THE SOLUTIONS OF INITIAL BOUNDARY PROBLEMS NONLINEAR ELLIPTIC-PARABOLIC EQUATIONS

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Abstract. In the present paper boundary value problem for second order elliptic-parabolic equations is investigated. In degenerated case the priori estimates and qualitative properties of solutions are proved.

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1. Introduction

We consider problems which arise as mathematical models of various applied problems. For instance reaction-drift-diffusion processes of electrically charged species phase transition processes in porous media. Investigation of boundary value problems for second order elliptic-parabolic equations ascend to the work by Keldysh [9], Fichera [4]. In work [6] boundary value problem for second order degenerate elliptic-parabolic equations is investigated. In works many authors [1,2,5,7,8,10] similarly problem is considered. Now we consider case degenerate head part of nonlinear equation. Let Ω be a bounded domain in R^n, $Q_T = \Omega \times (0,T)$. We consider the initial boundary value problem

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,l}(x,t,u)\frac{\partial}{\partial x_j}\right) - \psi(x,t)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u = 0 \ , \tag{1}$$

$$u(x,t) = f(x,t), \quad (x,t) \in \Gamma = \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0)=h(x), \quad x\in\Omega\ . \tag{3}$$

We assume the following regularity condition on the boundary $\partial\Omega$ of the set Ω is fulfilled. There exist numbers k, R_0 such that for an arbitrary point $x\in\partial\Omega$ the following inequality holds $\{B(x,R)\setminus\Omega\}\geq kR^n$ for any $0<R<R_0$ where $B(x,R)$ is a ball of radius R with center x.

Let the coefficients of problem (1)-(3) satisfy the following assumptions: $\|a_{i,j}(x,t,u)\|$ is a real symmetrical matrix and for any $(x,t)\in Q_T$ and $\xi\in R_n$ the following inequality are true

$$\gamma\,\omega(x)|\xi|^2\leq\sum_{i,j=1}^{n}a_{i,j}(x,t,u)\xi_i\xi_j\leq\gamma^{-1}\omega(x)|\xi|^2, \tag{4}$$

where $\gamma\in(0,1], a_{i,j}(x,t,u), c(x,t), b_i(x,t), i,j=\overline{1,n}$ are measurable functions with respect to x,t to for every $(x,t)\in Q_T$, where $\omega(x)$ satisfy Mackenhoupt condition [3]. We another papers will be consider case when coefficients $a_{i,j}(x,t,u)$ have growth by u. Also

$$c(x,t)\leq 0\ ,\ c(x,t)\in L_{n+1}(\Omega) \tag{5}$$

$$|b_i(x,t)|\in L_{n+2}(\Omega)\ ,\ |b_i(x,t)|^2+Kc(x,t)\leq 0 \tag{6}$$

Assume that the following conditions are true for the weighted functions

$$\psi(x,t)=\omega(x)\cdot\lambda(t)\varphi(T-t)$$

$$\lambda(t)\in C^1[0,T], \varphi(z)\geq 0, \varphi'(z)\geq 0, \varphi'(z)\in C^1[0,T],$$

$$\varphi(0)=\varphi'(0)=0, \varphi(z)=\beta\cdot z\cdot\varphi'(z), \tag{7}$$

where β are positive constants. We consider the problem (1)-(3) with data satisfying

$$f(x,t)\in L^\infty(Q_T)\cap L^\infty\left(0,T,W_\infty^1(\Omega)\right)\cap L_1\left(0,T,W_\infty^1(\Omega)\right)$$

$$\frac{\partial f}{\partial t}\in L_1\left(0,T,L_\infty(\Omega)\right) \tag{8}$$

$$h(x)\in L_\infty(\Omega). \tag{9}$$

We introduce some space of function in Q_T with finite norm

$$\|u\|_{W_{2,\psi}^{1,1}(\Omega)}=\left(\left(\int_{Q_T}\omega(x)\left(u^2+\sum_{i=1}^{n}u_{x_i}^2+u_t^2\right)+\psi^2(x,t)u_{tt_i}^2\right)dxdt\right)^{1/2}$$

