CR- SUBMANIFOLDS OF A NEARLY TRANS-HYPERBOLIC SASAKIAN MANIFOLD WITH A QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. We define a quarter symmetric metric connection in a nearly trans-hyperbolic sasakian manifold and we study CR-submanifolds of a nearly trans-hyperbolic sasakian manifold with a quarter symmetric metric connection. Moreover, we discuss the parallel distribution relating to ξ-vertical CR-submanifolds of a nearly trans-hyperbolic sasakian manifold with a quarter symmetric metric connection

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1. Introduction

A. Bejancu introduced the notion of CR-submanifolds of a Kaehler manifold in [5]. Latter, CR-submanifold have been studied by Kobayashi[1], Shahid et al. [18, 19], Yano and Kon [22] and others. Upadhyay and Dube [21] have studied almost contact hyperbolic \((f, g, \eta, \xi)\)-structure, Dube and Mishra [9] have considered Hypersurfaces im-mersed in an almost hyperbolic Hermitian manifold also Dube and Niwas [10] worked with almost r-contact hyperbolic structure in a product manifold. Gherghe studied on harmonicity on nearly trans-Sasaki manifolds [11]. Bhatt and Dube [7] studied on CR-submanifolds of trans- hyperbolic Sasakian manifold. Joshi and Dube [14] studied on Semi-invariant submanifold of an almost r-contact hyperbolic metric manifold. Gill and Dube have also worked on CR submanifolds of trans-hyperbolic Sasakian manifolds [12].

Let \(\nabla\) be a linear connection in an \(n\)-dimensional differentiable manifold \(\overline{M}\). The torsion tensor \(T\) and the curvature tensor \(R\) of \(\nabla\) are given respectively by \([8]\)

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]
\]

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z
\]

The connection \(\nabla\) is symmetric if the torsion tensor \(T\) vanishes, otherwise it is non-symmetric. The connection \(\nabla\) is metric if there is a Riemannian metric \(g\) in \(\overline{M}\) such that \(\nabla g = 0\), otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [13], S. Golab introduced the idea of a quarter-symmetric connection. A linear connection is said to be a quarter-symmetric connection if its torsion tensor \(T\) is of the form

\[
T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y.
\]
where $\eta$ is a 1-form. In [2, 4], M. Ahmad et al. studied some properties of hypersurfaces of an almost $r$-paracontact Riemannian manifold with connections and also in [1, 3, 15, 20] studied properties of CR-submanifolds of a nearly trans-Sasakian hyperbolic manifolds with connections.

2. Preliminaries

Let $\overline{M}$ be an $n$ dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure $(\phi, \xi, \eta, g)$ where a tensor $\phi$ of type $(1, 1)$, a vector field $\xi$, called structure vector field and $\eta$, the dual 1-form of is a 1-form $\xi$ satisfying the following

\begin{align}
\phi^2 X &= X - \eta(X)\xi, \quad g(X, \xi) = \eta(X) \\
\phi(\xi) &= 0, \quad \eta \phi = 0, \quad \eta(\xi) = -1 \\
g(\phi X, \phi Y) &= -g(X, Y) - \eta(X)\eta(Y),
\end{align}

for any $X, Y$ tangents to $\overline{M}$ [4]. In this case

\[ g(\phi X, Y) = -g(X, \phi Y) \]

An almost hyperbolic contact metric structure $(\phi, \xi, \eta, g)$ on $\overline{M}$ is called trans-hyperbolic Sasakian [6] if and only if

\[ (\overline{\nabla}_X \phi) Y = \alpha (g(X, Y)\xi - \eta(Y)\phi X) + \beta \{ g(\phi X, Y)\xi - \eta(Y)\phi X \} \]

for all $X, Y$ tangents to $\overline{M}$ and $\alpha, \beta$ are functions on $\overline{M}$. On a trans-hyperbolic Sasakian manifold $M$, we have

\[ \overline{\nabla}_X \xi = -\alpha(\phi X) + \beta[X - \eta(X)\xi] \]

a Riemannian metric $g$ and Riemannian connection $\overline{\nabla}$.

