SOME GENERALIZED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION OVER $n$-NORMED SPACES

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Abstract. In the present paper we introduce some sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$ over $n$-normed spaces. We study some topological properties and prove some inclusion relations between these spaces.

Keywords: Paranorm space, difference sequence space, Orlicz function, Musielak-Orlicz function, $n$-normed space.

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1. Introduction

The concept of 2-normed spaces was initially developed by Gähler [2] in the mid of 1960’s, while that of $n$-normed spaces one can see in Misiak [12]. Since then, many others have studied this concept and obtained various results, see Gunawan ([3], [4]) and Gunawan and Mashadi [5] and references therein.

Let $w$ be the set of all sequences of real or complex numbers and $l_\infty$, $c$ and $c_0$ be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$, respectively.

A sequence $x \in l_\infty$ is said to be almost convergent if all Banach limits of $x$ coincide. Lorentz [8] proved that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s \right\}.$$   

Maddox ([9], [10]) has defined $x$ to be strongly almost convergent to a number $L$ if

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$   

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Nanda [14] has defined the following sequence spaces:

$$[\hat{c}, p] = \left\{ x = (x_k) : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L|^{p_k} = 0, \text{ uniformly in } s \right\},$$   

$$[\hat{c}, p]_0 = \left\{ x = (x_k) : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \text{ uniformly in } s \right\}$$   

and

$$[\hat{c}, p]_\infty = \left\{ x = (x_k) : \sup_{s, n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} < \infty \right\}.$$  

The notion of difference sequence spaces was introduced by Kizmaz [6], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et

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by introducing the spaces \( l_\infty(\Delta^n) \), \( c(\Delta^n) \) and \( c_0(\Delta^n) \). Let \( m, r \) be non-negative integers, then for \( Z = l_\infty \), \( c \) and \( c_0 \), we have sequence spaces,

\[
Z(\Delta^m) = \{ x = (x_k) \in w : (\Delta^m_k x_k) \in Z \},
\]

where \( \Delta^m_r x = (\Delta^m_r x_k) = (\Delta^{m-1}_r x_k - \Delta^{m-1}_r x_{k+r}) \) and \( \Delta^0_r x_k = x_k \) for all \( k \in \mathbb{N} \), which is equivalent to the following binomial representation

\[
\Delta^m_r x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+rv}.
\]

Taking \( m = r = 1 \), we get the spaces \( l_\infty(\Delta) \), \( c(\Delta) \) and \( c_0(\Delta) \) introduced and studied by Kizmaz [6].

An Orlicz function \( M \) is a function, which is continuous and convex with \( M(0) = 0 \), \( M(q) > 0 \) for \( q > 0 \) and \( M(q) \to \infty \) as \( q \to \infty \).

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the sequence space, then the space

\[
\ell_M = \{ x = (x_k) \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \},
\]

which is called as an Orlicz sequence space. The space \( \ell_M \) is a Banach space with the norm

\[
||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.
\]

It is shown in [10] that every Orlicz sequence space \( \ell_M \) contains a subspace isomorphic to \( \ell_p(p \geq 1) \). The \( \Delta_2 \)-condition is equivalent to \( M(Lx) \leq kLM(x) \) for all values of \( x \geq 0 \), and for \( L > 1 \).

A sequence \( M = (M_k) \) of Orlicz functions is called a Musielak-Orlicz function see ([11],[13]). A sequence \( N = (N_k) \) defined by

\[
N_k(v) = \sup \{|v|u - (M_k)(u) : u \geq 0\}, \quad k = 1, 2, \cdots
\]

is called the complementary function of a Musielak-Orlicz function \( M \). For a given Musielak-Orlicz function \( M \), the Musielak-Orlicz sequence space \( t_M \) and its subspace \( h_M \) are defined as follows

\[
t_M = \{ x \in w : I_M(cx) < \infty \text{ for some } c > 0 \},
\]

\[
h_M = \{ x \in w : I_M(cx) < \infty \text{ for all } c > 0 \},
\]

where \( I_M \) is a convex modular defined by

\[
I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_M.
\]

We consider \( t_M \) equipped with the Luxemburg norm

\[
||x|| = \inf \left\{ k > 0 : I_M \left( \frac{x}{k} \right) \leq 1 \right\}
\]

or equipped with the Orlicz norm

\[
||x||^0 = \inf \left\{ \frac{1}{k} \left( 1 + I_M(kx) \right) : k > 0 \right\}.
\]

