SECOND HANKEL DETERMINANT FOR A CERTAIN SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

B.A. FRASIN, K. VIJAYA, M. KASTHURI

Abstract. In the present paper, we consider a subclass of the function class $\Sigma$ of bi-univalent analytic functions in the open unit disk $\Delta$ and we obtain the functional $j_{a_2 a_4 a_2}$ for the function class. Also we give upper bounds for $|a_2 a_4 - a_2^3|$. Our result gives corresponding functional for the subclasses of $\Sigma$ defined in the literature.

Keywords: Univalent functions, analytic functions, Bi-univalent functions, coefficient bounds, Hankel determinant.

AMS Subject Classification: 30C45

1. Introduction

Let $A$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $S$ denote the class of all functions in $A$ which are univalent in $\Delta$. Some of the important and well-investigated subclasses of the univalent function class $S$ include (for example) the class $S^\ast(\alpha)$ of starlike functions of order $\alpha$ $(0 \leq \alpha < 1)$ in $\Delta$ and the class $K(\alpha)$ of convex functions of order $\alpha$ $(0 \leq \alpha < 1)$ in $\Delta$. It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\Sigma$ denote the class of bi-univalent functions in $\Delta$ given by (1). Earlier, Brannan and Taha [4] introduced certain subclasses of bi-univalent function class $\Sigma$, namely bi-starlike functions of order $\alpha$ denoted by $S^\ast(\alpha)$ and bi-convex function of order $\alpha$ denoted by $K(\alpha)$ corresponding to the function classes $S^\ast(\alpha)$ and $K(\alpha)$ respectively.

A function $f \in A$ is in the class of strongly bi-starlike(and strongly bi-convex) functions $S^\ast(\alpha)$ and $K(\alpha)$ of order $\alpha$ [4, 19] of order $\alpha$ $(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad \text{and} \quad \left| \arg \left( \frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2}$$

1 Department of Mathematics, Al al-Bayt University, Mafraq, Jordan
2 School of Advanced Sciences, VIT University, Vellore, Tamilnadu, India
3 e-mail: bafrasin@yahoo.com, kvijaya@vit.ac.in, kasthuri.m@vit.ac.in
4 Manuscript received February 2015.
and

\[ \left| \arg \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right| < \frac{\alpha \pi}{2} \quad \text{and} \quad \left| \arg \left( 1 + \frac{w g''(w)}{g'(w)} \right) \right| < \frac{\alpha \pi}{2} \]

where \( g \) is the extension of \( f^{-1} \) to \( \Delta \). For each of the function classes \( S_\Sigma^c(\alpha) \) and \( K_\Sigma(\alpha) \) non-sharp estimates on the first two Taylor-Maclaurin coefficients \(|a_2|\) and \(|a_3|\) were found \([4, 19]\). But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

\[ |a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} := \{1, 2, 3, \ldots \}) \]

is still an open problem (see \([3, 4, 13, 16, 19]\)).

An analytic function \( f \) is subordinate to an analytic function \( g \), written \( f(z) \prec g(z) \), provided there is an analytic function \( w \) defined on \( \Delta \) with \( w(0) = 0 \) and \(|w(z)| < 1\) satisfying \( f(z) = g(w(z)) \). Ma and Minda \([14]\) unified various subclasses of starlike and convex functions for which either of the quantity \( \frac{z f''(z)}{f'(z)} \) or \( 1 + \frac{z f''(z)}{f'(z)} \) is subordinate to a more general superordinate function. For this purpose, they considered an analytic function \( \phi \) with positive real part in the unit disk \( \Delta \), \( \phi(0) = 1 \), \( \phi'(0) > 0 \) and \( \phi \) maps \( \Delta \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions \( f \in \mathcal{A} \) satisfying the subordination \( \frac{z f'(z)}{f(z)} \prec \phi(z) \). Similarly, the class of Ma-Minda convex functions consists of functions \( f \in \mathcal{A} \) satisfying the subordination \( 1 + \frac{z f''(z)}{f'(z)} \prec \phi(z) \).

A function \( f \) is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both \( f \) and \( f^{-1} \) are respectively Ma-Minda starlike or convex. These classes are denoted respectively by \( S_\Sigma^c(\phi) \) and \( K_\Sigma(\phi) \) (see \([1]\)). In the sequel, it is assumed that \( \phi \) is an analytic function with positive real part in the unit disk \( \Delta \), satisfying \( \phi(0) = 1 \), \( \phi'(0) > 0 \) and \( \phi(\Delta) \) is symmetric with respect to the real axis. Such a function has a series expansion of the form

\[ \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{3} \]

We need the following lemma for our investigation.

**Lemma 1.1.** (see \([7]\), p. 41) Let \( \mathcal{P} \) be the class of all analytic functions \( p(z) \) of the form

\[ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{4} \]

satisfying \( \Re(p(z)) > 0 \) \((z \in \Delta)\) and \( p(0) = 1 \). Then

\[ |p_n| \leq 2 \quad (n = 1, 2, 3, \ldots). \]

This inequality is sharp for each \( n \). In particular, equality holds for all \( n \) for the function

\[ p(z) = \frac{1 + z}{1 - z} = 1 + \sum_{n=1}^{\infty} 2 z^n. \]

In 1976, Noonan and Thomas \([17]\) defined the \( q \)th Hankel determinant of \( f \) for \( q \geq 1 \) by

\[ H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \]

Further, Fekete and Szegö \([8]\) considered the Hankel determinant of \( f \in \mathcal{A} \) for \( q = 2 \) and \( n = 1 \), \( H_2(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \). They made an early study for the estimates of \(|a_3 - \mu a_2^2|\) when
$a_1 = 1$ with $\mu$ real. The well known result due to them states that if $f \in \mathcal{A}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3 & \text{if } \mu \geq 1, \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if } \mu \leq 0. \end{cases}$$

Furthermore, Hummel [10, 11] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when $f$ is convex functions and also Keogh and Merkes [12] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when $f$ is close-to-convex, starlike and convex in $\Delta$.