We denote by $\overset{\circ}{W}{}^{1,1}_{2,\psi}(Q_T)$ the closure of $C_0^\infty\left(\overline{Q_T}\right)$ with respect to the norm of $W^{1,1}_{2,\psi}(Q_T)$. A functions $u(x,t)$ from $\overset{\circ}{W}{}^{1,1}_{2,\psi}(Q_T)$ is called solution of problem (1)-(3) the integral identities

$$\int_0^T\int_\Omega\left(\frac{\partial u}{\partial t}\phi\, dxdt+\int_{Q_T}\left[\sum_{i,j=1}^n a_{i,j}(x,t,u)\frac{\partial u}{\partial x_j}\frac{\partial \phi}{\partial x_i}+\sum_{i,l=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}\phi+c(x,t)u\phi\right]dx\right)dt+$$

$$+\int_0^T\int_\Omega \psi^2(x,t)\frac{\partial^2 u}{\partial t^2}\phi\, dxdt=0 \tag{10}$$

hold for arbitrary functions $C^\infty\left(\overline{Q_T}\right)$ vanishing near Γ and $\tau\in(0,T)$,

$u-f(x,t)\in L_2\left(0,\tau,\overset{\circ}{W}{}^{1,1}_{2,\psi}(Q)\right)$.

2. Main results

Theorem 2.1. Let the conditions (4)-(9) be satisfied. Then there exists a constant M_1 depending only on known parameters such that each solution of problems (1)-(3) satisfies

$$\underset{t\in(0,T)}{ess}\int_\Omega\{\Lambda_1(u(x,t))+\Lambda_2(u(x,t))\}dx+\int_{Q_T}\omega(x)\left|\frac{\partial u}{\partial t}\right|^2dxdt+$$
$$+\int_{Q_T}\psi^2(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2dxdt\le M_1, \tag{11}$$

where

$\Lambda_1(u)=\int_0^u s\omega(s)ds$, $\Lambda_2(u)=\int_0^u s\psi(s,t)ds$, for almost everywhere by t.

Proof . Let $u(x,t)$ be the solution regularized problem (1)-(3). We extend function $u(x,t)$ by setting $u(x,t)=\phi(x)$ for $t<0, x\in\Omega.$ Denote

$$\overline{u}(x,t)=u(x,t)-f(x,t).$$

Testing (10) with $\hat{\phi}(x,t)=\overline{u}(x,t+s)-\overline{u}(x,t)$. We obtain for $\tau\in(0,T).s\in(0.T-\tau)$

$$\int_{-s}^{\tau}\int_{\Omega}\left\{\frac{\partial \overline{u}}{\partial t}\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]+\sum_{i.j=1}^{n}a_{ij}(x,t)\frac{\partial \overline{u}}{\partial x_j}\frac{\partial}{\partial x_j}\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]+\right.$$

$$\left.+\sum_{i=1}^{n}b_i(x,t)\frac{\partial u}{\partial x_i}\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]+c(x,t)\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]\right\}dxdt+$$

$$+\int_{-s}^{\tau}\int_{\Omega}\psi^2(x,t)\frac{\partial^2 u}{\partial t^2}\left[\left|\overline{u}(x,t+s)-\overline{u}(x,t)\right|\right]dxdt=0.$$

Hence we get by simple calculation

$$\int_{\tau}^{\tau+s}\int_{\Omega}\frac{\partial u}{\partial t}\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]dxdt+\int_{\tau}^{\tau+s}a_{ij}(x,t)\left|\frac{\partial \overline{u}}{\partial x}\right|^2dxdt-$$

$$-s\int_{\Omega}a_{ij}(x,t)\left|\frac{\partial v_0}{\partial x}\right|^2dxdt+\int_{-s}^{T}\left(\int_{\Omega}\sum_{i=1}^{n}b_i(x,t)\frac{\partial}{\partial x_i}\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]dx\right)dt+$$

$$+\int_{-s}^{T}\int_{\Omega}c(x,t)\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]dxdt+\int_{-s}^{T}\int_{\Omega}\psi^2(x,t)\frac{\partial^2\overline{u}}{\partial t^2}\left[\overline{u}(x,t+s)-\overline{u}(x,t)\right]dxdt=0,$$

where denote by $v_0(x)$ the solution of problem (1)-(3) for $t=0$ with $u(x,0)$ defined by (3).