Further, an almost contact metric manifold $\overline{M}$ on $(\phi, \xi, \eta, g)$ is called nearly trans-hyperbolic Sasakian if [5]

\[ (\overline{\nabla}_X \phi) Y + (\overline{\nabla}_Y \phi) X = \alpha \{ 2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y \} - \beta \{ \eta(X)\phi Y + \eta(Y)\phi X \} \]

On other hand, a quarter symmetric metric connection $\overline{\nabla}$ on $M$ is defined by

\[ \overline{\nabla}_X Y = \overline{\nabla}_X^* Y + \eta(Y)\phi X - g(\phi X, Y)\xi \]

Using (2.1), (2.2) and (2.6) in (2.5) and (2.6), we get respectively

\[ (\overline{\nabla}_X \phi) Y = \alpha \{ g(X, Y)\xi - \eta(Y)\phi X \} + \beta \{ g(\phi X, Y)\xi - \eta(Y)\phi X \} + g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \]

\[ \overline{\nabla}_X \xi = -(\alpha + 1)\phi X + \beta[X - \eta(X)\xi] \]

In particular, an almost contact metric manifold $\overline{M}$ on $(\phi, \xi, \eta, g)$ is called nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric metric connection if

\[ (\overline{\nabla}_X \phi) Y + (\overline{\nabla}_Y \phi) X = \alpha \{ 2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y \} - \beta \{ \eta(X)\phi Y + \eta(Y)\phi X \} \]

Now, let $M$ be a submanifold immersed in $\overline{M}$. The Riemannian metric induced on $M$ is denoted by the same symbol $g$. Let $TM$ and $T^\perp M$ be the Lie algebras of vector fields tangential to $M$ and normal to $M$ respectively and $\nabla$ be the induced Levi-Civita connection on $M$, then the Gauss and Weingarten formulas for the quarter symmetric metric connection are given by
\[ \bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \]  
for any \( X, Y \in TM \) and \( V \in T^\perp M \), where \( \nabla^\perp \) is the connection on the normal bundle \( T^\perp M \), \( h \) is the second fundamental form and \( A_N \) is the Weingarten map associated with \( N \) as

\[ g(A_N X, Y) = g(h(X, Y), N) \]  
for any \( x \in M \) and \( X \in T_x M \), we write

\[ X = PX + QX \]  
where \( PX \in D \) and \( QX \in D^\perp \).

Similarly for \( N \) normal to \( M \), we have

\[ \phi N = BN + CN \]  
where \( BN \) (respectly \( CN \)) is the tangential component (respectly normal component) of \( \phi N \).

**Definition.** An \( m \) dimensional Riemannian submanifold \( M \) of \( \bar{M} \) is called a CR-submanifold of \( M \) if there exists a differentiable distribution \( D : x \to D_x \) on \( M \) satisfying the following conditions:

(i) \( D \) is invariant, that is \( \phi D_x \subset D_x \) for each \( x \in M \),

(ii) The complementary orthogonal distribution \( D^\perp : X \to D^\perp_x \subset T_x M \) of \( D \) is anti-invariant, that is, \( \phi D^\perp_x \subset T^\perp_x M \) for each \( x \in M \). If \( \dim D^\perp_x = 0 \) (respectly \( \dim D_x = 0 \) ), then the CR-submanifold is called an invariant (respectly, anti-invariant) submanifold. The distribution \( D \) (respectly, \( D^\perp \)) is called the horizontal (respectly, vertical) distribution. Also, the pair \( (D, D^\perp) \) is called \( \xi \)-horizontal (respectly, vertical) if \( \xi \in D_x \) (respectly, \( \xi \in D^\perp_x \)).

### 3. Some basic lemmas

**Lemma 1.** If \( M \) be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \bar{M} \) with a quarter symmetric metric connection, then

\[ P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY} X - PA_{\phi QX} Y \]

\[ = 2(\alpha + 1)g(X,Y)P\xi - \alpha \eta(Y)\phi PX - \alpha \eta(X)\phi PY - \beta \eta(Y)\phi PX - \beta \eta(X)\phi PY \]

\[ -\eta(X)PY - \eta(Y)PX + 4\eta(\xi)\eta(Y)P\xi + \phi P\nabla_X Y + \phi PV_Y X \]

\[ Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - QA_{\phi QY} X - QA_{\phi QX} Y \]

\[ = 2Bh(X,Y) + 2(\alpha + 1)g(X,Y)Q\xi - \alpha \eta(Y)Q\xi - \alpha \eta(X)Q\xi \]

\[ -\eta(X)QY - \eta(Y)QX + 4\eta(\xi)\eta(Y)Q\xi \]

\[ h(X, \phi PY) + h(Y, \phi PX) + \nabla^\perp_X \phi QY + \nabla^\perp_Y \phi QX \]

\[ = \phi Q\nabla_X + \phi Q\nabla_Y + 2Ch(X,Y) - \beta \eta(Y)Q\xi - \beta \eta(X)Q\xi \]

for any \( X, Y \in TM \).