Here we consider the Hankel determinant of $f \in \mathcal{A}$ for $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$ 

To prove our main result, we need the following lemmas.

**Lemma 1.2.** If the function $p(z) \in \mathcal{P}$ is given by the series (4), then

$$2p_2 = p_1^2 + x(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2(1 - p_1^2)p_1x - p_1(1 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some $x, z$ with $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

**Lemma 1.3.** [9] The power series for $p(z)$ given in (4) converges in $\Delta$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_1 & 2 & p_1 & \cdots & p_{n-1} \\ p_2 & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 2 & p_{n-1} \\ p_{n-1} & p_{n-1+1} & p_{n-1+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots$$

and $p_{-k} = \overline{p_k}$, are all nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k e^{i\mu_k z}, \quad \rho_k > 0, \quad \tau_k \text{ real}$$

and $\tau_k \neq \tau_j$ for $k \neq j$ in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

Many researchers (see [1, 2, 5, 18, 20, 21]) have introduced and investigated several interesting subclasses (by generalization or in associated with certain linear operators see [15]) of the bi-univalent function class $\Sigma$ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The object of the present paper is to determine the functional $|a_2a_4 - a_3^2|$ for a subclass of the function class $\Sigma$. Also we give upper bounds for $|a_2a_4 - a_3^2|$. Our result gives corresponding $|a_2a_4 - a_3^2|$ for the subclasses of $\Sigma$ defined in the literature.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $F_{\Sigma}(\phi, \alpha)$

For $\alpha \geq 0$ we let a function $f \in \Sigma$ given by (1) is said to be in the class $F_{\Sigma}(\phi, \alpha)$, if the following conditions are satisfied:

$$(1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec \phi(z)$$

and

$$(1 - \alpha) \left( \frac{wg'(w)}{g(w)} \right) + \alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) \prec \phi(w),$$

where $g$ is the inverse of $f$ given by (2).
Definition 2.1. Taking $\alpha = 0$, we let $\mathcal{F}_\Sigma(\phi, \alpha) \equiv \mathcal{S}_\Sigma(\phi)$ and if $f \in \mathcal{S}_\Sigma(\phi)$, then

$$\frac{zf'(z)}{f(z)} < \phi(z)$$

and

$$\frac{wg'(w)}{g(w)} < \phi(w),$$

where the function $g$ is the inverse of $f$ given by (2).

Definition 2.2. Taking $\alpha = 1$, we let $\mathcal{F}_\Sigma(\phi, \alpha) \equiv \mathcal{K}_\Sigma(\phi)$ and if $f \in \mathcal{K}_\Sigma(\phi)$, then

$$1 + \frac{zf''(z)}{f'(z)} < \phi(z)$$

and

$$1 + \frac{wg''(w)}{g'(w)} < \phi(w),$$

where the function $g$ is the inverse of $f$ given by (2).

Theorem 2.1. Let $f$ given by (1) be in the class $\mathcal{F}_\Sigma(\phi, \alpha)$. Then

$$|a_2 a_4 - a_3^2| \leq \begin{cases} 
H(2), & f(\alpha, B_1, B_2, B_3) \geq 0, C(\alpha, B_1, B_2, B_3) \geq 0 \\
\max \left\{ \frac{B_1^2}{4(1+2\alpha)^2}, H(2) \right\}, & f(\alpha, B_1, B_2, B_3) > 0, C(\alpha, B_1, B_2, B_3) < 0 \\
\frac{B_1^2}{4(1+2\alpha)^2}, & f(\alpha, B_1, B_2, B_3) \leq 0, C(\alpha, B_1, B_2, B_3) \leq 0 \\
\max \{H(t_0), H(2)\}, & f(\alpha, B_1, B_2, B_3) < 0, C(\alpha, B_1, B_2, B_3) > 0,
\end{cases}$$

where

$$H(2) = B_1|2B_1 - 2B_2 + B_3|(1 + \alpha)^2(1 + 2\alpha)^2 + B_1^4(1 + 2\alpha)^2 + 2B_1|B_2 - B_1(1 + \alpha)^3(1 + 2\alpha)|$$

$$= 3(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)$$

and

$$H(t_0) = \sqrt{\frac{2C(\alpha, B_1, B_2, B_3)}{f(\alpha, B_1, B_2, B_3)}} = \frac{B_1^2}{4(1+2\alpha)^2} - \frac{[C(\alpha, B_1, B_2, B_3)]^2}{48(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.$$ 