For more details about sequence spaces see ([15], [16], [17], [18]) and many others.
2. Preliminaries

Let \( n \in \mathbb{N} \) and \( X \) be a linear space over the field \( \mathbb{K} \), where \( \mathbb{K} \) is field of real or complex numbers of dimension \( d \), where \( d \geq n \geq 2 \). A real valued function \( ||\cdot, \cdots, \cdot|| \) on \( X^n \) satisfying the following four conditions:

1. \( ||x_1, x_2, \cdots, x_n|| = 0 \) if and only if \( x_1, x_2, \cdots, x_n \) are linearly dependent in \( X \);
2. \( ||x_1, x_2, \cdots, x_n|| \) is invariant under permutation;
3. \( ||\alpha x_1, x_2, \cdots, x_n|| = \alpha ||x_1, x_2, \cdots, x_n|| \) for any \( \alpha \in \mathbb{K} \), and
4. \( ||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n|| \)

is called a \( n \)-norm on \( X \) and the pair \((X, ||\cdot, \cdots, \cdot||)\) is called a \( n \)-normed space over the field \( \mathbb{K} \).

For example, we may take \( X = \mathbb{R}^n \) being equipped with the \( n \)-norm \( ||x_1, x_2, \cdots, x_n||_E = \) the volume of the \( n \)-dimensional parallelepiped spanned by the vectors \( x_1, x_2, \cdots, x_n \) which may be given explicitly by the formula

\[
||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,
\]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots, n \) and \( ||\cdot||_E \) is the \( n \)-norm on Euclidean space \( \mathbb{R}^n \). Let \((X, ||\cdot, \cdots, \cdot||)\) be an \( n \)-normed space of dimension \( d \geq n \geq 2 \) and \( \{a_1, a_2, \cdots, a_n\} \) be linearly independent set in \( X \). Then the following function \( ||\cdot, \cdots, \cdot||_\infty \) on \( X^{n-1} \) defined by

\[
||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}
\]
defines an \((n-1)\)-norm on \( X \) with respect to \( \{a_1, a_2, \cdots, a_n\} \).

A sequence \((x_k)\) in a \( n \)-normed space \((X, ||\cdot, \cdots, \cdot||)\) is said to converge to some \( L \in X \) if

\[
\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \quad \text{for every} \quad z_1, \cdots, z_{n-1} \in X.
\]

A sequence \((x_k)\) in a \( n \)-normed space \((X, ||\cdot, \cdots, \cdot||)\) is said to be Cauchy if

\[
\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \quad \text{for every} \quad z_1, \cdots, z_{n-1} \in X.
\]

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be \( n \)-Banach space.

Let \( X \) be a linear metric space. A function \( p : X \to \mathbb{R} \) is called paranorm, if

1. \( p(x) \geq 0 \), for all \( x \in X \);
2. \( p(-x) = p(x) \), for all \( x \in X \);
3. \( p(x + y) \leq p(x) + p(y) \), for all \( x, y \in X \);
4. if \((\sigma_n)\) is a sequence of scalars with \( \sigma_n \to \sigma \) as \( n \to \infty \) and \((x_n)\) is a sequence of vectors with \( p(x_n - x) \to 0 \) as \( n \to \infty \), then \( p(\sigma_n x_n - \sigma x) \to 0 \) as \( n \to \infty \).

A paranorm \( p \) for which \( p(x) = 0 \) implies \( x = 0 \) is called total paranorm and the pair \((X, p)\) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19], Theorem 10.4.2, p.183).

Let \( M = (M_k) \) be a Musielak-Orlicz function, \((X, ||\cdot, \cdots, \cdot||)\) be a \( n \)-normed space, \( p = (p_k) \) be bounded sequence of strictly positive real numbers and \( u = (u_k) \) be any sequence of strictly positive real numbers. By \( S(n - X) \) we denote the space of all sequences defined over \((X, ||\cdot, \cdots, \cdot||)\). In the present paper we define the following sequence spaces:

\[
\left[ \hat{c}, M, u, p, ||\cdot, \cdots, \cdot|| \right] (\Delta^m_p) = \left\{ x = (x_k) \in S(n - X) : \lim_{s \to \infty} \frac{\sum_{k = 1}^{n} M_k \left( \frac{u_k \Delta^m_p x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1} \right)}{p_k} = 0 \right\}_0 \quad \text{uniformly in} \ s, \ \text{for some} \ L \ \text{and} \ \rho > 0 \ ,
\]

