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SECOND HANKEL DETERMINANT FOR A GENERAL SUBCLASS OF BI-UNIVALENT FUNCTIONS

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Abstract. Making use of the Hankel determinant, in this work, we consider a general subclass of bi-univalent functions. Moreover, we investigate the bounds of initial coefficients of this class.

Keywords: analytic and univalent functions, bi-univalent functions, Hankel determinant.

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1. Introduction

Let A denote the class of functions f which are analytic in the open unit disk $U = \{z : |z| < 1\}$ with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be the subclass of A consisting of the form (1) which are also univalent in U. The Koebe one-quarter theorem [10] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z\ ,\ \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w\ ,\ \left(|w| < r_0(f)\ \ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w\ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

For a brief history and interesting examples in the class Σ, see [26]. Examples of functions in the class Σ are

$$\frac{z}{1-z},\quad -\log(1-z),\quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familier Koebe function is not a member of Σ. Other common examples of functions in S such as

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ (see [26]).

Lewin [16] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Netanyahu [18] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [6] introduced

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certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $S^{\star}(\beta)$ and $K(\beta)$ of starlike and convex function of order β $(0 \leq \beta < 1)$ respectively (see [18]). By definition, we have

$$S^{\star}(\beta) = \left\{ f \in S : Re \left(\frac{zf'(z)}{f(z)} \right) > \beta; \quad 0 \leq \beta < 1, \ z \in U \right\}$$

and

$$K(\beta) = \left\{ f \in S : Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta; \quad 0 \leq \beta < 1, \ z \in U \right\}.$$

The classes $S^{\star}_{\Sigma}(\beta)$ and $K_{\Sigma}(\beta)$ of bi-starlike functions of order α and bi-convex functions of order β, corresponding to the function classes $S^{\star}(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S^{\star}_{\Sigma}(\beta)$ and $K_{\Sigma}(\beta)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [1, 4, 12, 17, 26, 27, 29]. Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions [2, 8, 14, 15]. The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \backslash \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

The Fekete-Szegö functional $\left| a_3 - \mu a_2^2 \right|$ for normalized univalent functions

$$f(z) = z + a_2 z^2 + \cdots$$

is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Parley that the coefficients of odd univalent functions are bounded by unity (see [11]). The functional has since received great attention, particularly in many subclasses of the family of univalent functions. Nowadays, it seems that this topic had become an interest among the researchers (see, for example, [3,21,30]).

The q^{th} Hankel determinant for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas ([20]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

This determinant has also been considered by several authors. For example, Noor [20] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles [20,22] for different classes of functions.

It is interesting to note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is well-known as Fekete-Szegö functional. Very recently, the upper bounds of $H_2(2)$ for some classes were discussed by Deniz et al. [9].

Let $f \in A$. We define the differential operator $D^n,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by (see [25])

$$\begin{aligned} D^0 f(z) &= f(z); \\ D^1 f(z) &= Df(z) = zf'(z); \\ &\vdots \\ D^n f(z) &= D\left(D^{n-1} f(z)\right). \end{aligned}$$

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

The study of Salagean differential operator plays an important role in Geometric Function Theory in Complex Analysis and its related fields (for example, see [5], [28]).

Definition 1.1 (24). *A function $f \in \Sigma$ is said to be $H_\Sigma^\lambda(n,\beta)$, if the following conditions are satisfied:*

$$Re\left(\frac{(1-\lambda)D^n f(z) \ + \ \lambda D^{n+1} f(z)}{z}\right) > \beta; \quad 0 \le \beta < 1, \quad \lambda \ge 0, \quad z \in U \tag{2}$$

and

$$Re\left(\frac{(1-\lambda)D^n g(w) \ + \ \lambda D^{n+1} g(w)}{w}\right) > \beta; \quad 0 \le \beta < 1, \quad \lambda \ge 0, \quad w \in U \tag{3}$$

where $g(w) = f^{-1}(w)$.

We note that for $n = 0$ and $n = 0$, $\lambda = 1$, the class $H_\Sigma^\lambda(n,\beta)$ reduce to the classes $H_\Sigma^\lambda(\beta)$ and $H_\Sigma(\beta)$ studied by Frasin and Aouf [12] and Srivastava et al. [26], respectively.

In this paper, we get upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for functions f belongs to the class $H_\Sigma^\lambda(n,\beta)$.

In order to derive our main results, we require the following lemma.

Lemma 1.1 (23). *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is an analytic function in U with positive real part, then*

$$|p_n| \le 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \le 2 - \frac{|p_2|^2}{2}.$$

Let P denote the class of functions consisting of p.