**Proof.** Using (4), (9) and (10) in (11) we get

\[ (\nabla_X \phi PY) + h(X, \phi PY) - A_{\phi QY} X + \nabla^\perp_X \phi QY - \phi(\nabla_X Y) - \phi h(X, Y) \]

\[ +(\nabla_Y \phi PX) + h(Y, \phi PX) - A_{\phi QX} Y + \nabla^\perp_Y \phi QX - \phi(\nabla_Y X) - \phi h(Y, X) \]

\[ = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi Y\} \]

\[ -\eta(X)Y - \eta(Y)X + 4\eta(\xi)\eta(Y)\xi + 2\eta(X)\xi \]
Again using (15) we get
\[ P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y - \phi P\nabla_X Y \quad (20) \]
\[ -\phi Q \nabla_X Y - \phi P \nabla_Y X - Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) \]
\[ -QA_{\phi QY}X - QA_{\phi QX}Y + h(X, \phi PX) + h(Y, \phi PX) + \nabla_X \phi QY \]
\[ +\nabla_Y \phi QX - 2Bh(X, Y) - 2C(h(X, Y) = 2\alpha g(X, Y)P\xi \]
\[ +2\alpha g(X, Y)Q\xi - \alpha \eta(Y)\phi PX - \alpha \eta(X)\phi QX - \alpha \eta(X)\phi PY \]
\[ -\alpha \eta(X)\phi QY - \beta \eta(Y)\phi PX - \beta \eta(Y)\phi QX - \beta \eta(X)\phi PY \]
\[ -\beta \eta(X)\phi QY - \eta(X)PY - \eta(X)QY - \eta(Y)PX - \eta(Y)QX \]
\[ +4\eta(X)\eta(Y)P\xi + 4\eta(X)\eta(Y)Q\xi + 2g(X, Y)P\xi + 2g(X, Y)Q\xi \]
for any \( X, Y \in TM \).

Now equating horizontal, vertical, and normal components in (20), we get the desired result.

**Lemma 2.** If \( M \) be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a quarter symmetric metric connection, then
\[ 2(\nabla_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y] \quad (21) \]
\[ +\alpha \{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta \{\eta(X)\phi Y + \eta(Y)\phi X\} \]
\[ -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi \]
\[ 2(\nabla_Y \phi)X = \alpha \{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta \{\eta(X)\phi Y + \eta(Y)\phi X\} \quad (22) \]
\[ -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi - \nabla_X \phi Y + \nabla_Y \phi X \]
\[ -h(X, \phi Y) + h(Y, \phi X) + \phi [X, Y] \]

**Proof.** From Gauss formula (12), we have
\[ \nabla_X \phi Y - \nabla_Y \phi X = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) \quad (23) \]

Also we have
\[ \nabla_X \phi Y - \nabla_Y \phi X = (\nabla_X \phi)Y - (\nabla_Y \phi)X + \phi [X, Y] \quad (24) \]

From (22) and (23), we get
\[ (\nabla_X \phi)Y - (\nabla_Y \phi)X = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y] \quad (25) \]

Also for nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a quarter symmetric metric connection, we have
\[ (\nabla_X \phi)Y + (\nabla_Y \phi)X \]
\[ = \alpha \{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta \{\eta(X)\phi Y + \eta(Y)\phi X\} \]
\[ -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi \quad (26) \]

Adding (3.9) and (3.10), we get
\[ 2(\nabla_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi [X, Y] \]
\[ +\alpha \{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta \{\eta(X)\phi Y + \eta(Y)\phi X\} \]
\[ \eta(Y)\phi X \]
\[ -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi \]

Subtracting (25) from (26) we get
\[ 2(\nabla_Y \phi)X = \alpha \{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta \{\eta(X)\phi Y + \eta(Y)\phi X\} \]
\[ -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi - \nabla_X \phi Y + \nabla_Y \phi X \]
\[ -h(X, \phi Y) + h(Y, \phi X) + \phi [X, Y] \]

Hence Lemma is proved.
Lemma 3. If $M$ be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric metric connection, then

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y - \nabla^Z_2\phi Y + \nabla^Y_2\phi Z - \phi[Y, Z]$$

$$+ \alpha\{2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\}$$

$$- \eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y, Z)\xi$$

$$2(\overline{\nabla}_Z\phi)Y = \alpha\{2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\}$$

$$- \eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y, Z)\xi - A_{\phi Y}Z + A_{\phi Z}Y$$

$$+ \nabla^Z_2\phi Y - \nabla^Y_2\phi Z + \phi[Y, Z]$$

for any $Y, Z \in D^\perp$.