Proof. Let $f \in \mathcal{F}_\Sigma(\phi, \alpha)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \Delta \to \Delta$, with $u(0) = v(0) = 0$, satisfying

$$\left(1 - \alpha\right) \left(\frac{zf'(z)}{f(z)}\right) + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) = \phi(u(z))$$

and

$$\left(1 - \alpha\right) \left(\frac{wg'(w)}{g(w)}\right) + \alpha \left(1 + \frac{wg''(w)}{g'(w)}\right) = \phi(v(w)).$$
Define the functions \( p(z) \) and \( q(z) \) by
\[
p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots
\]
and
\[
q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots.
\]
It follows that,
\[
u(z) := \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) z^3 + \cdots\right] \tag{17}
\]
and
\[
v(z) := \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) z^3 + \cdots\right]. \tag{18}
\]
Then \( p(z) \) and \( q(z) \) are analytic in \( \Delta \) with \( p(0) = 1 = q(0) \). Using (17) and (18), it is clear that,
\[
\phi(u(z)) = 1 + \frac{B_1 p_1}{2} z + \left[B_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2\right] z^2
\]
\[+ \left[B_1 \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + B_2 p_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_3 p_1^3}{8}\right] z^3 + \cdots \tag{19}
\]
and
\[
\phi(v(z)) = 1 + \frac{B_1 q_1}{2} w + \left[B_1 \left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4} B_2 q_1^2\right] w^2
\]
\[+ \left[B_1 \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) + B_2 q_1 \left(q_2 - \frac{q_1^2}{2}\right) + \frac{B_3 q_1^3}{8}\right] w^3 + \cdots. \tag{20}
\]
Equating the coefficients in (15) and (16), we get,
\[
(1 + \alpha) a_2 = \frac{B_1 p_1}{2}, \tag{21}
\]
\[
2(1 + 2\alpha) a_3 - (1 + 3\alpha) a_2^2 = \frac{B_1}{2} \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2, \tag{22}
\]
\[
3(1 + 3\alpha) a_4 - 3(1 + 5\alpha) a_2 a_3 + (1 + 7\alpha) a_2^3
\]
\[= \frac{B_1}{2} \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + B_2 p_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_3 p_1^3}{8} \tag{23}
\]
and
\[
-(1 + \alpha) a_2 = \frac{B_1 q_1}{2}, \tag{24}
\]
\[
-2(1 + 2\alpha) a_3 + (3 + 5\alpha) a_2^2 = \frac{B_1}{2} \left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4} B_2 q_1^2, \tag{25}
\]
\[
-3(1 + 3\alpha) a_4 + 6(2 + 5\alpha) a_2 a_3 - 2(5 + 11\alpha) a_2^3
\]
\[= \frac{B_1}{2} \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) + B_2 q_1 \left(q_2 - \frac{q_1^2}{2}\right) + \frac{B_3 q_1^3}{8}. \tag{26}
\]
From (21) and (24) gives
\[
a_2 = \frac{B_1 p_1}{2(1 + \alpha)} = - \frac{B_1 q_1}{2(1 + \alpha)}. \tag{27}
\]
which implies
\[ p_1 = -q_1. \] (28)

Now from (22), (25) and by using (27), we obtain
\[ a_3 = \frac{B_4^3 p_4}{4(1 + \alpha)^2} + \frac{B_4^3 p_1 (p_2 - q_2)}{8(1 + 2\alpha)}. \] (29)

On the other hand, subtracting (26) from (23) and by using (27), (29), we get
\[
a_4 = \frac{5B_3^3 p_3}{16(1 + \alpha)^3} + \frac{5B_3^3 p_1 (p_2 - q_2)}{32(1 + \alpha)(1 + 2\alpha)} + \frac{B_4^1 p_1 (p_3 - q_3)}{12(1 + 3\alpha)}
+ \frac{B_3^3 (p_1 - 2B_2 + B_3)}{24(1 + 3\alpha)} + \frac{(B_2 - B_1) p_1 (p_2 + q_2)}{12(1 + 3\alpha)} - \frac{p_1^3 B_3^3 (11 + 29\alpha)}{48(1 + \alpha)^3}. \] (30)

Thus we establish that
\[
a_{2a4} - a_3^3 = \frac{2B_1 (B_1 - 2B_2 + B_3)(1 + \alpha)^3}{96(1 + \alpha)^3(1 + 3\alpha)} - \frac{2(1 + \alpha) B_4^1 p_1 - B_4^2 (p_2 - q_2)^2}{64(1 + 2\alpha)^2}
+ \frac{B_3^3 p_1 (p_2 - q_2)}{64(1 + \alpha)^2(1 + 2\alpha)} + \frac{B_4^1 (B_2 - B_1) p_1^3 (p_2 + q_2)}{24(1 + 3\alpha)} + \frac{B_3^3 p_1 (p_3 - q_3)}{24(1 + \alpha)(1 + 3\alpha)}. \] (31)

According to Lemma 1.2, we have
\[ 2p_2 = p_1^2 + x(4 - p_1^2) \quad \text{and} \quad 2q_2 = q_1^2 + y(4 - q_1^2), \] (32)

hence by (28), we have
\[
p_2 - q_2 = \frac{4 - p_1^2}{2} (x - y) \] (33)
\[
p_2 + q_2 = p_1^2 + \frac{4 - p_1^2}{2} (x + y) \] (34)

and further
\[
4p_3 = p_1^4 + 2(4 - p_1^2)p_1 x - p_1 (4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,
4q_3 = q_1^4 + 2(4 - q_1^2)q_1 y - q_1 (4 - q_1^2)y^2 + 2(4 - q_1^2)(1 - |y|^2)w,
\]

for some \( x, y, z, w \) with \( |x| \leq 1, |y| \leq 1, |z| \leq 1, |w| \leq 1 \) and \( p_1, q_1 \in [0, 2] \).