Lemma 1.2 (13). *If the function $p \in P$, then*

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2) \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \end{aligned}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

2. Main result

Theorem 2.1. *Let f given by (1) be in the class $H_{\Sigma}^{\lambda}(n,\beta)$ and $0 \leq \beta < 1$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \left[\frac{4(1-\beta)^2}{2^n(1+\lambda)^3} + \frac{1}{1+3\lambda}\right] \frac{4(1-\beta)^2}{(1+\lambda)8^n}, \\ \qquad\qquad \beta \in \left[0, 1 - \frac{1}{2}\sqrt{\frac{2^{n-1}(1+\lambda)^3}{1+3\lambda}}\right] \\ \frac{9(1+\lambda)^2(1-\beta)^2}{2^{2n+1}(1+3\lambda)[2^n(1+\lambda)^3 - 2(1+3\lambda)(1-\beta)^2]}, \\ \qquad\qquad \beta \in \left[1 - \frac{1}{2}\sqrt{\frac{2^{n-1}(1+\lambda)^3}{1+3\lambda}}, 1\right) \end{cases}.$$

Proof. Let $f \in H_{\Sigma}^{\lambda}(n,\beta)$. Then

$$\frac{(1-\lambda)\, D^n f(z) \; + \; \lambda D^{n+1} f(z)}{z} = \beta + (1-\beta)p(z) \tag{4}$$

$$\frac{(1-\lambda)\, D^n g(w) \; + \; \lambda D^{n+1} g(w)}{w} = \beta + (1-\beta)q(w) \tag{5}$$

where $p, q \in P$ and $g = f^{-1}$.

It follows from (4) and (5) that

$$(1+\lambda)\, 2^n a_2 = (1-\beta)\, p_1, \tag{6}$$

$$(1+2\lambda)3^n a_3 = (1-\beta)\, p_2, \tag{7}$$

$$(1+3\lambda)4^n a_4 = (1-\beta)\, p_3 \tag{8}$$

$$-(1+\lambda)\, 2^n a_2 = (1-\beta)\, q_1, \tag{9}$$

$$(1+2\lambda)3^n \left(2a_2^2 - a_3\right) = (1-\beta)\, q_2 \tag{10}$$

$$-(1+3\lambda)4^n \left(5a_2^3 - 5a_2a_3 + a_4\right) = (1-\beta)q_3. \tag{11}$$

From (6) and (9) we obtain

$$p_1 = -q_1. \tag{12}$$

and

$$a_2 = \frac{(1-\beta)}{(1+\lambda)\, 2^n} p_1. \tag{13}$$

Subtracting (7) from (10), we have

$$a_3 = \frac{(1-\beta)^2}{(1+\lambda)^2\, 4^n} p_1^2 + \frac{(1-\beta)}{2\,(1+2\lambda)\, 3^n}\left(p_2 - q_2\right). \tag{14}$$

Also, subtracting (8) from (11), we have

$$a_4 = \frac{5(1-\beta)^2}{4(1+\lambda)(1+2\lambda)6^n} p_1 \left(p_2 - q_2\right) + \frac{(1-\beta)}{2(1+3\lambda)4^n}\left(p_3 - q_3\right). \tag{15}$$

Then, we can establish that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Big| &-\frac{(1-\beta)^4}{(1+\lambda)^4 16^n} p_1^4 + \frac{5(1-\beta)^3}{4(1+\lambda)^2(1+2\lambda)12^n} p_1^2 \left(p_2 - q_2\right) - \frac{(1-\beta)^3}{(1+\lambda)^2(1+2\lambda)12^n} p_1^2 \left(p_2 - q_2\right) \\ &+ \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)8^n} p_1 \left(p_3 - q_3\right) - \frac{(1-\beta)^2}{4(1+2\lambda)^2 9^n}\left(p_2 - q_2\right)^2 \Big|. \end{aligned} \tag{16}$$

According to Lemma 3 and (12), we write

$$\left.\begin{array}{l} 2p_2 = p_1^2 + x(4-p_1^2) \\ 2q_2 = q_1^2 + x(4-q_1^2) \end{array}\right\} \Rightarrow p_2 = q_2 \tag{17}$$

and

$$p_3 - q_3 = \frac{p_1^3}{2} - p_1(4-p_1^2)x - \frac{p_1}{2}(4-p_1^2)x^2. \tag{18}$$

