$.*
1. If B_1, B_2 and B_3 satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| \geq 0, 3B_1^2 - 2B_1 + 12|B_2| \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1(B_1^3 + 4|2B_1 - 4B_2 + B_3|)}{96}.$$

2. If B_1, B_2 and B_3 satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| > 0, 3B_1^2 - 2B_1 + 12|B_2| < 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \max\left\{\frac{B_1^2}{36}, \frac{B_1(B_1^3 + 4|2B_1 - 4B_2 + B_3|)}{96}\right\}.$$

3. If B_1, B_2 and B_3 satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| \leq 0, 3B_1^2 - 2B_1 + 12|B_2| \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{36}.$$

4. If B_1, B_2 and B_3 satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| < 0, 3B_1^2 - 2B_1 + 12|B_2| > 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \max\left\{\frac{B_1(B_1^3 + 4|2B_1 - 4B_2 + B_3|)}{96},\right.$$
$$\left.\frac{B_1^2}{36} - \frac{B_1(3B_1^2 - 2B_1 + 12|B_2|)^2}{288[3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2|]}\right\}.$$

Corollary 2.5. *By choosing $\phi(z)$ of the form (??), we state the following results for functions $f \in \mathcal{F}_\Sigma(\phi,\alpha)$,*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{16(1-\beta)^4+4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}, & \beta \in [0, 1-\beta_0\ ] \\ \max\left\{H(t_0), \frac{16(1-\beta)^4+4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}\right\}, & \beta \in (1-\beta_0, 1)\,, \end{cases} \tag{54}$$

where
$\beta_0 = 1 - \frac{3(1+\alpha)(1+2\alpha)(1+3\alpha)-\sqrt{9(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)^2-16(1+2\alpha)^2[3(1+\alpha)^3(1+3\alpha)-8(1+\alpha)^2(1+2\alpha)^2]}}{16(1+2\alpha)^2}$,

$$\begin{aligned} H(t_0) &= \frac{(1-\beta)^2}{(1+2\alpha)^2} - \frac{[C(\alpha,\beta)]^2}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)f(\alpha,\beta)}, \\ f(\alpha,\beta) &= 4(1-\beta)^2\left\{16(1+2\alpha)^2(1-\beta)^2 - 6(1+\alpha)(1+2\alpha)(1+3\alpha)(1-\beta)\right. \\ &\quad \left. +3(1+\alpha)^3(1+3\alpha) - 8(1+\alpha)^2(1+2\alpha)^2\right\}, \\ C(\alpha,\beta) &= 24(1-\beta)^2\left\{(1+\alpha)(1+2\alpha)(1+3\alpha)(1-\beta)\right. \\ &\quad \left. +2(1+\alpha)^2(1+2\alpha)^2 - (1+\alpha)^3(1+3\alpha)]\right\}. \end{aligned}$$

Proof. Let $f \in \mathcal{F}_\Sigma(\phi,\alpha)$, with $\phi(z)$ of the form (??). We need to maximize function $F(\lambda,\mu)$, definition by the formula (37), in the closed square $S = \{(\lambda,\mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\}$. This proof will be completed as proof of Theorem 2.1.

1. For $t = 0$,

$$F(\lambda,\mu) = \frac{(1-\beta)^2}{4(1+2\alpha)^2}(\lambda+\mu)^2.$$

This function has no critical point in square S, so it has no maximum point. Then

$$\max\{F(\lambda,\mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = F(1,1) = \frac{(1-\beta)^2}{(1+2\alpha)^2}. \tag{55}$$

2. If $t = 2, F(\lambda,\mu)$ is a constant function: $F(\lambda,\mu) = C_1(2)$.
According to this,

$$\max\{F(\lambda,\mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \frac{16(1-\beta)^4 + 4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}. \tag{56}$$