Thus,
\[
4p_3 - q_3 = \frac{p_1^4}{2} + \frac{p_1 (4 - p_1^2)}{2} (x + y) - \frac{p_1 (4 - p_1^2)}{4} (x^2 + y^2) + \frac{4 - p_1^2}{2} [(1 - |x|^2)z - (1 - |y|^2)w]. \] (35)

Using (33) - (35) in (31), we get,
\[
a_{2a4} - a_3^3 = \left( \frac{B_1 (B_2 - B_1 - 2B_2 + B_3)(1 + \alpha)^3}{48(1 + \alpha)^3(1 + 3\alpha)} - \frac{B_1^2 (B_2 - B_1 - 2B_2 + B_3)(1 + \alpha)^3}{48(1 + \alpha)^3(1 + 3\alpha)} \right) p_1^4 + \frac{B_3^3 p_1 (p_2 - q_2)}{64(1 + \alpha)^2(1 + 2\alpha)} + \frac{(B_2 - B_1) p_1^3 (p_2 + q_2)}{24(1 + 3\alpha)} + \frac{B_3^3 p_1 (p_3 - q_3)}{24(1 + \alpha)(1 + 3\alpha)}. \] (36)

Since \( p(z) \in P, \) so \( |p_1| \leq 2. \) Thus, letting \( p_1 = t \) and applying triangle inequality on (36), with \( \lambda = |x| \leq 1 \) and \( \mu = |y| \leq 1, \) we obtain
\[
|a_{2a4} - a_3^3| \leq C_1 + C_2(\lambda + \mu) + C_3(\lambda^2 + \mu^2) + C_4(\lambda + \mu)^2 = F(\lambda, \mu), \] (37)
where
\[ C_1 = C_1(t) = \frac{t}{48(1 + \alpha)^3(1 + 3\alpha)} \{ B_1(2B_1 - 2B_2 + B_3)((1 + \alpha)^2 t^3 + B_1^4 t^3 \\
+ 2B_1|B_2 - B_1|(1 + \alpha)^3 t^3 + 2B_1^2(1 + \alpha)^2(4 - t^2) \} \geq 0, \]
\[ C_2 = C_2(t) = \frac{B_1(4 - t^2)t^2}{384(1 + \alpha)^2(1 + 2\alpha)(1 + 3\alpha)} \{ 3B_1^2(1 + 3\alpha) \\
+ 8|B_2 - B_1|(1 + \alpha)^2(1 + 2\alpha) + B_1(1 + \alpha)(1 + 2\alpha) \} \geq 0, \]
\[ C_3 = C_3(t) = \frac{B_1^2(4 - t^2)^2 t(t - 2)}{96(1 + \alpha)(1 + 3\alpha)} \leq 0, \]
\[ C_4 = C_4(t) = \frac{B_1^2(4 - t^2)^2}{256(1 + 2\alpha)^2} \geq 0. \]

Now, we need to maximize function \( F(\lambda, \mu) \) in the closed square, \( S = \{ (\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1 \} \). Since, coefficients of the function \( F(\lambda, \mu) \) has dependent variable \( t \), we need to maximize \( F(\lambda, \mu) \) in the cases \( t = 0, t = 2 \) and \( t \in (0, 2) \).

1. Firstly, let \( t = 0 \). Therefore, from (37), we write
\[ F(\lambda, \mu) = \frac{B_1^2}{16(1 + 2\alpha)^2}(\lambda + \mu)^2. \]
We can see easily the maximum of function \( F(\lambda, \mu) \) occurs at \( \lambda = \mu = 1 \) and
\[ \max \{ F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1 \} = \frac{B_1^2}{4(1 + 2\alpha)^2} \quad (38) \]

2. Secondly, let \( t = 2 \). In this case, \( F(\lambda, \mu) \) is a constant function as follows
\[ F(\lambda, \mu) = \frac{|2B_1 - 2B_2 + B_3|B_1(1 + \alpha)^2 + B_1^4 + 2B_1|B_2 - B_1|}{3(1 + \alpha)^3(1 + 3\alpha)}. \quad (39) \]

3. Thirdly, let \( t \in (0, 2) \). In this case, if we change \( \lambda + \mu = \xi \) and \( \lambda, \mu = \eta \), then
\[ F(\lambda, \mu) = C_1(t) + C_2(t)\xi + [C_3(t) + C_4(t)]\xi^2 - 2C_3(t)\eta = G(\xi, \eta), \quad 0 \leq \xi \leq 2, 0 \leq \eta \leq 1. \quad (40) \]

Now, we investigate maximum of \( G(\xi, \eta) \) in \( D = \{ (\xi, \eta) : 0 \leq \xi \leq 2, 0 \leq \eta \leq 1 \} \).

From definition of function \( G(\xi, \eta) \), we have
\[ G'_{\xi}(\xi, \eta) = C_2(t) + 2[C_3(t) + C_4(t)]\xi = 0, \]
\[ G'_{\eta}(\xi, \eta) = -2C_3(t) = 0. \]

From this, it is clear that, the function has no critical point in \( D \). Thus, \( F(\lambda, \mu) \) has no critical point in square \( S \). Then, the function can not take maximum value in square \( S \).

Now, we investigate maximum of \( F(\lambda, \mu) \) on the boundary of the square \( S \).

3.1. Firstly, let \( \lambda = 0, 0 \leq \mu \leq 1 \) (similarly, \( \mu = 0, 0 \leq \lambda \leq 1 \)). In this case, we write
\[ F(0, \mu) = C_1(t) + C_2(t)\mu + [C_3(t) + C_4(t)]\mu^2 = \varphi_1(\mu). \]
Then,
\[ \varphi'_1(\mu) = C_2(t) + 2[C_3(t) + C_4(t)]\mu. \]
Case (i) If \( C_3(t) + C_4(t) \geq 0 \), then \( \varphi'_1(\mu) > 0 \) and the function is increasing and the maximum occurs at \( \mu = 1 \).