Then, using (17) and (18), in (16),

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Big| & -\tfrac{(1-\beta)^4}{(1+\lambda)^4 16^n} p_1^4 + \tfrac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)8^n} p_1^4 \\ & -\tfrac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)8^n} p_1^2(4-p_1^2)x - \tfrac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)8^n} p_1^2(4-p_1^2)x^2 \Big|. \end{aligned} \tag{19}$$

Since $p \in P$, so $|p_1| \leq 2$. Letting $|p_1| = p$, we may assume without restriction that $p \in [0,2]$. Then, applying the triangle inequality on (19), with $\mu = |x| \leq 1$, we get

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \leq & \tfrac{(1-\beta)^4}{(1+\lambda)^4 16^n} p^4 + \tfrac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)8^n} p^4 \\ & + \tfrac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)8^n} p^2(4-p^2)\mu + \tfrac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)8^n} p^2(4-p^2)\mu^2 = F(\mu). \end{aligned}$$

Differentiating $F(\mu)$, we obtain

$$F'(\mu) = \tfrac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)8^n} p^2(4-p^2) + \tfrac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)8^n} p^2(4-p^2)\mu.$$

Furthermore, for $F'(\mu) > 0$ and $\mu > 0$, F is an increasing function and thus, the upper bound for $F(\mu)$ corresponds to $\mu = 1$;

$$F(\mu) \leq \tfrac{(1-\beta)^4}{(1+\lambda)^4 16^n} p^4 - \tfrac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)8^n} p^4 + \tfrac{3(1-\beta)^2}{(1+\lambda)(1+3\lambda)8^n} p^2 = G(p).$$

Assume that $G(p)$ has a maximum value in an interior of $p \in [0,2]$, then

$$G'(p) = \left[\tfrac{2(1-\beta)^2}{(1+\lambda)^3 2^n} - \tfrac{1}{1+3\lambda}\right] \tfrac{2(1-\beta)^2}{(1+\lambda)8^n} p^3 + \tfrac{6(1-\beta)^2}{(1+\lambda)(1+3\lambda)8^n} p.$$

Then,

$$G'(p) = 0 \Rightarrow \begin{cases} p_{01} = 0 \\ \\ p_{02} = \sqrt{\tfrac{3(1+\lambda)^3 2^n}{(1+\lambda)^3 2^n - 2(1+3\lambda)(1-\beta)^2}} \end{cases}.$$

□

Case 1. When $\beta \in \left[0, 1 - \frac{1}{2}\sqrt{\frac{2^{n-1}(1+\lambda)^3}{1+3\lambda}}\right]$, we observe that $p_{02} > 2$ and G is an increasing function in the interval $[0,2]$, so the maximum value of $G(p)$ occurs at $p = 2$. Thus, we have

$$G(2) = \left[\tfrac{4(1-\beta)^2}{2^n(1+\lambda)^3} + \tfrac{1}{1+3\lambda}\right] \tfrac{4(1-\beta)^2}{(1+\lambda)8^n}.$$

Case 2. When $\beta \in \left[1 - \frac{1}{2}\sqrt{\frac{2^{n-1}(1+\lambda)^3}{1+3\lambda}}, 1\right)$, we observe that $p_{02} < 2$ and since $G''(p_{02}) < 0$, the maximum value of $G(p)$ occurs at $p = p_{02}$. Thus, we have

$$G(p_{02}) = \frac{9(1+\lambda)^2(1-\beta)^2}{2^{2n+1}(1+3\lambda)[(1+\lambda)^3 2^n - 2(1+3\lambda)(1-\beta)^2]}.$$

This completes the proof.

Remark 2.1. *Putting* $n = 0$ *in Theorem 4 we have the second Hankel determinant for the well-known class* $H_\Sigma^\lambda(n, \beta) = N_\Sigma^{1,\lambda}(\beta)$ *as in [9].*

Remark 2.2. *Let* f *given by (1) be in the class* $N_\Sigma^{1,\lambda}(\beta)$ *and* $0 \leq \beta < 1$. *Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{4(1-\beta)^2}{(1+\lambda)}\left[\frac{4(1-\beta)^2}{(1+\lambda)^3} + \frac{1}{1+3\lambda}\right] & \beta \in \left[0, 1 - \frac{1}{2}\sqrt{\frac{(1+\lambda)^3}{2(1+3\lambda)}}\right] \\ \\ \frac{9(1+\lambda)^2(1-\beta)^2}{2(1+3\lambda)[(1+\lambda)^3 - 2(1+3\lambda)(1-\beta)^2]} & \beta \in \left[1 - \frac{1}{2}\sqrt{\frac{(1+\lambda)^3}{2(1+3\lambda)}}, 1\right) \end{cases}.$$

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