**Proof.** From Weingarten formula (13), we have

$$\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^Y_2\phi Z - \nabla^Z_2\phi Y$$

(27)

Also, we have

$$\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = (\overline{\nabla}_Y\phi)Z - (\overline{\nabla}_Z\phi)Y + \phi[Y, Z]$$

(28)

From (27) and (28), we get

$$(\overline{\nabla}_Y\phi)Z - (\overline{\nabla}_Z\phi)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^Y_2\phi Z - \nabla^Z_2\phi - \phi[Y, Z]$$

(29)

Also for nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric metric connection, we have

$$\overline{\nabla}_Y\phi Z + (\overline{\nabla}_Z\phi)Y = \alpha\{2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\}$$

$$- \eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y, Z)\xi$$

(30)

Adding (29) and (30), we get

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y - \nabla^Z_2\phi Y + \nabla^Y_2\phi Z - \phi[Y, Z]$$

$$+ \alpha\{2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\}$$

$$- \eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y, Z)\xi$$

Subtracting (29) from (30) we get

$$2(\overline{\nabla}_Z\phi)Y = \alpha\{2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\}$$

$$- \eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y, Z)\xi - A_{\phi Y}Z$$

$$+ A_{\phi Z}Y + \nabla^Y_2\phi Y - \nabla^Z_2\phi Z + \phi[Y, Z]$$

This proves our assertions.

Lemma 4. If $M$ be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric metric connection, then

$$2(\overline{\nabla}_X\phi)Y = \alpha\{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}$$

$$- \eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi - A_{\phi Y}X + \nabla^Y_2\phi Y$$

$$- \nabla^X_2\phi Y - h(Y, \phi X) - \phi[X, Y]$$

$$2(\overline{\nabla}_Y\phi)X = \alpha\{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}$$

$$- \eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi + A_{\phi Y}X - \nabla^X_2\phi Y$$

$$+ \nabla^Y_2\phi X + h(Y, \phi X) + \phi[X, Y]$$

for any $X \in D$ and $Y \in D^\perp$. 
Proof. By using Gauss equation and Weingarten equation for \( X \in D \) and \( Y \in D^\perp \) respectively we get
\[
\nabla_X \phi Y - \nabla_Y \phi X = -A_{\phi Y} X + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X)
\] (31)

Also, we have
\[
\nabla_X \phi Y - \nabla_Y \phi X = (\nabla_X \phi) Y - (\nabla_Y \phi) X + \phi [X, Y]
\] (32)

From (31) and (32), we get
\[
(\nabla_X \phi) Y - (\nabla_Y \phi) X = -A_{\phi Y} X + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y]
\] (33)

Also for nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a quarter symmetric metric connection, we have
\[
(\nabla_X \phi) Y + (\nabla_Y \phi) X = \alpha \{2g(X, Y) \xi - \eta(Y) \phi X - \eta(X) \phi Y\} - \beta \{\eta(Y) \phi X + \eta(Y) \phi X\}
\] (34)

Adding (33) and (34), we get
\[
2(\nabla_X \phi) Y = \alpha \{2g(X, Y) \xi - \eta(Y) \phi X - \eta(Y) \phi X\} - \beta \{\eta(Y) \phi X + \eta(X) \phi Y\}
\]
\[
- \eta(Y)X - \eta(Y)X + 4\eta(X)\eta(Y) \xi + 2g(X, Y) \xi - A_{\phi Y} X + \nabla_X \phi Y
\]
\[
- \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y]
\] Subtracting (25) from (26) we get
\[
2(\nabla_Y \phi) X = \alpha \{2g(X, Y) \xi - \eta(Y) \phi X - \eta(X) \phi Y\} - \beta \{\eta(Y) \phi X + \eta(X) \phi Y\}
\]
\[
- \eta(Y)X - \eta(Y)X + 4\eta(X)\eta(Y) \xi + 2g(X, Y) \xi - A_{\phi Y} X - \nabla_X \phi Y
\]
\[
+ \nabla_Y \phi X + h(Y, \phi X) + \phi [X, Y]
\] Hence Lemma is proved.