Case (ii) Let \( C_3(t) + C_4(t) < 0 \). Since \( C_2(t) + 2[C_3(t) + C_4(t)] > 0 \), we have,
\[ C_2(t) + 2[C_3(t) + C_4(t)]\mu \geq C_2(t) + 2[C_3(t) + C_4(t)] \]
is true for all \( \mu \in [0, 1] \). So, \( \varphi_1'(\mu) > 0 \). Therefore, \( \varphi_1(\mu) \) is an increasing function. Thus, maximum occurs at \( \mu = 1 \),
\[
\max\{F(0, \mu) : 0 \leq \mu \leq 1\} = C_1(t) + C_2(t) + C_3(t) + C_4(t).
\]

3.2. Secondly, let \( \lambda = 1, 0 \leq \mu \leq 1 \) (similarly, \( \mu = 1, 0 \leq \lambda \leq 1 \)). Then
\[
F(1, \mu) = C_1(t) + C_2(t) + C_3(t) + C_4(t) + [C_2(t) + 2C_4(t)]\mu + [C_3(t) + C_4(t)]\mu^2 = \varphi_2(\mu).
\]
We can show that \( \varphi_2(\mu) \) is an increasing function as similar to previous case. Therefore,
\[
\max\{F(1, \mu) : 0 \leq \mu \leq 1\} = C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t).
\]
Also, for every \( t \in (0, 2) \), we can see easily that
\[
C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t) > C_1(t) + C_2(t) + C_3(t) + C_4(t).
\]
Therefore we obtain,
\[
\max\{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t).
\]
Since \( \varphi_1(1) \leq \varphi_2(1) \) for \( t \in [0, 2] \), \( \max F(\lambda, \mu) = F(1, 1) \) on the boundary of the square \( S \). Thus the maximum of \( F \) occurs at \( \lambda = 1 \) and \( \mu = 1 \) in the closed square \( S \).

Let us define \( H : (0, 2) \to \mathbb{R} \) as
\[
H(t) = \max F(\lambda, \mu) = F(1, 1) = C_1(t) + 2[C_2(t) + C_3(t)] + 4C_4(t).
\]
On substituting the value of \( C_1(t), C_2(t), C_3(t) \) and \( C_4(t) \) in the above function, we obtain
\[
H(t) = \frac{B_1^2}{4(1 + 2\alpha)^2} + \frac{f(\alpha, B_1, B_2, B_3)}{192(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)} t^2,
\]
where
\[
f(\alpha, B_1, B_2, B_3) = 4B_1|2B_1 - 2B_2 + B_3|(1 + \alpha)^2(1 + 2\alpha)^2 + 4B_1^4(1 + 2\alpha)^2 - 3B_3^2(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha) - 12B_1^2(1 + \alpha)^2(1 + 2\alpha)^2 + 3B_3^2(1 + \alpha)^3(1 + 3\alpha),
\]
\[
C(\alpha, B_1, B_2, B_3) = 3B_3^3(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha) + 8B_1|B_2 - B_1|(1 + \alpha)^3(1 + 2\alpha)^2 + 12B_1^2(1 + \alpha)^2(1 + 2\alpha)^2 - 6B_3^2(1 + \alpha)^3(1 + 3\alpha).
\]
Now, we investigate the maximum value of \( H(t) \) in the interval \((0, 2)\).

By simple calculation, we obtain
\[
H'(t) = \frac{[f(\alpha, B_1, B_2, B_3)t^3 + C(\alpha, B_1, B_2, B_3)]t}{48(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.
\]
Let us examine the different cases of \( f(\alpha, B_1, B_2, B_3) \) and \( C(\alpha, B_1, B_2, B_3) \) as follows:

**Case 1:** Let \( f(\alpha, B_1, B_2, B_3) \geq 0 \) and \( C(\alpha, B_1, B_2, B_3) \geq 0 \), then \( H'(t) \geq 0 \), so the function is increasing. Thus, maximum point must be on the boundary of \( t \in [0, 2] \), that is, \( t = 2 \).

Thus,
\[
\max\{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = H(2) = \frac{B_3^4(2B_1 - 2B_2 + B_3)^2(1 + \alpha)^2 + 2B_1^4(1 + 2\alpha)^2 + 2B_1|B_2 - B_1|(1 + \alpha)^3(1 + 2\alpha)^2}{3(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.
\]
Case 2: If \( f(\alpha, B_1, B_2, B_3) > 0 \) and \( C(\alpha, B_1, B_2, B_3) < 0 \), \( t_0 = \sqrt{-\frac{2C(\alpha, B_1, B_2, B_3)}{f(\alpha, B_1, B_2, B_3)}} \) is critical point of \( H(t) \). Since \( H''(t_0) < 0 \), the maximum value of function \( H(t) \) occurs at \( t = t_0 \) and

\[
H(t_0) = \frac{B_1^2}{4(1 + 2\alpha)^2} - \frac{|C(\alpha, B_1, B_2, B_3)|^2}{48(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)f(\alpha, B_1, B_2, B_3)}.
\]

In this case,

\[ H(t_0) < \frac{B_1^2}{4(1 + 2\alpha)^2}. \]