\[
\left[ \hat{c}, M, u, p, ||\cdot, \cdots, \cdot|| \right] (\Delta^m_p) =
\]
\[
\begin{align*}
\{ x = (x_k) \in S(n - X) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\eta} M_k \left( \frac{\| u_k \Delta^m x_{k+s} - z_1, \cdots, z_{n-1} \|}{\rho} \right) \} & = \emptyset, \\
\text{uniformly in } s, \text{ for } \rho > 0, \\
\{ x = (x_k) \in S(n - X) : \sup_{s, \eta} \frac{1}{\eta} \sum_{k=1}^{\eta} M_k \left( \frac{\| u_k \Delta^m x_{k+s} - z_1, \cdots, z_{n-1} \|}{\rho} \right)^p & < \infty, \text{ for } \rho > 0 \}.
\end{align*}
\]

If we take \( \mathcal{M}(x) = \rho \), we get
\[
\{ x = (x_k) \in S(n - X) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\eta} \left( \frac{\| u_k \Delta^m x_{k+s} - L, z_1, \cdots, z_{n-1} \|}{\rho} \right) \} = 0,
\]

uniformly in \( s \), for some \( L \) and \( \rho > 0 \),
\[
\{ x = (x_k) \in S(n - X) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\eta} \left( \frac{\| u_k \Delta^m x_{k+s} - z_1, \cdots, z_{n-1} \|}{\rho} \right)^p \} = 0,
\]

uniformly in \( s \), for \( \rho > 0 \),
\[
\{ x = (x_k) \in S(n - X) : \sup_{s, \eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \left( \frac{\| u_k \Delta^m x_{k+s} - z_1, \cdots, z_{n-1} \|}{\rho} \right)^p < \infty, \text{ for } \rho > 0 \}.
\]

If we take \( p = (p_k) = 1 \) for all \( k \in \mathbb{N} \), we get
\[
\{ x = (x_k) \in S(n - X) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\eta} M_k \left( \frac{\| u_k \Delta^m x_{k+s} - L, z_1, \cdots, z_{n-1} \|}{\rho} \right) \} = 0,
\]

uniformly in \( s \), for some \( L \) and \( \rho > 0 \),
\[
\{ x = (x_k) \in S(n - X) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\eta} M_k \left( \frac{\| u_k \Delta^m x_{k+s} - z_1, \cdots, z_{n-1} \|}{\rho} \right)^p \} = 0,
\]

uniformly in \( s \), for \( \rho > 0 \),
\[
\{ x = (x_k) \in S(n - X) : \sup_{s, \eta} \frac{1}{\eta} \sum_{k=1}^{\eta} M_k \left( \frac{\| u_k \Delta^m x_{k+s} - z_1, \cdots, z_{n-1} \|}{\rho} \right)^p < \infty, \text{ for } \rho > 0 \}.
\]
The following inequality will be used throughout the paper. If \( 0 \leq p_k \leq \sup p_k = H, K = \max(1, 2H^{-1}) \) then
\[
|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\}
\]
for all \( k \) and \( a_k, b_k \in \mathbb{C} \). Also \( |a|^{p_k} \leq \max(1, |a|^H) \) for all \( a \in \mathbb{C} \).

The main aim of this paper is to study some sequence spaces defined by a Musielak-Orlicz function over \( n \)-normed spaces. We also make an effort to study some topological properties and some inclusion relations between these spaces.

3. **Main Results**

**Theorem 3.1.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function, \( p = (p_k) \) be bounded sequence of positive real numbers and \( u = (u_k) \) be any sequence of strictly positive real numbers. Then the spaces \( \left[ \mathcal{M}, u, p, ||\cdot, \cdot, \cdot|| \right] \left( \Delta^m \right) \) and \( \left[ \mathcal{M}, u, p, ||\cdot, \cdot, \cdot|| \right] \left( \Delta^m \right) \) are linear spaces.

