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THE MULTISTEP MULTIDERIVATIVE METHODS FOR THE NUMERICAL SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS

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Abstract. In a recent paper, Shokri [14] introduce a new class of hybrid Obrechkoff methods for the numerical solution of second order initial value problems. In this work, we will derive the new class of, higher order, multistep methods with multiderivatives for the numerical solution of first order initial value problems. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.

Keywords: Hybrid methods, P-stable, off-step points, Obrechkoff methods.

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1. Introduction

Consider the initial value problems for a single first order ordinary differential equation

$$y'(x) = f(x, y(x)), \quad y(a) = \eta. \tag{1}$$

Initial value problems occur frequently in applications. The numerical solution of these kind of problems is a central task in all simulation environments for mechanical, electrical, chemical systems. There are special purpose simulation programs for application in these fields, which often require from their users a deep understanding of the basic properties of the underlying numerical methods. From discussion in some papers and books on the relative merits of linear multistep and Runge-Kutta methods, it emerged that the former class of methods, though generally the more efficient in terms of accuracy and weak stability properties for a given number of functions evaluations per step, suffered the disadvantage of requiring additional starting values and special procedures for changing steplength. These difficulties would be reduced, without sacrifice, if we could lower the stepnumber of the linear multistep methods without reducing their order. The difficulty here lies in satisfying the essential condition of zero-stability. This 'zero-stability barrier' was circumvented by the introduction, in 1964-5, of modified linear multistep formula which incorporate a function evaluation at on off-step point. Such formula, simultaneously proposed by Gragg and Stetter [5], Butcher [1], Shokri [12] and Gear [4] were christened 'hybrid' by the last author an apt name since, whilst retaining certain linear multistep characteristics, hybrid methods share with Runge-Kutta methods the property of utilizing data at points other than the step points. Thus, we may regard the introduction of hybrid formula as an important step into the no man's land described by Kopal.

The k-step classical hybrid methods formula are as follows

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_v f_{n+v}, \tag{2}$$

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where $\alpha_k = +1$, α_0 and β_0 are not both zero, $v \notin \{0, 1, \ldots, k\}$, and also $f_{n+v} = f(x_{n+v}, y_{n+v})$. These methods are similar to linear multistep methods in predictor-corrector mode, but with one essential modification: an additional predictor is introduced at an off-step point. This means that the final (corrector) stage has an additional derivative approximation to work from. This greater generality allows the consequences of the Dahlquist barrier to be avoided and it is actually possible to obtain convergent k-step methods with order $2k+1$ up to $k=7$. Even higher orders are available if two or more off-step points are used. The other independent discoveries of this approach were reported in [2,3,4]. Although a flurry of activity by other authors followed, these methods have never been developed to the extent that they have been implemented in general purpose software. Recall that The formula (2) is zero-stable if the polynomial $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$, has no any roots with modulus of greater than one and every roots with modulus one are simple. Thus Gragg and Stetter's results showed that [5], with certain exceptions, we can utilize both of new parameters v and β_v we have introduced, to raise the order of (2) to two above attained by linear multistep methods having the same left-hand side and the same value for k'.

The one of the other important class of linear multistep methods for the numerical solution of first order ordinary differential equation is classical Obrechkoff methods. The k-step classical Obrechkoff method using the first l derivatives of y, for solution of (1), are given by (see, e.g, [5, pp. 199-204, 4-6])

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)}, \quad \alpha_k = 1. \tag{3}$$

According to [9], the error constant decreases more rapidly with increasing l rather than the step k. It is difficult to satisfy the zero-stability for large k. The weak stability interval appears to be small. The advantage of classical Obrechkoff methods is the fact that are k-step high order methods and as such do not require additional starting values. A list of classical Obrechkoff methods for $l = 1, 2, \ldots, 5-k$, $k = 1, 2, 3, 4$ are given in [9]. For example, for $k=1$, and $l=2$, we get an implicit method of order 4 with an error constant $C_5 = \frac{1}{720}$, and the method is

$$y_{n+1} - y_n = \frac{h}{2}(y'_{n+1} + y'_n) - \frac{h^2}{12}(y''_{n+1} - y''_n). \tag{4}$$

For many problems, such explicit differentiation is intolerably complicated, but when it is feasible to evaluate the first few total derivatives of y, then generalizations of linear multistep methods which employ such derivatives can be very efficient. Although the original work of Obrechkoff was concerned only with numerical quadrature, and it would appear that [6-23] was the first to advocate the use of Obrechkoff formula for the numerical solution of differential equations.