Therefore,

\[
\max\{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \max \left\{ \frac{B_1^2}{4(1 + 2\alpha)^2}, \right. \\
\frac{B_1|2B_1 - 2B_2 + B_3|(1 + \alpha)^2(1 + 2\alpha)^2 + B_1^3(1 + 2\alpha)^2 + 2B_1|B_2 - B_1|(1 + \alpha)^3(1 + 2\alpha)^2}{3(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)} \left. \right\}.
\]

Case 3: If \( f(\alpha, B_1, B_2, B_3) \leq 0 \) and \( C(\alpha, B_1, B_2, B_3) \leq 0 \), \( H(t) \) is a decreasing function on the interval \( (0, 2) \). Thus,

\[
\max\{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \frac{B_1^2}{4(1 + 2\alpha)^2}.
\]

Case 4: If \( f(\alpha, B_1, B_2, B_3) < 0 \) and \( C(\alpha, B_1, B_2, B_3) > 0 \), \( t_0 \) is a critical point of \( H(t) \). Since \( H''(t_0) < 0 \), the maximum value of \( H(t) \) occurs at \( t = t_0 \). In this case,

\[
\frac{B_1^2}{4(1 + 2\alpha)^2} < H(t_0).
\]

Therefore,

\[
\max\{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \max \{H(t_0) \},
\]

\[
\frac{B_1|2B_1 - 2B_2 + B_3|(1 + \alpha)^2(1 + 2\alpha)^2 + B_1^3(1 + 2\alpha)^2 + 2B_1|B_2 - B_1|(1 + \alpha)^3(1 + 2\alpha)^2}{3(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.
\]

Thus, from (44),(46),(47) and (48), the proof is completed. \( \square \)

**Corollary 2.1.** Let \( f \) given by (1) be in the class \( F_{\Sigma}(\phi, \alpha) \) and \( B_1 < 1, B_1 = 2|B_2| \). Then

\[
|a_2a_4 - a_3^2| \leq \frac{B_1|2B_1 - 2B_2 + B_3|(1 + \alpha)^2(1 + 2\alpha)^2 + B_1^3(1 + 2\alpha)^2 + 2B_1|B_2 - B_1|(1 + \alpha)^3(1 + 2\alpha)^2}{3(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.
\]

In particular, if \( B_1 = 1/2, B_2 = 1/4 \) and \( B_3 = 1 \), then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{4(1 + \alpha)(1 + 3\alpha)} + \frac{1}{48(1 + \alpha)^3(1 + 3\alpha)} + \frac{1}{12(1 + 3\alpha)}.
\]
Corollary 2.2. Let \( f \) given by (1) be in the class \( \mathcal{F}_2(\phi, \alpha) \) and \( B_1 < 1, B_1 \neq 2|B_2| \). Then

\[
|a_2a_4 - a_3^2| \leq \max \left\{ \frac{B_1^2}{4(1 + 2\alpha)^2}, \frac{B_1|2B_1 - 2B_2 + B_3|(1 + \alpha)^2(1 + 2\alpha)^2 + B_1^3(1 + 2\alpha)^2 + 2B_1|B_2 - B_1|(1 + \alpha)^3(1 + 2\alpha)^2}{3(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)} \right\}.
\]

(51)

In particular, if \( B_1 = 1/2, B_2 = 1/2 \) and \( B_3 = 2 \), then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{3(1 + \alpha)(1 + 3\alpha)} + \frac{1}{48(1 + \alpha)^3(1 + 3\alpha)}.
\]

Corollary 2.3. Let \( f \) given by (1) be in the class \( \mathcal{F}_2(\phi, \alpha) \) and \( B_1 \geq 1 \). Then

\[
|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4(1 + 2\alpha)^2}.
\]

In particular, if \( B_1 = 1 \), then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{4(1 + 2\alpha)^2}.
\]

Corollary 2.4. Let \( f \) given by (1) be in the class \( \mathcal{F}_2(\phi, \alpha) \) and \( B_1 \geq 1, B_1 = 2|B_2| \). Then

\[
|a_2a_4 - a_3^2| \leq \max \left\{ H(t_0), \frac{B_1|2B_1 - 2B_2 + B_3|(1 + \alpha)^2(1 + 2\alpha)^2 + B_1^3(1 + 2\alpha)^2 + 2B_1|B_2 - B_1|(1 + \alpha)^3(1 + 2\alpha)^2}{3(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)} \right\}.
\]

(52)

In particular, if \( B_1 = 1, B_2 = 1/2 \) and \( B_3 = 4 \), then

\[
|a_2a_4 - a_3^2| \leq \frac{5}{3(1 + \alpha)(1 + 3\alpha)} + \frac{1}{3(1 + \alpha)^3(1 + 3\alpha)} + \frac{1}{3(1 + 3\alpha)}.
\]

(53)

The following theorems are results of Theorem 2.1.

Theorem 2.2. Let \( f \) given by (1) be in the class \( \mathcal{S}_2(\phi) \).