3. Now let $t \in (0,2)$. In this case, $F(\lambda,\mu)$ will take a maximum value depend on t :

$$\max\{F(\lambda,\mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = H(t),$$

where $H(t)$ is given in (43). If we write $B_1 = B_2 = B_3 = 2(1-\beta)$ in value of $C_1(t), C_2(t), C_3(t), C_4(t)$ and we consider these in $H(t)$, we obtain

$$H(t) = \frac{(1-\beta)^2}{(1+2\alpha)^2} + \frac{f(\alpha,\beta)\ t^4 + 4\ C(\alpha,\beta)\ t^2}{192(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)},$$

where

$$\begin{aligned} f(\alpha,\beta) &= 4(1-\beta)^2\left\{16(1+2\alpha)^2(1-\beta)^2 - 6(1+\alpha)(1+2\alpha)(1+3\alpha)(1-\beta)\right. \\ &\quad \left. +3(1+\alpha)^3(1+3\alpha) - 8(1+\alpha)^2(1+2\alpha)^2\right\}, \\ C(\alpha,\beta) &= 24(1-\beta)^2\left\{(1+\alpha)(1+2\alpha)(1+3\alpha)(1-\beta)\right. \\ &\quad \left. +2(1+\alpha)^2(1+2\alpha)^2 - (1+\alpha)^3(1+3\alpha)]\right\}. \end{aligned}$$

Now, we investigate maximum of $H(t)$ in the open interval $(0, 2)$.
The derivative of $H(t)$ is as follows:

$$H'(t) = \frac{[f(\alpha,\beta)t^2 + 2C(\alpha,\beta)]t}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)}.$$

For all values of $\alpha \in [0,1]$ and $\beta \in [0,1), C(\alpha,\beta) > 0$. Moreover, for all $\alpha \in [0,1]$ and $\beta \in [0, 1-\beta_0], f(\alpha,\beta) \geq 0$. In here,

$\beta_0 = 1 - \frac{3(1+\alpha)(1+2\alpha)(1+3\alpha)-\sqrt{9(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)^2-16(1+2\alpha)^2[3(1+\alpha)^3(1+3\alpha)-8(1+\alpha)^2(1+2\alpha)^2]}}{16(1+2\alpha)^2}$, In this case, $H'(t) > 0$, so $H(t)$ is an increasing function in $(0, 2)$. However, this function doesn't take maximum value in $(0, 2)$.
Thus, for $\beta \in [0, 1-\beta_0]$,

$$\max\{F(\lambda,\mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \frac{16(1-\beta)^4 + 4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}. \tag{57}$$

If $\beta \in (1-\beta_0, 1], f(\alpha,\beta) < 0$. In this case,

$$t_0 = \sqrt{\frac{-2C(\alpha,\beta)}{f(\alpha,\beta)}}$$

is a critical point of $H(t)$. We observe that $t_0 < 2$, that is, t_0 is interior point of the interval $(0, 2)$. Since $H''(t_0) < 0$, the maximum value of $H(t)$ occurs at $t = t_0$ and

$$\max\{H(t) : 0 < t < 2\} = H(t_0) = \frac{(1-\beta)^2}{(1+2\alpha)^2} - \frac{[C(\alpha,\beta)]^2}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)f(\alpha,\beta)}.$$

In this case,

$$\frac{(1-\beta)^2}{(1+2\alpha)^2} < H(t_0).$$

Therefore,

$$\begin{aligned}\max\{F(\lambda,\mu) &: 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} \\ &= \max\left\{\frac{(1-\beta)^2}{(1+2\alpha)^2} - \frac{[C(\alpha,\beta)]^2}{48(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)f(\alpha,\beta)},\right. \\ &\qquad\qquad \left.\frac{16(1-\beta)^4 + 4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}\right\}. \end{aligned} \tag{58}$$

Thus, from (57) and (58), the proof is completed.