**Proof.** Let \( x = (x_k), y = (y_k) \in \left[ \mathcal{M}, u, p, ||\cdot, \cdot, \cdot|| \right] \left( \Delta^m \right) \) and \( \alpha, \beta \) be any scalars. Then there exist positive numbers \( \rho_1 \) and \( \rho_2 \) such that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\alpha u_k \Delta^m x_{k+s} + \beta u_k y_{k+s}}{\rho_3}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} = 0
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\alpha u_k \Delta^m x_{k+s} + \beta u_k y_{k+s}}{\rho_3}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} = 0.
\]
Let \( \rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2) \). Since \( \mathcal{M} = (M_k) \) is non-decreasing convex function and so by using inequality (1), we have
\[
\frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\alpha u_k \Delta^m x_{k+s} + \beta u_k y_{k+s}}{\rho_3}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} \leq
\]
\[
\leq \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\alpha u_k \Delta^m x_{k+s}}{\rho_3}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} + \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\beta u_k y_{k+s}}{\rho_3}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} \leq
\]
\[
\leq \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\alpha u_k \Delta^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} + \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\beta u_k y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} \leq
\]
\[
K \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\alpha u_k \Delta^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} + K \frac{1}{n} \sum_{k=1}^{n} \left| M_k \left( \frac{\beta u_k y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1} \right) \right|^{p_k} \to 0
\]
as \( n \to \infty \), uniformly in \( s \).

So that \( \alpha x + \beta y \in \left[ \mathcal{M}, u, p, ||\cdot, \cdot, \cdot|| \right] \left( \Delta^m \right) \). Thus \( \left[ \mathcal{M}, u, p, ||\cdot, \cdot, \cdot|| \right] \left( \Delta^m \right) \) is a linear space. Similarly, we can prove that \( \left[ \mathcal{M}, u, p, ||\cdot, \cdot, \cdot|| \right] \left( \Delta^m \right) \) and \( \left[ \mathcal{M}, u, p, ||\cdot, \cdot, \cdot|| \right] \left( \Delta^m \right) \) are linear spaces.

**Theorem 3.2.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function, \( p = (p_k) \) be a bounded sequence of positive real numbers and \( u = (u_k) \) be any sequence of strictly positive real numbers. Then
\[ [c, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||]_0 (\Delta^m) \text{ is a paranormed space with respect to the paranorm defined by} \]
\[
g(x) = \inf \left\{ \rho^m : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\rho} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1 \right\},
\]
where \( H = \max(1, \sup_k p_k < \infty) \).

**Proof.** Clearly \( g(x) \geq 0 \) for \( x = (x_k) \in [c, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||]_0 (\Delta^m) \). Since \( M_k(0) = 0 \), we get \( g(0) = 0 \).

Conversely, suppose that \( g(x) = 0 \), then
\[
\inf \left\{ \rho^m : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\rho} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1 \right\} = 0.
\]
This implies that for a given \( \epsilon > 0 \), there exists some \( \rho_\epsilon (0 < \rho_\epsilon < \epsilon) \) such that
\[
\left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\rho_\epsilon} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1.
\]
Thus
\[
\left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\epsilon} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\rho_\epsilon} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1,
\]
for each \( n \). Suppose that \( x_k \neq 0 \) for each \( k \in N \). This implies that \( u_k \Delta^m x_{k+s} \neq 0 \), for each \( k, s \in N \). Let \( \epsilon \to 0 \), then \( || \frac{u_k \Delta^m x_{k+s}}{\epsilon} \cdot z_1, \cdots, z_{n-1} || \to \infty \). It follows that
\[
\left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\epsilon} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \to \infty
\]
which is a contradiction. Therefore, \( u_k \Delta^m x_{k+s} = 0 \) for each \( k \) and thus \( x_k = 0 \) for each \( k \in N \). Let \( \rho_1 > 0 \) and \( \rho_2 > 0 \) be such that
\[
\left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\rho_1} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1
\]
and
\[
\left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m x_{k+s}}{\rho_2} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1
\]
for each \( n \). Let \( \rho = \rho_1 + \rho_2 \). Then by using Minkowski’s inequality, we have
\[
\left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( || \frac{u_k \Delta^m (x_{k+s} + y_{k+s})}{\rho} \cdot z_1, \cdots, z_{n-1} || \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq
\]
\begin{align*}
&\leq \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{\| u_k \Delta^{m}_{r} x_{k+s} + \Delta^{m}_{r} y_{k+s} \|}{\rho_1 + \rho_2}, z_1, \ldots, z_{n-1} \| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq \\
&\leq \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ \frac{\rho_1}{\rho_1 + \rho_2} M_k \left( \frac{\| u_k \Delta^{m}_{r} x_{k+s} \|}{\rho_1}, z_1, \ldots, z_{n-1} \| \right) \right]^{\frac{1}{p}} \right) + \\
&+ \left( \frac{\rho_2}{\rho_1 + \rho_2} M_k \left( \frac{\| u_k \Delta^{m}_{r} y_{k+s} \|}{\rho_2}, z_1, \ldots, z_{n-1} \| \right) \right) \leq 1.
\end{align*}