1. If \( B_1, B_2 \) and \( B_3 \) satisfy the conditions

\[
4B_1^3 - 3B_1^2 - B_1 + 4|2B_1 - 4B_2 + B_3| - 8|B_2| \geq 0, 3B_1^2 - 2B_1 + 8|B_2| \geq 0,
\]

then the second Hankel determinant satisfies

\[
|a_2a_4 - a_3^2| \leq \frac{B_1(B_1^3 + |2B_1 - 4B_2 + B_3|)}{3}.
\]

2. If \( B_1, B_2 \) and \( B_3 \) satisfy the conditions

\[
4B_1^3 - 3B_1^2 - B_1 + 4|2B_1 - 4B_2 + B_3| - 8|B_2| > 0, 3B_1^2 - 2B_1 + 8|B_2| < 0,
\]

then the second Hankel determinant satisfies

\[
|a_2a_4 - a_3^2| \leq \max \left\{ \frac{B_1^2}{4}, \frac{B_1(B_1^3 + |2B_1 - 4B_2 + B_3|)}{3} \right\}.
\]

3. If \( B_1, B_2 \) and \( B_3 \) satisfy the conditions

\[
4B_1^3 - 3B_1^2 - B_1 + 4|2B_1 - 4B_2 + B_3| - 8|B_2| \leq 0, 3B_1^2 - 2B_1 + 8|B_2| \leq 0,
\]
then the second Hankel determinant satisfies

$$|a_{2a4} - a_{33}| \leq \frac{B_1^2}{4}.$$ 

4. If $B_1, B_2$ and $B_3$ satisfy the conditions

$$4B_1^3 - 3B_1^2 - B_1 + 4|2B_1 - 4B_2 + B_3| - 8|B_2| < 0, 3B_1^2 - 2B_1 + 8|B_2| > 0,$$

then the second Hankel determinant satisfies

$$|a_{2a4} - a_{33}| \leq \max \left\{ \frac{B_1(B_1^2 + |2B_1 - 4B_2 + B_3|)}{3}, \frac{B_1^2}{4} - \frac{B_1(3B_1^2 - 2B_1 + 8|B_2|)^2}{48[4B_1^2 - 3B_1^2 - B_1 + 4|2B_1 - 4B_2 + B_3| - 8|B_2|]} \right\}.$$ 

**Theorem 2.3.** Let $f$ given by (1) be in the class $K_\Sigma(\phi)$.

1. If $B_1, B_2$ and $B_3$ satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| \geq 0, 3B_1^2 - 2B_1 + 12|B_2| \geq 0,$$

then the second Hankel determinant satisfies

$$|a_{2a4} - a_{33}| \leq \frac{B_1(4B_1 - 4B_2 + B_3)}{96}.$$ 

2. If $B_1, B_2$ and $B_3$ satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| > 0, 3B_1^2 - 2B_1 + 12|B_2| < 0,$$

then the second Hankel determinant satisfies

$$|a_{2a4} - a_{33}| \leq \max \left\{ \frac{B_1^2}{36}, \frac{B_1(B_1^2 + 4|2B_1 - 4B_2 + B_3|)}{96} \right\}.$$ 

3. If $B_1, B_2$ and $B_3$ satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| \leq 0, 3B_1^2 - 2B_1 + 12|B_2| \leq 0,$$

then the second Hankel determinant satisfies

$$|a_{2a4} - a_{33}| \leq \frac{B_1^2}{36}.$$ 

4. If $B_1, B_2$ and $B_3$ satisfy the conditions

$$3B_1^3 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2| < 0, 3B_1^2 - 2B_1 + 12|B_2| > 0,$$

then the second Hankel determinant satisfies

$$|a_{2a4} - a_{33}| \leq \max \left\{ \frac{B_1(B_1^2 + 4|2B_1 - 4B_2 + B_3|)}{96}, \frac{B_1^2}{36} - \frac{B_1(3B_1^2 - 2B_1 + 12|B_2|)^2}{288[3B_1^2 - 6B_1^2 - 4B_1 + 12|2B_1 - 4B_2 + B_3| - 24|B_2|]} \right\}.$$
Corollary 2.5. By choosing \( \phi(z) \) of the form \((??)\), we state the following results for functions \( f \in \mathcal{F}_\Sigma(\phi, \alpha) \),

\[
|a_2a_4 - a_3^2| \leq \begin{cases} 
\frac{16(1-\beta)^4 + 4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}, & \beta \in [0, 1 - \beta_0] \\
\max \left\{ H(t_0), \frac{16(1-\beta)^4 + 4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)} \right\}, & \beta \in (1 - \beta_0, 1), 
\end{cases}
\]  

where \( \beta_0 = 1 - \frac{3(1+\alpha)(1+2\alpha)(1+3\alpha) - \sqrt{9(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)^2 - 16(1+2\alpha)^2(1+3\alpha)^4(1+3\alpha)^2 - 8(1+\alpha)^2(1+2\alpha)^2}}{16(1+2\alpha)^2} \).

Proof. Let \( f \in \mathcal{F}_\Sigma(\phi, \alpha) \), with \( \phi(z) \) of the form \((??)\). We need to maximize function \( F(\lambda, \mu) \), definition by the formula \((37)\), in the closed square \( S = \{(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} \). This proof will be completed as proof of Theorem 2.1.

1. For \( t = 0 \),

\[
F(\lambda, \mu) = \frac{(1-\beta)^2}{(1+2\alpha)^2}(\lambda + \mu)^2.
\]

This function has no critical point in square \( S \), so it has no maximum point. Then

\[
\max \{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = F(1, 1) = \frac{(1-\beta)^2}{(1+2\alpha)^2}.
\]  

(55)

2. If \( t = 2 \), \( F(\lambda, \mu) \) is a constant function: \( F(\lambda, \mu) = C_1(2) \).

According to this,

\[
\max \{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \frac{16(1-\beta)^4 + 4(1+\alpha)^2(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)}.
\]  

(56)