2. Construction of new method

For the numerical integration of (1), we consider k-step methods with first l derivatives of y, and ν off-step points of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} + h \sum_{j=1}^{v} \gamma_j f_{n+\theta_j} \quad , \quad \alpha_k = 1, \tag{5}$$

where α_j, β_{ij}, γ_j, $0 < \theta_j < k$ such that $\theta_j \notin \{0, 1, 2, \cdots, k\}$, $j = 1, 2, \ldots, v$ are $(k + l(k+1) + 2\nu)$ arbitrary parameters. Formula (5) can only be used if we know the values of the solution $y(x)$ and associated derivatives at k successive points. These k values will be assumed to be given. Further, if $\beta_{ik} = 0$, this equation is referred to as an explicit or predictor formula since y_{n+1}

occurs only on left hand side of the equation. In other words the unknown y_{n+1} can be calculated directly. If $\beta_{ik} \neq 0$, the equation is referred to as an implicit or corrector formula since y_{n+1} occurs on both sides of the equation. In other words the unknown y_{n+1} cannot be calculated directly since it is contained within $y^{(i)}_{n+1}$.

Now with the difference equation (5), we can associate the difference operator L defined next.

Definition 2.1. *Let the differential equation* (1) *have a unique solution* $y(x)$ *on* $[a,b]$ *and suppose that* $y(x) \in C^{(p+1)}[a,b]$ *for* $p \geq 1$*. Then the deference operator* L *for method (5) can be written as*

$$\begin{aligned} L[y(x),h] &= \sum_{j=1}^{k}\Big[\alpha_j y(x+jh) - \sum_{i=1}^{l} h^i \beta_{ij} y^{(i)}(x+jh)\Big] - \\ &- h\sum_{j=1}^{v}\Big[\gamma_j f(x+\theta_j h, y(x+\theta_j h))\Big]. \end{aligned} \tag{6}$$

In order that the difference equation (8) be useful for numerical integration, it is necessary that it be satisfied to high accuracy by the solution of the differential equation $y' = f(x,y)$, when h is small for an arbitrary function $f(x,y)$. This imposes restrictions on the coefficients α_j, β_{ij}, γ_j and θ_j. We assume that the function $y(x)$ has continuous derivatives at least of order 10.

3. PECE mode and stability analysis

The explicit and implicit form of the new methods can be combined predictor - corrector modes. Thus if we indicate by $y^{(i)}(x,y)$ the ith total derivative, then with the usual convention that k is the steplenght of the overall method and that coefficients of the predictor are marked *, we may define a general multistep multiderivatives with off-step points PECE method. Applying Newton's interpolation formula for evaluate off-step terms, gives us the following scheme

$$\begin{aligned} [y_{n+\theta_j}]^{[0]} &= [y_{n+1}]^{[0]} + (\theta_j - 1)h[f_{n+1}]^{[0]} + \\ &+ (s-1)^2(h[f_{n+1}]^{[0]} - \nabla[y_{n+1}]^{[0]}) + \\ &+ (\theta_j - 1)^2 \sum_{i=0}^{k} \frac{\binom{-s}{i}}{k!}\left(h[f_{n+1}]^{[0]} - \sum_{p=1}^{i}\frac{1}{p}\nabla^p[y_{n+1}]^{[0]}\right), \qquad (7) \\ y^{[0]}_{n+k} &+ \sum_{j=0}^{k-1}\alpha^*_j y^{[1]}_{n+j} = \sum_{i=1}^{l^*} h^i \sum_{j=0}^{k-1}\beta^*_{ij}[y^{(i)}_{n+j}]^{[1]} + h\sum_{j=1}^{v}\gamma^*_j[f_{n+\theta_j}]^{[1]}, \\ [y^{(i)}_{n+k}]^{[0]} &= y^{(i)}(x_{n+k}, y^{[0]}_{n+k}) \quad , \quad i = 1,2,\ldots,l\ , \\ \sum_{j=0}^{k}\alpha_j y^{[1]}_{n+j} &= \sum_{i=1}^{l} h^i\Big\{\beta_{ik}[y^{(i)}_{n+k}]^{[0]} + \sum_{j=0}^{k-1}\beta_{ij}[y^{(i)}_{n+j}]^{[1]} + h\sum_{j=1}^{v}\gamma_j[f_{n+\theta_j}]^{[1]}\Big\}, \\ [y^{(i)}_{n+k}]^{[1]} &= y^{(i)}(x_{n+k}, y^{[1]}_{n+k}), \quad i = 1,2,\ldots,\max(l,l^*). \qquad (8) \end{aligned}$$