Dividing this equality by s and passing to the limit $s\to 0$, we obtain for almost every $\tau\in(0,T)$ and doing some calculations

$$\int_{\Omega}a_{ij}(x,t)\left|\frac{\partial u(x,t)}{\partial x}\right|^2-\int_{\Omega}a_{ij}(x,t)\left|\frac{\partial v_0(x)}{\partial x}\right|^2dxdt+$$

$$+\int_0^{\tau}\int_{\Omega}\sum_{i=1}^{n}b_i(x,t)\frac{\partial u(x,t)}{\partial x_i}dxdt\le\int_0^{\tau}\int_{\Omega}c(x,t)\left|\frac{\partial u}{\partial x}\right|^2dxdt. \tag{12}$$

Using (10) we can write in (12)

$$\int_{\Omega}\varpi_{\varepsilon}(x)\left|\frac{\partial u(x,\tau)}{\partial x}\right|^2dx+\int_0^{\tau}\frac{\partial u}{\partial t}\overline{u}(x,t)dt+\int_{\Omega}\psi_{\varepsilon}^2(x,t)\left|\frac{\partial^2 u(x,\tau)}{\partial t^2}\right|^2dx\le$$

$$\le C_1\left\{1+\int_0^{\tau}\int_{\Omega}\left|\frac{\partial u(x,t)}{\partial x}\right|^2dxdt\right\}, \tag{13}$$

Here and in what follows C_i denote constants depending only on known parameters. The conditions (8), (9) allow us to substitute $\phi=\overline{u}$ in the regularized identity (10).

By (13) this gives

$$\int_0^\tau \frac{\partial u}{\partial t}\left(\overline{u}(x,t)-f(x,t)\right)dt+$$

$$+\int_0^\tau\int_\Omega\left\{\sum_{i.j=1}^n a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}+c(x,t)\overline{u}+\sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}\overline{u}\right\}dxdt\le$$

$$\le\int_0^\tau\int_\Omega\left\{\sum_{i.j=1}^n a_{ij}(x.t)\frac{\partial u}{\partial x_i}\frac{\partial f}{\partial x_i}-c(x,t)\overline{u}\right\}dxdt+$$

$$+C_1\left\{1+\int_0^\tau\int_\Omega\left|\frac{\partial u(x,t)}{\partial x}\right|^2dxdt\right\}. \tag{14}$$

We write first integral from (14) in the form

$$\int_0^\tau \frac{\partial u}{\partial t}\left(u-f(x,t)\right)dt=\int_0^\tau \frac{\partial u}{\partial t}\left(|u|_{-m}^m-f(x,t)\right)dt+\int_0^\tau \frac{\partial u}{\partial t}\left(u-|u|_{-m}^m\right)dt, \tag{15}$$

with

$$m\ge\|f(x,t)\|_{L_\infty(Q_T)},\ |u|_{-m}^m=\max\{\min[u,m],-m\}.$$

Then we can evaluate the first and the second integral of the right hand side of (15) by using Lemmas respectively [9]. So we obtain

$$\int_0^\tau \frac{\partial u}{\partial t}\left(u-f(x,t)\right)dt=\int_0^\tau\left\{\int_0^{u(x,\tau)}s\varpi(s)ds-\int_0^{h(x)}s\varpi(s)ds\right\}dx+$$

$$+\int_0^\tau\left\{\int_0^{u(x,\tau)}s\Psi(s,t)ds-\int_0^{h(x)}s\Psi(s,t)ds\right\}dx+\int_0^\tau\int_\Omega|u-h(x)|\frac{\partial f}{\partial t}dxdt-$$

$$-\int_\Omega[u(x,\tau)-h(x)]f(x,\tau)dx, \tag{16}$$

Immediately from the definition of $\Lambda_1(u),\Lambda_2(u)$. We deduce

$$u<\varepsilon_1\left(\Lambda_1(u)+\Lambda_2(u)\right)+C_{\varepsilon_1} \tag{17}$$

for $u\ge 0$ with arbitrary positive number ε and constant C_ε depending only on ε_1 and the function $\omega(x),\Psi(x,t)$. Using the condition (4)-(6), (8)-(9) and the conditions on $\omega(x),\Psi(x,t)$ and the inequality (17), we obtain with arbitrary positive number ε_1 and some function $\mu(t)\in L_1(0,T)$

$$\left|\int_0^\tau\int_\Omega \omega_\delta(x)\left|\frac{\partial u}{\partial x}\right|^2 \frac{\partial f(x,t)}{\partial x_j}dxdt\right| + \left|\int_0^\tau\int_\Omega \Psi_\varepsilon^2(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\left|\frac{\partial f}{\partial x_j}\right|dxdt\right| \le$$