4. Parallel distributions

Definition. The horizontal (respectively, vertical) distribution \( D \) (respectively, \( D^\perp \)) is said to be parallel [1] with respect to the connection on \( M \) if \( \nabla_X Y \in D \) (respectively, \( \nabla_Z W \in D^\perp \)) for any vector field \( X, Y \in D \) (respectively, \( W, Z \in D^\perp \)).

Proposition 1. If \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a quarter symmetric metric connection and the horizontal distribution \( D \) is parallel, then
\[
h(X, \phi Y) = h(Y, \phi X)
\] (35)

for all \( X, Y \in D \).

Proof. Using parallelism of horizontal distribution \( D \), we have
\[
\nabla_X \phi Y \in D, \ \nabla_Y \phi X \in D \quad \text{for any} \ X, Y \in D
\] (36)

Thus using the fact that \( X = QY = 0 \) for \( Y \in D \), (18) gives
\[
Bh(X, Y) = g(X, Y)Q \xi \quad \text{for any} \ X, Y \in D
\] (37)

Also, since
\[
\phi h(X, Y) = Bh(X, Y) + Ch(X, Y),
\] (38)

then
\[
\phi h(X, Y) = g(X, Y)Q \xi + Ch(X, Y) \quad \text{for any} \ X, Y \in D
\] (39)

Next from (19), we have
\[
h(X, \phi Y) + h(Y, \phi X) = 2Ch(X, Y) = 2\phi h(X, Y) - 2g(X, Y)Q \xi,
\] (40)

for any \( X, Y \in D \). Putting \( X = \phi X \in D \) in (40), we get
\[
h(\phi X, \phi Y) + h(Y, \phi X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q \xi
\] (41)
or
\[ h(\phi X, \phi Y) - h(Y, X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q\xi \] (42)
Similarly, putting \( Y = \phi Y \in D \) in (4.6), we get
\[ h(\phi Y, \phi X) - h(X, Y) = 2\phi h(X, \phi Y) - 2g(X, \phi Y)Q\xi. \] (43)
Hence from (42) and (43), we have
\[ \phi h(X, \phi Y) - \phi h(Y, \phi X) = g(X, \phi Y)Q\xi - g(\phi X, Y)Q\xi \] (44)
Operating \( \phi \) on both sides of (4.10) and using \( \phi \xi = 0 \), we get
\[ h(X, \phi Y) = h(Y, \phi X) \] (45)
for all \( X, Y \in D \).
Now, for the distribution \( D^\perp \), we prove the following proposition.

**Proposition 2.** If \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold \( \bar{M} \) with a quarter symmetric metric connection and the distribution \( D^\perp \) is parallel with respect to the connection on \( M \), then
\[ A_{\phi Y}Z + A_{\phi Z}Y \in D^\perp \text{ for any } Y, Z \in D^\perp. \] (46)

**Proof.** Let \( Z \in D^\perp \), then using Gauss and Weingarten formula (2.10), we obtain
\[ -A_{\phi Z}Y + \nabla_Y^\perp \phi Z - \theta_{\phi Z}Y = \phi \nabla_Y Z + \phi h(Y, Z) + \phi \nabla_Z Y + \phi h(Z, Y) \]
\[ + \alpha(2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y) \]
\[ - \eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y, Z)\xi \] (47)
for any \( Y, Z \in D^\perp \). Taking inner product with \( X \in D \) in (47), we get
\[ g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X) \] (48)
But the distribution \( D^\perp \) is parallel, then \( \nabla_Y Z \in D^\perp \) and \( \nabla_Z Y \in D^\perp \), for any \( Y, Z \in D^\perp \). Thus from (48) we have
\[ g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = 0 \text{ or } g(A_{\phi Y}Z + A_{\phi Z}Y, X) = 0 \] (49)
which is equivalent to
\[ A_{\phi Y}Z + A_{\phi Z}Y \in D^\perp \text{ for any } Y, Z \in D^\perp \]
and this completes the proof.