The weak stability of new methods my be investigated very easily. Thus, if we define the characteristic polynomials

$$\rho(r) = \sum_{j=0}^{k}\alpha_j r^j \quad , \quad \sigma(r) = \sum_{j=0}^{k}\beta_{ij} r^j \quad i = 1,2,\ldots,l$$

$$\rho^*(r)=\sum_{j=0}^{k}\alpha_j^* r^j \quad , \quad \sigma^*(r)=\sum_{j=0}^{k}\beta_{ij}^* r^j \quad i=1,2,\dots,l^*,$$

then the stability polynomial of PECE mode for the new method can be written as

$$\pi(r,\bar{h})=\rho(r)-\sum_{i=1}^{l}\bar{h}^i\sigma_i(r)+\Big(\sum_{i=1}^{l}\bar{h}^i\beta_{ik}\Big)\Big[\rho^*(r)-\sum_{i=1}^{l^*}\bar{h}^i\sigma_i^*(r)\Big]. \tag{9}$$

We can assume that the functions $\rho(\xi)$ and $\sigma(\xi)$ have no common factors since, otherwise, (8) can be reduced to an equation of lower order. We firstly use the Taylor series expansion to determine all the coefficients of (5) which we have

$$L[y(x),h]=C_0y(x_n)+C_1hy^{(1)}(x_n)+\cdots+C_qh^qy^{(q)}(x_n)+\cdots. \tag{10}$$

Definition 3.1. *The new multistep method (5) are said to be of order p if,*

$$C_0=C_1=C_2=\cdots=C_p=0 \quad , \quad C_{p+1}\neq 0$$

thus for any function $y(x)\in C^{(p+2)}$ and for some nonzero constant C_{p+1}, we have

$$L[y(x),h]=C_{p+1}h^{p+1}y^{(p+1)}(x_n)+O(h^{p+2}) \tag{11}$$

where C_{p+1} is called an error constant.

In particular, $L[y(x),h]$ vanishes identically when $y(x)$ is polynomial whose degree is less than or equal to p.

Lemma 3.1. *The new multistep method (5) is consistent if and only if*

$$\rho(1)=0 \quad , \quad \rho'(1)=\sigma(1)+\sum_{j=1}^{k}\gamma_j \tag{12}$$

Proof. We know that the general linear multistep methods are consistent if and only if they have the order of $p\geq 1$. This implies $C_0=C_1=0$. Therefore by a simple calculation, we get (11). □

Theorem 3.1. *Assume that formula (7) is of order $k+l(k+1)+2\nu-3$. Then, the method (5) has order $k+l(k+1)+2(\nu-1)$.*

3.1. **One-step new method with one off-step point.** If we take $k=v=1$ and $l=2$ in (5), we get

$$y_{n+1}-y_n=h(b_0f_n+b_1f_{n+1})+h^2(b_2f'_n+b_3f'_{n+1})+hd_1f_{n+\theta_1} \tag{13}$$

where b_0, b_1, b_2, b_3, d_1, and $0<\theta_1<1$ are 6 arbitrary parameters and $f'_i=y''_i$. In order to implement such a formula, a special predictor to estimate $y_{n+\theta_1}$ is necessary, we suppose that θ_1 is free parameter and by substituting $C_i=0$, $i=0,1,2,3,4,5$, we have