$$\le \varepsilon_1\int_0^\tau\int_\Omega \omega_\delta(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt + \varepsilon_1\int_0^\tau\int_\Omega \Psi_\varepsilon^2(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt +$$

$$+\frac{C_2}{\varepsilon_1}\int_0^\tau\int_\Omega (\Lambda_1(u)+\Lambda_2(u))\mu(t)dxdt + \int_0^\tau\int_\Omega u\frac{\partial f}{\partial t}dxdt \le$$

$$\le C_2\left\{1+\int_0^\tau\int_\Omega (\Lambda_1(u)+\Lambda_2(u))\mu(t)dxdt\right\},$$

$$\int u(x,\tau)f(x,\tau)dx \le C_2\left\{\varepsilon_1\int \Lambda_1(u(x,\tau))+\Lambda_2(u(x,\tau))dx + C_{\varepsilon_1}\right\} \quad (18)$$

We estimate terms (14) involving the function α in standard way by using (4)-(6), (8)-(9). Now from (14), (16), (18) and evident estimates for another terms in (16), we obtain

$$\int_\Omega (\Lambda_1(u(x,\tau))+\Lambda_2(u(x,\tau)))dx + \int_0^\tau\int_\Omega\left[\omega_\delta(x)\left|\frac{\partial u}{\partial x}\right|^2 + \Psi_\varepsilon^2(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt \le$$

$$\le C_3\left\{1+\int_0^\tau\int_\Omega [1+\mu(t)](\Lambda_1(u)+\Lambda_2(u))dxdt\right\}. \quad (19)$$

Now the last inequality and Gronwalls lemma complete the proof theorem 2.1.

Theorem 2.2. Let the assumptions of Theorem 2.1 be satisfied. Then there exists a constant M_2 depending only on know parameters, such that each solution of problem (1)-(3) satisfies

$$\int_{Q_T}\left[\omega(x)\left|\frac{\partial u}{\partial t}\right|^2 + \psi^2(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt \le M_2\,. \quad (20)$$

In order to prove theorem 2.2 we need auxiliary estimates.

Lemma 2.1. Assume that the conditions of Theorem 2.1 are satisfied and the following inequality

$$\underset{t\in(0,T)}{ess}\int_\Omega \omega(x)u^q(x,t)dx \le K_1\,, \quad (21)$$

holds with a number $q\in\left(\frac{2n}{n+2},\frac{n}{2}\right), K_1$ which defined by the equality

$$\underset{t\in(0,T)}{ess}\left\{\int_\Omega \omega(x)|u(x,t)|^{\frac{pn}{n-2}}dx+\int_\Omega \omega(x)|u(x,t)|^{p-2}\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx\right\}\le K_2 \qquad (22)$$

$$p\frac{n}{n-2}=(p-1)\frac{q}{q-1} \qquad (23)$$

and with a const K_2 depending only on known parameters.

Proof of Theorem 2.2 We assume firstly that $\frac{2+\gamma}{1+\gamma}<\frac{n}{2}$. It is simple to check

$$|u|<c_0 \text{ for } u<0. \qquad (24)$$

For proving regularity properties of the function u we need the following growth condition

$$\rho_1^{-1}(u^\gamma+1)\le u\le \rho_1(u^\gamma+1),\ u>0,\ 0<\gamma<\frac{2}{n-2} \qquad (25)$$

with some positive constants ρ_1. From (24) and (25) we find

$$|u|^{q_0}\le C_7[\Lambda_1(u)+\Lambda_2(u)+1] \qquad (26)$$

with $q_0=\frac{2+\gamma}{1+\gamma}$.

Using (26), (11) and Lemma 2.1 we obtain (22).

Lemma 2.2. Assume that the conditions of theorem 2.2 are satisfied and

$$\underset{t\in(0,T)}{ess}\int_\Omega \omega(x)u^q(x,t)dx+\iint_{\{u>1\}}\left[\omega^2(x)u^{q-2}\left|\frac{\partial u}{\partial x}\right|^2+\psi^2(x,t)u^{q-2}\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt\le K_3, \qquad (27)$$

holds with numbers $q\in\left[\frac{2+\gamma}{1+\gamma},\frac{n}{2}\right], K_3$ depending only on known parameters.