**Definition:** A CR-submanifold \( M \) of a nearly trans-hyperbolic Sasakian Manifold \( \bar{M} \) with a quarter symmetric metric connection is said to be totally geodesic if
\[ h(X, Y) = 0 \text{ for } X \in D \text{ and } Y \in D^\perp. \]
It follows immediately that a CR-submanifold is mixed totally geodesic if and only if \( A_N X \in D \) for each \( X \in D \) and \( N \in T^\perp M \).
Let \( X \in D \) and \( Y \in \phi D^\perp \). For a mixed totally geodesic \( \xi \)-vertical CR-submanifold \( M \) of a nearly trans hyperbolic Sasakian Manifold \( \bar{M} \) with a quarter symmetric metric connection then from (9), we have
\[ (\bar{\nabla}_X \phi)N = 0 \]
Since \( \bar{\nabla}_X \phi N = (\bar{\nabla}_X \phi)N + \phi(\bar{\nabla}_X N) \) so that \( \bar{\nabla}_X \phi N = \phi(\bar{\nabla}_X N) \).
Hence in view of (2.13), we get
\[ \bar{\nabla}_X \phi N = -A_{\phi N}X + \nabla^\perp_X \phi N = -\phi A_N X + \phi \bar{\nabla}^\perp_X N \]
As \( A_N X \in D \), \( \phi A_N X \in D \), so \( \phi \nabla^\perp_X N = 0 \) if and only if \( \bar{\nabla}_X \phi N \in D \).
Thus we have the following proposition.
Proposition 3. If $M$ be a mixed totally geodesic $\xi$-vertical CR-submanifold of a nearly trans hyperbolic Sasakian Manifold $\tilde{M}$ with a quarter symmetric metric connection, then the normal section $N \in \phi D^\perp$ is $D$ parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.

5. Integrability conditions of distributions

Lemma 5.1. If $M$ be a CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\tilde{M}$ with a quarter symmetric metric connection, then

$$\nabla_{\phi X}(Y) = 2(\alpha + 1)g(X,Y)\xi - (\alpha + \beta)\eta(Y)X + (\alpha + \beta)\eta(X)\eta(Y)\xi$$

for any $X,Y \in TM$.

Proof. For nearly trans-hyperbolic Sasakian Manifold $\tilde{M}$ with a quarter symmetric metric connection, we have

$$\nabla_{\phi X}(Y) = 2(\alpha + 1)g(X,Y)\xi - (\alpha + \beta)\eta(Y)X + (\alpha + \beta)\eta(X)\eta(Y)\xi - \eta(Y)\phi X$$

and we have

$$\nabla_{\phi X}(Y) = \nabla_X \phi_2 X - \phi(\nabla_X \phi X) = \nabla_X \phi_2 X - \phi(\nabla_X \phi_2 X) + \phi(\nabla_Y \phi X) - \phi(\phi \nabla_Y X)$$

$$= \nabla_Y X - \eta(X)\nabla_Y \xi - \phi(\nabla_X \phi X) - \phi(\phi \nabla_X X)$$

for any $X,Y \in TM$, which completes the proof of the lemma. On a nearly trans-hyperbolic Sasakian Manifold $\tilde{M}$ with a quarter symmetric metric connection, Nijenhuis tensor is given by

$$N_{\phi}(X,Y) = \nabla_{\phi X}(Y) - \nabla_{\phi Y}(X) - \phi(\nabla_X \phi X) - \phi(\nabla_Y \phi Y)$$

for any $X,Y \in TM$.

As of (50) and (54), we have

$$N_{\phi}(X,Y) = 4(\alpha + 1)g(X,Y)\xi + (\alpha + \beta)\eta(Y)X + 3(\alpha + \beta)\eta(X)Y + \eta(Y)\phi X$$

$$+ 3\eta(X)\phi Y - 4(\alpha + \beta)\eta(X)\eta(Y)\xi + \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi$$

Proposition 5.2. If $M$ be a CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\tilde{M}$ with a quarter symmetric metric connection, then

$$\nabla_{A\phi Y Z}(A\phi Z Y) = \phi P[Y,Z] + (2\alpha + 3)[\eta(Y)Z - \eta(Z)Y]$$

for any $Y,Z \in D^\perp$.