$$\begin{aligned}
C_0 &= 0=0,\\
C_1 &= 1-b_0-b_1-d_1=0,\\
C_2 &= \frac{1}{2!}-b_1-b_2-b_3-\theta_1d_1=0,\\
C_3 &= \frac{1}{3!}-\frac{1}{2!}b_1-b_3-\frac{1}{2!}\theta_1^2d_1=0,\\
C_4 &= \frac{1}{4!}-\frac{1}{3!}b_1-\frac{1}{2!}b_3-\frac{1}{3!}\theta_1^3d_1=0,\\
C_5 &= \frac{1}{5!}-\frac{1}{4!}b_1-\frac{1}{3!}b_3-\frac{1}{4!}\theta_1^4d_1=0.
\end{aligned}$$

Now if we consider θ_1 is free parameter, then solving for the coefficients, we have

$$b_0 = \frac{(5\theta_1+1)(3\theta_1-1)}{30\theta_1^2} \quad , \quad b_1 = \frac{(3\theta_1-2)(5\theta_1-6)}{30(\theta_1-1)^2} \quad , \quad b_2 = \frac{5\theta_1-2}{60\theta_1} \tag{14}$$

$$b_3 = -\frac{5\theta_1-3}{60(\theta_1-1)} \quad , \quad d_1 = \frac{1}{30\theta_1^2(\theta_1-1)^2} \tag{15}$$

then the local truncation error is

$$\begin{aligned} E_6 &= \Big[\frac{1}{6!} - \frac{1}{5!}b_1 - \frac{1}{4!}b_3 - \frac{1}{5!}d_1\theta_1^5\Big]h^6 y^{(6)}(\xi) = \\ &= \frac{1}{7200}(1-2\theta_1)h^6 y^{(6)}(\xi). \end{aligned} \tag{16}$$

If we take $\theta_1 = \frac{1}{2}$ then obviously $E_6 = 0$, and then our new methods have order 6 with local truncation error

$$\begin{aligned} E_7 &= \Big[\frac{1}{7!} - \frac{1}{6!}b_1 - \frac{1}{5!}b_3 - \frac{1}{6!}d_1\theta_1^6\Big]h^6 y^{(6)}(\xi) = \\ &= \frac{1}{3048192000}h^7 y^{(7)}(\xi). \end{aligned} \tag{17}$$

If we take $\theta_1 = \frac{1}{2}$, we have

$$b_0 = \frac{7}{30} \quad , \quad b_1 = \frac{7}{30} \quad , \quad b_2 = \frac{1}{60} \quad , \quad b_3 = -\frac{1}{60} \quad , \quad d_1 = \frac{8}{15} \tag{18}$$

and the method is then

$$y_{n+1} - y_n = \frac{7h}{30}(f_n + f_{n+1}) + \frac{h^2}{60}(f'_n - f'_{n+1}) + \frac{8h}{15}f_{n+\frac{1}{2}} \tag{19}$$

which is the implicit one-step method of order 6 and its local truncation error is

$$E = \frac{1}{3048192000}h^7 y^{(7)}(\xi), \quad \xi \in (x_n, x_{n+1}).$$

By choosing $\theta_1 = \frac{1}{3}$, we have

$$b_0 = 0 \quad , \quad b_1 = \frac{13}{40} \quad , \quad b_2 = -\frac{1}{60} \quad , \quad b_3 = -\frac{1}{30} \quad , \quad d_1 = \frac{27}{40} \tag{20}$$

hence the method is

$$y_{n+1} - y_n = \frac{13h}{40}f_{n+1} - \frac{h^2}{60}(f'_n + 2f'_{n+1}) + \frac{27h}{40}f_{n+\frac{1}{3}} \tag{21}$$

is the implicit one-step method of order 5 and its local truncation error is

$$E = \frac{2807}{979776000}h^6 y^{(6)}(\xi).$$

3.2. **Two-step new methods with one off-step point.** Upon choosing $k = l = 2$ and $v = 1$ in (5), we get

$$\begin{aligned} y_{n+2} + a_1 y_{n+1} + a_0 y_n &= h(b_0 f_n + b_1 f_{n+1} + b_2 f_{n+2}) + \\ &+ h^2(c_0 f'_n + c_1 f'_{n+1} + c_2 f'_{n+2}) + \\ &+ h d_1 f_{n+\theta_1}, \end{aligned}$$