Corollary 2.6. *Taking $\alpha = 0$ and $\alpha = 1$ in the Corollary 2.5, we obtain the results for the classes $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$, which leads to the results obtained in Theorem 2.1 and 2.3 of* [6], *respectively.*

Corollary 2.7. *Putting $\beta = 0$ in the Corollary 2.6, we get the boundary estimates for the second Hankel determinant in the classes of bi-starlike and bi-convex functions as $|a_2a_4 - a_3^2| \leq 20/3$ and $|a_2a_4 - a_3^2| \leq 1/3$.*

The boundary estimates for the second Hankel determinant obtained in the Corollary 2.7 verifies to the Corollary 2.2 and 2.4 of [6], respectively.

3. Acknowledgment

We would like to thank the referee(s) for his comments and suggestions on the manuscript.

References

[1] Ali, R.M., Lee, S.K., Ravichandran, V., Supramaniam, S., (2012), Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl.Math. Lett, 25(3), pp.344–351.

[2] Altinkaya, S., Yalcin, S., Coefficient bounds for a subclass of bi-univalent functions, TWMS J. Pure Appl. Math., 6(2), 2015, pp.180-185.

[3] Brannan, D.A., Clunie(Eds.), J.G., (1980), Aspects of Contemporary Complex Analysis (Proceeding of the NATO Advanced study Institute held at the University of Durham, Durham: July 1-20, 1979), Academic Press, New York and London.

[4] Brannan, D.A., Taha, T.S., (1986), On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math., 31(2), pp.70-77.

[5] Deniz, E., (2013), Certain subclasses of bi-univalent functions satisfying subor dinate conditions, J.Class.Anal., 2(1), pp.49–60.

[6] Deniz, E., Caglar, M., Orhan, H., (2015), Second Hankel determinant for bi-starlike and bi-convex functions of order β, Appl. Math.Comp., 271, pp.301-307.

[7] Duren, P.L., (1983), Univalent Functions, in: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo.

[8] Fekete, M., Szegö,G., Eine Bemerkung ü, (1933), ber ungerade schlichte Funktionen, J. London. Math. Soc., 8, pp.85–89.

[9] Grenander, U. , Szegö,G.,(1958), Toeplitz Forms and Their Applications, Univ. of California Press, Berkeley and Los Angeles.

[10] Hummel, J., (1957), The coefficient regions of starlike functions, Pacific. J. Math., 7, pp.1381–1389.

[11] Hummel, J., (1960), Extremal problems in the class of starlike functions, Proc. Amer. Math. Soc., 11, pp.741–749.

[12] Keogh, F.R. Merkes, E.P., (1969), A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20, pp.8–12.

[13] Lewin, M., (1967), On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18, pp.63–68.

[14] Ma, W.C., Minda, D., (1992), A unified treatment of some special classes of functions, Proceedings of the Conference on Complex Analysis, Tianjin, pp.157-169, Conf. Proc.Lecture Notes Anal. 1. Int. Press, Cambridge, MA, 1994.

[15] Murugusundaramoorthy, G., Janani,T., Inclusion results associated with certain subclass of analytic functions involving calculus operator, TWMS J. Pure Appl. Math., 7(1), 2016, pp.63-75.

[16] Netanyahu, E., (1969), The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal., 32, pp.100-112.

[17] Noonan, J.W., Thomas, D.K., (1976), On the second Hankel determinant of areally mean $p-$valent functions, Trans. Amer. Math. Soc., 223(2), pp.337–346.

[18] Srivastava, H.M., Mishra, A.K., Gochhayat, P., (2010), Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23, pp.1188-1192.

[19] Taha, T.S., (1981), Topics in Univalent Function Theory, Ph.D Thesis, University of London.

[20] Xu, Q.-H, Gui, Y.-C., Srivastava, H.M., (2012), Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett., 25, pp.990–994.

[21] Xu, Q.-H., Xiao, H.-G., Srivastava, H.M., (2012), A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput., 218, pp.11461–11465.

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