Then there exist positive constants K_4 and β

$$\iint_{\{u>1\}}\left[\omega^2(x)u^{q-2+\beta}\left|\frac{\partial u}{\partial x}\right|^2+\psi^2(x,t)u^{q-2+\beta}\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt\le K_{.4}. \qquad (28)$$

Proof. From theorem 2.2 - it follows that (28) holds for $q=q_0=\frac{2+\gamma}{1+\gamma}$.. We shall prove (28) this value of q. If we use Lemma 2.1 and Theorem 2.2, after some calculations we prove Lemma 2.2.

Lemma 2.3. Assume that the conditions of Theorem 2.2 are satisfied. Then there exist numbers $\bar{q}, K_5$ depending only on known parameters, such that $\bar{q} > \frac{n}{2}$ and

$$\underset{t\in(0,T)}{ess}\int_{\Omega}\omega(x)u^{\bar{q}}(x,t)dx + \iint_{\{u>1\}}\left[\omega^2(x)u^{\bar{q}-2}\left|\frac{\partial u}{\partial x}\right|^2 + \psi^2(x,t)u^{\bar{q}-2}\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt \le K_5 . \quad (29)$$

Proof. We substitute the function

$$\varphi = [u - m_0]^2\left\{1 + [u - m_0]^3\right\}^r \ , \ r \in \left(-\frac{3}{2}, \infty\right)$$

in the integral identity (26), where

$$m_0 = \| f(x,t)\|_{L_\infty(Q_T)} + \| f(x,t)\|_{L_\infty(\Omega)} + 1.$$

If we use Lemma 2.1 and Lemma 2.2 after some calculations the proof is obtained.

Theorem 2.3. Let the assumptions of Theorem 2.2 be satisfied. Then the estimates

$$\|u(x,t)\|_{L_\infty(Q_T)} \le M_3 , \ |u(x',t)\omega(x') - u(x'',t)\omega(x'')| \le M_4|x' - x''|, \quad (30)$$

hold with positive constants depending only on known quantities.

Proof. The result of theorem follows immediately from the conditions (4)-(6), (7), (8), (9), (25) and $\omega'(z) \le \rho_2\omega(z), \rho_2 > 0$ constant is satisfied.

Then the initial-boundary value problem (1)-(3) has at least one solution in the sense of (10).

REFERENCES

1. Alt H.W., Luckhaus S. Quasilinear elliptic-parabolic differential equations, Math. Z., 183, 1983, pp.311-341.
2. Benilan P.H., Wittbold P. On mind and weak solutions of elliptic-parabolic systens, Adv. Differ. Equ., 1, 1996, pp.1053-1073.
3. Chanillo S., Wheeden R. Weighted Poincare and Sobolev inequalities. Amer. J. Math., 107, 1985, pp.1191-1226.
4. Fichera G. On a unified theory of boundary value problem for elliptic-parabolic equations of second order, Boundary problem in differential equations Madison, 1960, pp.97-120.
5. Gajewski H., Skrypnik I.V. To the uniqueness problem for nonllinear elliptic equations, Nonlinear Analysis, 52, 2003, pp.291-304.
6. Gadjiev T.S., Kerimova M.N. The solutions degenerate elliptic-parabolic, Equations, Journal of Advances in Mathematics, V.3, N.3, 2013, pp.218-235.
7. Gajewski H., Skrypnik I.V. On the uniqueness of solution for nonlinear parabolic equations, Preprint No 658, WIAS, 2001, 2003.

8. Gadjiev T.S., Gasimova E.R. On smoothness of solution of the first boundary value problem for second order degenerate elliptic-parabolic equations, Ukr. Mat. Journ., V.60, N.6, 2008, pp.723-736.
9. Keldysh M.V. On some cases of degeneration of equations of elliptic type on the boundary of a domain, DAN SSSR, V.77, 1951, pp.181-183 (in Russian).
10. Ladyzhenskaya O.A., Uraltseva N.N. Linear and quasilinear elliptic equations, Moscow, Nauka, 1973 (in Russian).

Решения начальных граничных проблем нелинейные эллиптические параболические уравнения

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РЕЗЮМЕ

В работе исследуется краевая задача для эллиптически-параболических уравнений второго порядка. В вырожденном случае априорные оценки и качественные свойства решений доказаны.

Ключевые слова: нелинейные эллиптические параболические уравнения, априорные оценки, краевая задача.