Abstract. The purpose of this paper is to prove some common fixed point theorems for four self maps in complete 2-metric spaces by employing the notion of weakly compatible mappings. Our results extend and generalize the results of Iseki (Fixed point theorems in 2-metric spaces, Math Seminar Notes, Kobe Uni. 3, 1975, 133 - 136) and several other authors.

Keywords: Fixed point, 2-metric space, weakly compatible, contractive modulus.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

Fixed point theory has many applications, including variational and linear inequalities, optimization, approximation theory and minimum norm problem. Banach [1] proved the famous and well known Banach contraction principle concerning the fixed point of contraction mappings defined on a complete metric space. This theorem has been generalized and extended by many authors (see: [7, 8]).

In 1963, Gahler [5] introduced the generalization of metric space and called it 2-metric space. Let $X$ be a set consisting at least three points. 2-metric on $X$ is a function $\rho : X \times X \times X \rightarrow IR^+$ which satisfies the following conditions:

1. To each pair of points $a, b \in X$ with $a \neq b$, there exists a point $c \in X$ such that $\rho(a, b, c) \neq 0$;
2. $\rho(a, b, c) = 0$, when at least two of points are equal;
3. $\rho(a, b, c) = \rho(b, c, a) = \rho(c, a, b), \forall a, b, c \in X$
4. $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c), \forall a, b, c, d \in X$.

Here the 2 metric $\rho(x, y, z)$ represents the area of triangle spanned by $x, y, z$ Examples of 2-metric space are:

Example 1. [5] A circle in the Euclidean space $R^2$ is a 2-metric space.

Example 2. [5] Define $d$ on $R^+ \times R^+ \times R^+$ as
Fixed Point Theory in 2-metric space has been proved initially by Iseki [9]. After that several authors ([12, 19, 22]) proved fixed point results in the setting of 2-metric space.


In 1992, Murthy [17] used compatible type mapping to prove fixed point results which is more general than commuting and semi-commuting maps.

After that in 1978, Khan [13] proved a result by taking a uniformly convergent sequence of 2-metrics in \( X \).

In 1977, Fisher [3] proved the following result in metric space:

**Theorem 1.** [3] Let \( f \) be a self map on complete metric space \((X, \rho)\) such that

\[
\rho^2(fx, fy) \leq \alpha \rho(x, fx) \rho(y, fy) - \beta \rho(x, fy) \rho(y, fx), \quad \forall x, y \in X
\]

and for some nonnegative constants \( \alpha, \beta \) with \( \alpha < 1 \). Then \( f \) has a fixed point in \( X \).

Moreover, if further \( \beta < 1 \), then \( f \) has a unique fixed point in \( X \).


Further, in 1989, Bijendra [2] introduced the concept of semi-compatibility in 2-metric space and prove some fixed point results which improves the results of Kang et al. [12]. Also, Gupta et al. [21], [20] proved a result by using the concept of weak compatibility and property \( \alpha \). Gupta [6] in 2012, proved fixed point results using A-contraction in the setting of 2-metric space.

In 2011, Mehta et al. [16] proved fixed point result using weakly contractive condition and contractive modulus property in the setting of metric space. Also in 2014, Gupta et al. [11] showed result employing the same property in complete metric space.

In this paper, we prove a common fixed point result for four mappings by using weakly compatible property and contractive modulus.

2. Preliminaries

**Definition 1.** [9] A sequence \( \{x_n\} \) said to be a Cauchy sequence in 2-metric space \( X \), if for each \( a \in X \) there exists \( n_0 \in X \), \( \lim_{n,m \to \infty} d(x_n, x_m, a) = 0 \), \( \forall n, m \geq n_0 \).

**Definition 2.** [9] A sequence \( \{x_n\} \) in 2-metric space \( X \) is convergent to an element \( x \in X \), if for each \( a \in X \), \( \lim_{n \to \infty} d(x_n, x, a) = 0 \).

**Definition 3.** [9] A complete 2-metric space is one in which every Cauchy sequence in \( X \) converges to an element of \( X \).
Definition 4. [4] Let \( A \) and \( S \) be self mappings on a 2-metric space then, \( A \) and \( S \) are said to be weakly compatible if they commute at their coincidence point. i.e. if \( Ax = Sx \) for some \( x \in X \), then \( ASx = SAx \).

Definition-5. [18] Two self maps \( f \) and \( g \) of a 2-metric space \((X, d)\) are called compatible if

\[
\lim_{n \to \infty} d(fgx_n, gfx_n, a) = 0 \quad \text{whenever} \quad \{x_n\} \quad \text{is a sequence in} \quad X
\]
such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \).

Definition-6. [18] Two self maps \( f \) and \( g \) of a 2-metric space \((X, d)\) are called non compatible if \( \exists \) at least one sequence \( \{x_n\} \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \). But \( \lim_{n \to \infty} d(fgx_n, gfx_n, a) \) is either non zero or non – existent.

Definition-7. [4] Two self maps \( f \) and \( g \) are said to be commuting if

\[
fgx = gfx \quad \forall x \in X.
\]

Definition-8. [4] Let \( f \) and \( g \) be two self maps on a set \( X \), if \( fx = gx \quad \forall x \in X \), then \( x \) is called coincidence point of \( f \) and \( g \).