where a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2, d_1, and $0 < \theta_1 < 1$ are 10 arbitrary parameters. In order to implement such a formula, a special predictor to estimate $y_{n+\theta_1}$ is necessary, we suppose that θ_1 is free parameter and by by substituting $C_i = 0$, $i = 0, 1, 2, \cdots, 9$, we have

$$\begin{aligned}
C_0 &= 1 + a_0 + a_1 = 0,\\
C_1 &= 2 + a_1 - b_0 - b_1 - b_2 - d_1 = 0,\\
C_2 &= 2 + \frac{1}{2!}a_1 - b_1 - 2b_2 - c_1 - c_2 - \theta_1 d_1 = 0,\\
C_3 &= \frac{2^3}{3!} + \frac{1}{3!}a_1 - \frac{1}{2!}b_1 - \frac{2^2}{2!}b_2 - c_1 - 2c_2 - \frac{1}{2!}\theta_1^2 d_1 = 0,\\
C_4 &= \frac{2^4}{4!} + \frac{1}{4!}a_1 - \frac{1}{3!}b_1 - \frac{2^3}{3!}b_2 - \frac{1}{2!}c_1 - \frac{2^2}{2!}c_2 - \frac{1}{3!}\theta_1^3 d_1 = 0,\\
C_5 &= \frac{2^5}{5!} + \frac{1}{5!}a_1 - \frac{1}{4!}b_1 - \frac{2^4}{4!}b_2 - \frac{1}{3!}c_1 - \frac{2^3}{3!}c_2 - \frac{1}{4!}\theta_1^4 d_1 = 0,\\
C_6 &= \frac{2^6}{6!} + \frac{1}{6!}a_1 - \frac{1}{5!}b_1 - \frac{2^5}{5!}b_2 - \frac{1}{4!}c_1 - \frac{2^4}{4!}c_2 - \frac{1}{5!}\theta_1^5 d_1 = 0,\\
C_7 &= \frac{2^7}{7!} + \frac{1}{7!}a_1 - \frac{1}{6!}b_1 - \frac{2^6}{6!}b_2 - \frac{1}{5!}c_1 - \frac{2^5}{5!}c_2 - \frac{1}{6!}\theta_1^6 d_1 = 0,\\
C_8 &= \frac{2^8}{8!} + \frac{1}{8!}a_1 - \frac{1}{7!}b_1 - \frac{2^7}{7!}b_2 - \frac{1}{6!}c_1 - \frac{2^6}{6!}c_2 - \frac{1}{7!}\theta_1^7 d_1 = 0.
\end{aligned}$$

Now if we consider θ_1 is free parameter, then solving for the coefficients, we have

$$a_0 = \frac{64\theta_1 - 99}{64\theta_1 - 29} \quad , \quad a_1 = -\frac{128(\theta_1 - 1)}{64\theta_1 - 29} \tag{22}$$

$$b_0 = -\frac{72\theta_1^3 - 121\theta_1^2 + 12\theta_1 + 4}{3(64\theta_1 - 29)\theta_1^2} \quad , \quad b_1 = \frac{16(7\theta_1^2 - 14\theta_1 + 6)}{3(64\theta_1 - 29)(\theta_1 - 1)^2}, \tag{23}$$

$$b_2 = -\frac{72\theta_1^3 - 311\theta_1^2 + 392\theta_1 - 120}{3(64\theta_1 - 29)(\theta_1 - 2)^2} \quad , \quad c_0 = -\frac{8\theta_1^2 - 15\theta_1 + 4}{3\theta_1(64\theta_1 - 29)}, \tag{24}$$

$$c_1 = \frac{16(2\theta_1 - 1)(2\theta_1 - 3)}{3(64\theta_1 - 29)(\theta_1 - 1)} \quad , \quad c_2 = -\frac{8\theta_1^2 - 17\theta_1 + 6}{3(\theta_1 - 2)(64\theta_1 - 29)} \tag{25}$$