Since \( \rho \)'s are non-negative, so we have

\begin{align*}
g(x + y) &= \inf \left\{ \rho_{\frac{M}{\eta}} : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{\| u_k \Delta^{m}_{r} x_{k+s} + \Delta^{m}_{r} y_{k+s} \|}{\rho}, z_1, \ldots, z_{n-1} \| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1 \right\} \\
&\leq \inf \left\{ \rho_{\frac{M}{\eta}} : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{\| u_k \Delta^{m}_{r} x_{k+s} \|}{\rho_1}, z_1, \ldots, z_{n-1} \| \right) \right]^{\frac{1}{p}} \right) \leq 1 \right\} + \\
&+ \inf \left\{ \rho_{\frac{M}{\eta}} : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{\| u_k \Delta^{m}_{r} y_{k+s} \|}{\rho_2}, z_1, \ldots, z_{n-1} \| \right) \right]^{\frac{1}{p}} \right) \leq 1 \right\}.
\end{align*}

Therefore,

\begin{align*}
g(x + y) \leq g(x) + g(y).
\end{align*}

Finally, we prove that the scalar multiplication is continuous. Let \( \lambda \) be any complex number. By definition,

\begin{align*}
g(\lambda x) &= \inf \left\{ \rho_{\frac{M}{\eta}} : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{\| u_k \Delta^{m}_{r} \lambda x_{k+s} \|}{\rho}, z_1, \ldots, z_{n-1} \| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1 \right\}.
\end{align*}

Then

\begin{align*}
g(\lambda x) &= \inf \left\{ \left( |\lambda| t \right)^{\frac{M}{\eta}} : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{\| u_k \Delta^{m}_{r} x_{k+s} \|}{t}, z_1, \ldots, z_{n-1} \| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1 \right\}.
\end{align*}

where \( t = \frac{\rho}{|\lambda|} \). Since \( |\lambda|^p \leq \text{max}(1, |\lambda|^\text{sup } p) \), we have

\begin{align*}
g(\lambda x) \leq \text{max}(1, |\lambda|^\text{sup } p) \inf \left\{ \left( |\lambda| t \right)^{\frac{M}{\eta}} : \left( \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{\| u_k \Delta^{m}_{r} x_{k+s} \|}{t}, z_1, \ldots, z_{n-1} \| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1 \right\}.
\end{align*}

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. \( \square \)

**Theorem 3.3.** Let \( M = (M_k) \) be a Musielak-Orlicz function. Then the following statements are equivalent

(i) \( \left[ \hat{c}, u, p, || \cdot, \cdots, || \right] \left( \Delta^m \right) \subseteq \left[ \hat{c}, M, u, p, || \cdot, \cdots, || \right] \left( \Delta^m \right), \)

(ii) \( \left[ \hat{c}, u, p, || \cdot, \cdots, || \right] \left( \Delta^m \right) \subseteq \left[ \hat{c}, M, u, p, || \cdot, \cdots, || \right] \left( \Delta^m \right), \)

(iii) \( \sup_{n} \frac{1}{\eta} \sum_{k=1}^{\eta} [M_k(t)]^{p_k} < \infty, \) where \( t = \frac{\| u_k \Delta^{m}_{r} x_{k+s} \|}{\rho}, z_1, \ldots, z_{n-1} \| > 0. \)
Proof. (i) $\implies$ (ii) is obvious, since 
$[\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m) \subseteq [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m).$

(ii) $\implies$ (iii). Suppose 
$[\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m) \subseteq [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m)$ and let (iii) does not hold. Then for some $t > 0$

$$\sup_{\eta_i} \frac{1}{\eta_i} \sum_{k=1}^{\eta_i} |M_k(t)|^p_k = \infty,$$

and therefore there is a sequence $(\eta_i)$ of positive integers such that

$$\frac{1}{\eta_i} \sum_{k=1}^{\eta_i} |M_k(i-1)|^p_k > i, \quad i = 1, 2, ... \quad (2)$$

Define $x = (x_k)$ by

$$x_k = \begin{cases} i-1, & 1 \leq k \leq \eta_i, \quad i = 1, 2, ... \\ 0, & k \geq \eta_i. \end{cases}$$

Then $x = (x_k) \in [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m)$ but $x = (x_k) \notin [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\implies$ (i). Suppose $x = (x_k) \in [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m)$ and $x = (x_k) \notin [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m)$.