3. Now let \( t \in (0, 2) \). In this case, \( F(\lambda, \mu) \) will take a maximum value depend on \( t \):

\[
\max \{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = H(t),
\]

where \( H(t) \) is given in \((43)\). If we write \( B_1 = B_2 = B_3 = 2(1-\beta) \) in value of \( C_1(t), C_2(t), C_3(t), C_4(t) \) and we consider these in \( H(t) \), we obtain

\[
H(t) = \frac{(1-\beta)^2}{(1+2\alpha)^2} + \frac{f(\alpha, \beta) t^4 + 4 C(\alpha, \beta) t^2}{192(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)},
\]

where

\[
f(\alpha, \beta) = 4(1-\beta)^2 \left\{ 16(1+2\alpha)^2(1-\beta)^2 - 6(1+\alpha)(1+2\alpha)(1+3\alpha)(1-\beta) + 3(1+\alpha)^3(1+3\alpha) - 8(1+\alpha)^2(1+2\alpha)^2 \right\},
\]

\[
C(\alpha, \beta) = 24(1-\beta)^2 \left\{ (1+\alpha)(1+2\alpha)(1+3\alpha)(1-\beta) + 2(1+\alpha)^2(1+2\alpha)^2 - (1+\alpha)^3(1+3\alpha) \right\}.
\]
Now, we investigate maximum of \( H(t) \) in the open interval \((0, 2)\). The derivative of \( H(t) \) is as follows:
\[
H'(t) = \frac{[f(\alpha, \beta)t^2 + 2C(\alpha, \beta)]t}{48(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.
\]
For all values of \( \alpha \in [0, 1] \) and \( \beta \in [0, 1], C(\alpha, \beta) > 0 \). Moreover, for all \( \alpha \in [0, 1] \) and \( \beta \in [0, 1 - \beta_0] \), \( f(\alpha, \beta) \geq 0 \). In here,
\[
\beta_0 = 1 - \frac{3(1+\alpha)(1+2\alpha)(1+3\alpha)-\sqrt{9(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)^2-16(1+2\alpha)^2[3(1+\alpha)^3(1+3\alpha)-8(1+\alpha)^2(1+2\alpha)^2]}16(1+2\alpha)^2}{16(1+2\alpha)^2},
\]
In this case, \( H'(t) > 0 \), so \( H(t) \) is an increasing function in \((0, 2)\). However, this function doesn’t take maximum value in \((0, 2)\).

Thus, for \( \beta \in [0, 1 - \beta_0] \),
\[
\max\{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \frac{16(1 - \beta)^4 + 4(1 + \alpha)^2(1 - \beta)^2}{3(1 + \alpha)^3(1 + 3\alpha)}.
\]  \( (57) \)

If \( \beta \in (1 - \beta_0, 1] \), \( f(\alpha, \beta) < 0 \). In this case,
\[
t_0 = \sqrt{\frac{-2C(\alpha, \beta)}{f(\alpha, \beta)}}
\]
is a critical point of \( H(t) \). We observe that \( t_0 < 2 \), that is, \( t_0 \) is interior point of the interval \((0, 2)\). Since \( H''(t_0) < 0 \), the maximum value of \( H(t) \) occurs at \( t = t_0 \) and
\[
\max\{H(t) : 0 < t < 2\} = H(t_0) = \frac{(1 - \beta)^2}{(1 + 2\alpha)^2} - \frac{[C(\alpha, \beta)]^2}{48(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)f(\alpha, \beta)}.
\]
In this case,
\[
\frac{(1 - \beta)^2}{(1 + 2\alpha)^2} < H(t_0).
\]
Therefore,
\[
\max\{F(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} = \max\left\{ \frac{(1 - \beta)^2}{(1 + 2\alpha)^2} - \frac{[C(\alpha, \beta)]^2}{48(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)f(\alpha, \beta)}, \frac{16(1 - \beta)^4 + 4(1 + \alpha)^2(1 - \beta)^2}{3(1 + \alpha)^3(1 + 3\alpha)} \right\}.
\]  \( (58) \)

Thus, from \( (57) \) and \( (58) \), the proof is completed.

**Corollary 2.6.** Taking \( \alpha = 0 \) and \( \alpha = 1 \) in the Corollary 2.5, we obtain the results for the classes \( S^*_\Delta(\phi) \) and \( K_S(\phi) \), which leads to the results obtained in Theorem 2.1 and 2.3 of [6], respectively.

**Corollary 2.7.** Putting \( \beta = 0 \) in the Corollary 2.6, we get the boundary estimates for the second Hankel determinant in the classes of bi-starlike and bi-convex functions as \( |a_2a_4 - a_3^2| \leq 20/3 \) and \( |a_2a_4 - a_3^2| \leq 1/3 \).

The boundary estimates for the second Hankel determinant obtained in the Corollary 2.7 verifies to the Corollary 2.2 and 2.4 of [6], respectively.
3. Acknowledgment

We would like to thank the referee(s) for his comments and suggestions on the manuscript.

REFERENCES

Dr. B.A. Frasin received his Ph.D. degree in complex analysis (Geometric Function Theory) from the National University of Malaysia (UKM), in 2002. He is having twenty five years of teaching and research experience. His research areas include special classes of univalent functions, special functions and harmonic functions.

Dr. K. Vijaya works as a Professor of mathematics at the School of Advanced Sciences, Vellore Institute of Technology, VIT University, Vellore-632014, Tamilnadu, India. She received her Ph.D. degree in Complex Analysis (Geometric Function Theory) from VIT University, Vellore, in 2007. Her research areas include special functions, harmonic functions.

M. Kasthuri is a Research Associate, pursuing Ph.D. in Mathematics, VIT University, Vellore-632014. She got Master of Science degree from Thiruvalluvar University, Vellore, TN, India. Her research areas include Bi-univalent, harmonic and Bessel functions.