and

$$d_1 = \frac{16}{3\theta_1^2(64\theta_1 - 29)(\theta_1 - 1)^2(\theta_1 - 2)^2}, \tag{26}$$

then the local truncation error is

$$\begin{aligned}
E_9 &= [\frac{2^9}{9!} + \frac{1}{8!}a_1 - \frac{1}{8!}b_1 - \frac{2^8}{8!}b_2 - \frac{1}{7!}c_1 - \frac{2^7}{7!}c_2 - \frac{1}{8!}\theta_1^8 d_1]h^9 y^{(9)}(\xi)\\
&= -\frac{3\theta_1^2 - 6\theta_1 + 2}{22680(64\theta_1 - 29)}h^9 y^{(9)}(\xi).
\end{aligned} \tag{27}$$

If we take $\theta_1 = \frac{1}{2}$, we have

$$a_0 = -\frac{67}{3} \quad , \quad a_1 = \frac{64}{3} \quad , \quad b_0 = 5 \quad , \quad b_1 = \frac{16}{3} \quad , \quad b_2 = \frac{29}{81}, \tag{28}$$

$$c_0 = \frac{1}{11619} \quad , \quad c_1 = 0 \quad , \quad c_2 = -\frac{1}{27} \quad , \quad d_1 = \frac{1024}{81}, \tag{29}$$

and the method is then

$$\begin{aligned} y_{n+2}+\frac{64}{3}y_{n+1}-\frac{67}{3}y_n &= h(5f_n+\frac{16}{3}f_{n+1}+\frac{29}{81}f_{n+2})+\\ &+ h^2(\frac{1}{11619}f'_n-\frac{1}{27}f'_{n+2})+\\ &+ h\frac{1024}{81}f_{n+\frac{1}{2}} \end{aligned} \tag{30}$$

which is the implicit two-step method of order 8 and its local truncation error is

$$E=\frac{1}{272160}h^9y^{(9)}(\xi), \quad \xi\in(x_n,x_{n+1}).$$

By choosing $\theta_1=\frac{1}{3}$, we have

$$a_0=\frac{233}{23}\quad,\quad a_1=-\frac{256}{23}\quad,\quad b_0=-\frac{25}{23}\quad,\quad b_1=-\frac{76}{23}\quad,\quad b_2=\frac{191}{575},$$

$$c_0=-\frac{1}{69}\quad,\quad c_1=\frac{56}{69}\quad,\quad c_2=-\frac{11}{345}\quad,\quad d_1=-\frac{2916}{575}$$

hence the method is

$$\begin{aligned} y_{n+2}-\frac{256}{23}y_{n+1}+\frac{233}{23}y_n &= -\frac{h}{23}(25f_n+76f_{n+1}-\frac{191}{25}f_{n+2})+\\ &+ h^2(-\frac{1}{69}f'_n+\frac{56}{69}f'_{n+1}-\frac{11}{345}f'_{n+2})-\\ &- \frac{2916h}{575}f_{n+\frac{1}{3}} \end{aligned} \tag{31}$$

is the implicit two-step method of order 8 and its local truncation error is

$$E=\frac{1}{521640}h^9y^{(9)}(\xi).$$

Finally if we take $\theta_1=1-\frac{\sqrt{3}}{3}$, we have

$$a_0=-\frac{4480}{11447}\sqrt{3}+\frac{9417}{11447}\quad,\quad a_1=\frac{4480}{11447}\sqrt{3}-\frac{20864}{11447}\quad,$$

$$b_0=\frac{16364}{103023}\sqrt{3}-\frac{33365}{103023}\quad,\quad b_1=\frac{5912}{34341}\sqrt{3}+\frac{416}{11447}\quad,$$

$$b_2=-\frac{3524}{34341}\sqrt{3}+\frac{2091}{11447}\quad,\quad c_0=\frac{2068}{103023}\sqrt{3}-\frac{1357}{34341}\quad,$$

$$c_1=-\frac{2984}{34341}\sqrt{3}+\frac{2888}{11447}\quad,\quad c_2=\frac{460}{34341}\sqrt{3}-\frac{169}{11447}\quad,$$

$$d_1=\frac{16792}{103023}\sqrt{3}+\frac{29072}{103023},$$

then the method of generalized by theses coefficients is implicit two-step method of order 9 and its local truncation error is

$$E_{10}=\left(\frac{1}{12276360}\sqrt{3}+\frac{8}{53709075}\right)h^{10}y^{(10)}(\xi).$$

4. Numerical examples

In this section we present some numerical results obtained by our new methods and compare them with those of other multistep methods.