Then

$$\sup_{s, \eta_i} \frac{1}{\eta_i} \sum_{k=1}^{\eta_i} \left| M_k\left(\|\frac{u_k \Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}\|\right)\right|_{\infty} = \infty. \quad (3)$$

Let $t = |\frac{u_k \Delta^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|$ for each $k$ and fixed $s$, then by (3)

$$\sup_{\eta_i} \frac{1}{\eta_i} \sum_{k=1}^{\eta_i} |M_k(t)|^p_k = \infty,$$

which contradicts (iii). Hence (i) must hold. \qed

**Theorem 3.4.** Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then the following statements are equivalent

(i) $[\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m) \subseteq [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m),$

(ii) $[\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m) \subseteq [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m),$

(iii) $\inf_{\eta_i} \frac{1}{\eta_i} \sum_{k=1}^{\eta_i} |M_k(t)|^p_k > 0, \quad t > 0.$

Proof. (i) $\implies$ (ii) is obvious.

(ii) $\implies$ (iii). Suppose $[\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m) \subseteq [\hat{c}, u, p, ||, \cdots, ||]_0 (\Delta^m)$ and let (iii) does not hold. Then

$$\inf_{\eta_i} \frac{1}{\eta_i} \sum_{k=1}^{\eta_i} |M_k(t)|^p_k = 0, \quad t > 0. \quad (4)$$

We can choose an index sequence $(\eta_i)$ such that

$$\frac{1}{\eta_i} \sum_{k=1}^{\eta_i} |M_k(i)|^p_k < i^{-1}, \quad i = 1, 2, ...$$
Define the sequence $x = (x_k)$ by

$$(x_k) = \begin{cases} 
  i, & 1 \leq k \leq \eta_i, \quad i = 1, 2, \ldots \\
  0, & k \geq \eta_i.
\end{cases}$$

Thus by (4), $x = (x_k) \in \left[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$ but $x = (x_k) \notin \left[\hat{c}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\implies$ (i) Let $x = (x_k) \in \left[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$. That is,

$$\lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{|u_k \Delta^m_r x_k + s, z_1, \cdots, z_{n-1}|}{\rho} \right) \right]^{p_k} = 0, \quad \text{uniformly in } s. \quad (5)$$

Suppose (iii) hold and $x = (x_k) \notin \left[\hat{c}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$. Then for some number $\epsilon_0 > 0$ and index $\eta_0$, we have $||u_k \Delta^m_r x_k + s, z_1, \cdots, z_{n-1}|| \geq \epsilon_0$, for some $s > s'$ and $1 \leq k \leq \eta_0$. Therefore $[M_k(\epsilon_0)]^{p_k} \leq \left[ M_k \left( \frac{|u_k \Delta^m_r x_k + s, z_1, \cdots, z_{n-1}|}{\rho} \right) \right]^{p_k}$ and consequently by (5)

$$\lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} [M_k(\epsilon_0)]^{p_k} = 0,$$

which contradicts (iii). Hence $\left[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r) \subseteq \left[\hat{c}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r).$ \hfill $\square$

**Theorem 3.5.** Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then

$$\left[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r) \subseteq \left[\hat{c}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$$

holds if and only if

$$\lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} [M_k(t)]^{p_k} = \infty, \quad t > 0. \quad (6)$$

**Proof.** Suppose $\left[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r) \subseteq \left[\hat{c}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$ and let (6) does not hold. Therefore there is a number $t_0 > 0$ and an index sequence $(\eta_i)$ such that

$$\frac{1}{\eta_i} \sum_{k=1}^{\eta_i} [M_k(t_0)]^{p_k} \leq N < \infty, \quad i = 1, 2, \ldots \quad (7)$$

Define the sequence $x = (x_k)$ by

$$(x_k) = \begin{cases} 
  t_0, & 1 \leq k \leq \eta_i, \quad i = 1, 2, \ldots \\
  0, & k \geq \eta_i.
\end{cases}$$

Clearly, $x = (x_k) \in \left[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$ but $x = (x_k) \notin \left[\hat{c}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r).$ Hence (6) must hold.