Proof: For $Y,Z \in D^\perp$ and $X \in T(M)$, we have

$$2g(A\phi_2 Y, X) = 2g(h(X,Y), \phi Z) = g(h(X,Y), \phi Z) + g(h(X,Y), \phi Z)$$

$$= g(\nabla_X Y, \phi Z) + g(\nabla_Y X, \phi Z) = g(\nabla_X Y + \nabla_Y X, \phi Z)$$
\[ (\bar{\nabla}_Y \phi)X, Z ) = g(\phi(\bar{\nabla}_X Y + \bar{\nabla}_Y X), Z) = -g(\bar{\nabla}_X \phi Y + \bar{\nabla}_Y \phi X - (\bar{\nabla}_X \phi)Y - \eta(X)\phi Y \]

\[ = -g(\bar{\nabla}_X \phi Y, Z) - g(\bar{\nabla}_Y \phi X, Z) + g(\alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) \]

\[ \eta(X)\phi Y \}

\[ - \beta \{ \eta(X)\phi Y + \eta(Y)\phi X \} - \eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi, Z) \]

\[ = g(A_{\phi Y} Z, X) - g(\phi(\bar{\nabla}_Y Z), X) + 2\alpha g(\eta(Y)\eta(Z), X) + \alpha g(\eta(Y)\phi Z, X) \]

\[ - \alpha g(\phi(Y, Z)\xi, X) - g(\eta(Y)\eta(Z), X) + 2g(\eta(Y)\eta(Z), X) \]

\[ 2g(\eta(Y)\eta(Z), X) \]

The above equation is true for all \( X \in T(M) \), therefore transvecting the vector field \( X \) both sides, we have

\[ A_{\phi Y} Z = A_{\phi Y} Z - \phi \bar{\nabla}_Z + 2\alpha \eta(Z) Y + \alpha \eta(Y) \phi Z - \alpha g(\phi Y, Z) \xi \]

\[ - \beta g(\phi Y, Z) \xi + \beta \eta(Y) \phi Z - g(Y, Z) \xi - \eta(Y)Z + 4\eta(Y)\eta(Z)\xi + 2\eta(Z) \]  

for any \( Y, Z \in D^\perp \). Interchanging the vector fields \( Y \) and \( Z \), we get

\[ 2A_{\phi Y} Z = A_{\phi Y} Z - \phi \bar{\nabla}_Y + 2\alpha \eta(Y) Y + \alpha \eta(Z) \phi Y - \alpha g(\phi Z, Y) \xi \]

\[ \beta g(\phi Z, Y) \xi + \beta \eta(Z) \phi Y - g(Z, Y) \xi - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2\eta(Y) \]

Subtracting (57) and (58), we get

\[ 3(A_{\phi Y} Z - A_{\phi Z} Y) = \phi P[Y, Z] + (2\alpha + 3)[\eta(Y)Z - \eta(Z)Y] \]

\[ + (\alpha + \beta)[\eta(Z)\phi P Y - \eta(Y)\phi Z] - 2(\alpha + \beta)g(\phi P Z, Y) P \xi \]

for any \( Y, Z \in D^\perp \).

**Theorem 5.1.** If \( M \) be a CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold \( \tilde{M} \) with a quarter symmetric metric connection, then the distribution \( D^\perp \) is integrable if and only if

\[ 3(A_{\phi Y} Z - A_{\phi Z} Y) = \left( \frac{2\alpha + 3}{3} \right) [\eta(Y)Z - \eta(Z)Y] \]

**Proof:** Primary suppose that the distribution \( D^\perp \) is integrable. Then \( [Y, Z] \in D \) for any \( Y, Z \in D^\perp \). Since \( P \) is a projection operator on \( D \), so \( P[Y, Z] = 0 \). Thus from (55) we get (60). Conversely, we suppose that (60) holds. Then using (55), we have \( \phi P[Y, Z] = 0 \) for any \( Y, Z \in D^\perp \). Since rank \( \phi = 2n \). Therefore, either \( P[Y, Z] = 0 \) or \( P[Y, Z] = k \xi \). But \( P[Y, Z] = k \xi \) is not possible as \( P \) is a projection operator on \( D \). Thus, \( P[Y, Z] = 0 \), which is equivalent to \( [Y, Z] \in D^\perp \) for any \( Y, Z \in D^\perp \) and hence \( D^\perp \) is integrable.

**Corollary 5.1.** If \( M \) be a \( \xi \)-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold \( \tilde{M} \) with a quarter symmetric metric connection, then the distribution \( D^\perp \) is integrable if and only if

\[ A_{\phi Y} Z - A_{\phi Z} Y = 0 \]

(61)
REFERENCES