Example 4.1. *Consider the initial value problem*

$$\begin{cases} y' = -5xy^2 + \frac{5}{x} - \frac{1}{x^2}, \\ y(1) = 1. \end{cases}$$

with the exact solution $y(x) = \frac{1}{x}$. We compared the results of two-step method (29) and Runge-Kutta methods of 4 stage with $h = 0.1$ and $h = 0.025$.

x_i	h	RK-4	Error in (30)
1.0	0.1	0	0
2.2		-0.001373	1.5e-8
3.4		-0.000321	9.3e-10
4.6		0.000121	1.4e-10
5.8		0.000058	3.3e-11
7.0		0.000033	1.1e-11
$\vdots$		$\vdots$	$\vdots$
25.0		0.000001	4.6e-15
1.0	0.025	0	0
2.2		-0.0002156	4.6e-10
3.4		-0.0000632	2.5e-11
4.6		0.0000021	3.8e-12
5.8		0.0000062	9.3e-13
7.0		0.0000003	2.9e-13
$\vdots$		$\vdots$	$\vdots$
25.0		0.0000008	1.3e-16

Example 4.2. *Consider the initial value problem*

$$\begin{cases} y_1' = -1002y_1 + 1000y_2^2, \\ y_2' = y_1 - y_2(1 + y_2), \\ y_1(0) = 1, \quad y_2(0) = 1. \end{cases}$$

with the exact solution

$$\begin{cases} y_1 = \exp(-2t), \\ y_2 = \exp(-t). \end{cases}$$

The numerical results are illustrated in follow table.

T	h	Y	Error of (30)	Error in [26]
50	0.05	y_1	2.3e-21	1.97e-15
		y_2	1.4e-18	2.02e-11

Example 4.3. *Consider the initial value problem*

$$\begin{cases} y_1' = -20y_1 - 0.25y_2 - 19.75y_3, \\ y_2' = 20y_1 - 20.25y_2 + 0.25y_3, \\ y_3' = 20y_1 - 19.75y_2 - 0.25y_3, \\ y_1(0) = 1, \quad y_2(0) = 0 \quad y_3(0) = -1. \end{cases}$$

The theoretical solution is

$$\begin{cases} y_1 = \frac{\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)+\sin(20t))\right]}{2}, \\ y_2 = \frac{\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)-\sin(20t))\right]}{2}, \\ y_3 = \frac{-\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)-\sin(20t))\right]}{2}. \end{cases}$$

The numerical results are illustrated in follow table.

T	h	Y	Error of (30)	Error in [26]
50	0.005	y_1	7.1e-26	1.38e-20
		y_2	7.1e-26	1.38e-20
		y_3	7.1e-26	1.38e-20
100	0.1	y_1	4.3e-33	3.57e-31
		y_2	4.3e-33	3.57e-31
		y_3	4.3e-33	3.57e-31

Example 4.4. *Consider the initial value problem*

$$\begin{cases} y_1' = -0.1y_1 - 49.9y_2, \\ y_2' = -50y_2, \\ y_3' = 70y_2 - 120y_3, \\ y_1(0) = 2, \quad y_2(0) = 1 \quad y_3(0) = 2. \end{cases}$$

The theoretical solution is

$$\begin{cases} y_1 = e^{-0.1t} + e^{-50t}, \\ y_2 = e^{-50t}, \\ y_3 = e^{-50t} + e^{-120t}. \end{cases}$$

The numerical results are illustrated in follow table.

T	h	Y	Error of (30)	Error in [26, 27]
0.1	.001	y_1	2.31e-9	1.75e-7
		y_2	6.81e-10	3.59e-8
		y_3	7.35e-10	3.72e-8
0.18	.01	y_1	8.12e-11	1.64e-5
		y_2	5.32e-12	2.79e-7
		y_3	7.95e-12	2.79e-7

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