Conversely, if $x = (x_k) \in \left[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdot, \cdot||\right]_0 (\Delta^m_r)$, then for each $s$ and $\eta$

$$\frac{1}{\eta} \sum_{k=1}^{\eta} \left[ M_k \left( \frac{|u_k \Delta^m_r x_k + s, z_1, \cdots, z_{n-1}|}{\rho} \right) \right]^{p_k} \leq N < \infty. \quad (8)$$
Suppose that \( x = (x_k) \not\in \left[\hat{c}, u, p, ||\cdot, \cdot, \cdot||\right]_0(\Delta^m_r) \). Then for some number \( \epsilon_0 > 0 \) there is a number \( s_0 \)

\[
\frac{\|u_k\Delta^m_r x_k + s\|}{s}, z_1, \cdots, z_{n-1} \| \geq \epsilon_0, \quad \text{for} \quad s \geq s_0.
\]

Therefore

\[
[M_k(\epsilon_0)]^{pk} \leq \left[ M_k\left(\frac{\|u_k\Delta^m_r x_k + s\|}{s}, z_1, \cdots, z_{n-1} \| \right) \right]^{pk},
\]

and hence for each \( k \) and \( s \) we get

\[
\frac{1}{\eta} \sum_{k=1}^\eta [M_k(\epsilon_0)]^{pk} \leq N < \infty,
\]

for some \( N > 0 \), which contradicts (6). Hence

\[
\left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_\infty(\Delta^m_r) \subseteq \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_0(\Delta^m_r).
\]

\( \square \)

**Theorem 3.6.** Suppose \( M = (M_k) \) be a Musielak-Orlicz function and let \( 1 \leq p_k \leq \sup_k p_k < \infty \).

Then

\[
\left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_\infty(\Delta^m_r) \subseteq \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_0(\Delta^m_r)
\]

holds if and only if

\[
\lim_{t \to 0} \frac{1}{\eta} \sum_{k=1}^\eta [M_k(t)]^{pk} = 0, \quad t > 0.
\]  \( (9) \)

**Proof.** Let \( \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_\infty(\Delta^m_r) \subseteq \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_0(\Delta^m_r) \). Suppose that (9) does not hold. Then for some \( t_0 > 0 \),

\[
\lim_{t \to 0} \frac{1}{\eta} \sum_{k=1}^\eta [M_k(t)]^{pk} = L \neq 0.
\]  \( (10) \)

Define \( x = (x_k) \) by

\[
(x_k) = t \sum_{v=0}^{k-m} (-1)^m \left( \begin{array}{c} m + k - v - 1 \\ k - v \end{array} \right)
\]

for \( k = 1, 2, \ldots \). Then \( x = (x_k) \not\in \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_0(\Delta^m_r) \) but

\[
x = (x_k) \in \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_\infty(\Delta^m_r).
\]

Hence (9) must hold.

Conversely, let \( x = (x_k) \in \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_\infty(\Delta^m_r) \). Then for every \( k \) and \( s \), we have

\[
\|u_k\Delta^m_r x_k + s, z_1, \cdots, z_{n-1} \| \leq N < \infty.
\]

Therefore

\[
\left[ M_k\left(\frac{\|u_k\Delta^m_r x_k + s\|}{s}, z_1, \cdots, z_{n-1} \| \right) \right]^{pk} \leq [M_k(N)]^{pk}
\]

and

\[
\lim_{t \to 0} \frac{1}{\eta} \sum_{k=1}^\eta [M_k\left(\frac{\|u_k\Delta^m_r x_k + s\|}{s}, z_1, \cdots, z_{n-1} \| \right)]^{pk} \leq \lim_{t \to 0} \frac{1}{\eta} \sum_{k=1}^\eta [M_k(N)]^{pk} = 0.
\]

Hence \( x = (x_k) \in \left[\hat{c}, M, u, p, ||\cdot, \cdot, \cdot||\right]_0(\Delta^m_r) \). This completes the proof. \( \square \)
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