Definition-9. [16] A function \( \phi : [0, \infty) \to [0, \infty) \) is said to be contractive modulus if \( \phi(t) < t \), for \( t > 0 \).

3. Main result

Theorem 2. Let \( F, G, S \) and \( T \) be four self mappings on 2-metric space \((X, d)\) satisfying the following conditions:

1. The pair \((F, S)\) and \((G, T)\) are weakly compatible,
2. \( F(X) \subseteq T(X) \) and \( G(X) \subseteq S(X) \) are closed subset of \( X \),
3. \( \phi(t) \leq \phi[\min\{d(Sx, Ty, t), d(Fx, Sx, t), d(Gy, Ty, t), d(Fx, Ty, t), d(Sx, Gy, t)\}] \)

where \( \phi \) is a contractive modulus.

Then the maps \( F, G, S \) and \( T \) have a unique common fixed point in \( X \).

Proof. Let \( \{y_n\} \) be a sequence in \( X \) such that \( y_n = Fx_n = Tx_{n+1} \)

and \( y_{n+1} = Gx_{n+1} = Sx_{n+2} \), by (3)

\[
d(y_n, y_{n+1}, t) = d(Fx_n, Gx_{n+1}, t)
\]

\[
\leq \phi[\min\{d(Sx_n, Tx_{n+1}, t), d(Fx_n, Sx_{n+1}, t), d(Gx_{n+1}, Tx_n, t), d(Fx_n, Tx_{n+1}, t), d(Sx_n, Gx_{n+1}, t)\}]
\]

\[
\leq \phi[\min\{d(y_n, y_{n-1}, t), d(y_n, y_{n-1}, t), d(y_{n+1}, y_n, t), d(y_n, y_{n+1}, t), d(y_{n-1}, y_{n+1}, t)\}]
\]

Thus \( d(y_n, y_{n+1}, t) \leq \phi[\min\{d(y_n, y_{n-1}, t)\}] \).
But \( \phi \) is a contractive module therefore \( \phi(d(y_n, y_{n+1}, t)) < d(y_n, y_{n+1}, t) \) and this is possible only if \( \lim_{n \to \infty} d(y_n, y_{n+1}, t) = 0 \).

Now we show that \( \{y_n\} \) is a Cauchy sequence in \( X \). If not \( \exists \varepsilon > 0 \) such that \( m < n < N, d(y_n, y_m, t) \geq \varepsilon \), but \( d(y_{n-1}, y_m, t) < \varepsilon \) and \( \varepsilon \leq d(y_{n-1}, y_m, t) = d(F_{x_m}, G_{x_n}, t) \)
\[
\leq \phi[\min\{d(S_{x_n}, T_{x_n}, t), d(F_{x_n}, S_{x_n}, t), d(G_{x_n}, T_{x_n}, t), d(F_{x_n}, T_{x_n}, t), d(S_{x_n}, G_{x_n}, t)\}]
\leq \phi[\min\{d(Y_{n-1}, Y_{n-1}, t), d(Y_{n-1}, Y_{n-1}, t), d(y_{n-1}, y_{n-1}, t), d(y_{n-1}, y_{n-1}, t), d(y_{n-1}, y_{n-1}, t)\}]
\leq \phi[\min\{\varepsilon, \varepsilon, 0, \varepsilon, \varepsilon\}] \text{. This gives } \varepsilon \leq \phi(\varepsilon) \text{.}
\]

But \( \phi \) is a contractive module therefore \( \phi(\varepsilon) < \varepsilon \), from this one can get \( \varepsilon < \varepsilon \), this is a contradiction, hence \( \{y_n\} \) is a Cauchy sequence. Since \( X \) is complete there exists a point \( z \) in \( X \) such that \( \lim_{n \to \infty} y_n = z \), this gives
\[
\lim_{n \to \infty} G_{x_n} = \lim_{n \to \infty} S_{x_n} = z = \lim_{n \to \infty} F_{x_n} = \lim_{n \to \infty} T_n \text{. Since } F(X) \subseteq T(X), \exists \text{ a point } \alpha \in X \text{ s.t. } z = T\alpha.
\]

If \( z \neq G\alpha \), using (3) we get \( d(G\alpha, z, t) = d(G\alpha, F_{x_n}, t) \)
\[
\leq \phi[\min\{d(S_{x_n}, T_{x_n}, t), d(F_{x_n}, S_{x_n}, t), d(G\alpha, T\alpha, t), d(F_{x_n}, T\alpha, t), d(S_{x_n}, G\alpha, t)\}]
\leq \phi[\min\{d(z, z, t), d(z, z, t), d(G\alpha, z, t), d(z, z, t), d(z, G\alpha, t)\} \leq \phi[d(G\alpha, z, t)] \text{.}
\]

This implies \( d(G\alpha, z, t) \leq \phi[d(G\alpha, z, t)] \). But \( \phi \) is a contractive modulus, this gives \( \phi[d(G\alpha, z, t)] < d(G\alpha, z, t) \), this is a contradiction. Thus \( G\alpha = z = T\alpha \).

Thus, \( \alpha \) is a co-occurrence point of \( G \) and \( T \) and \( (G, T) \) is weakly compatible, we get, \( GT\alpha = TG\alpha \Rightarrow Gz = Tz \). Now \( G(X) \subseteq S(X) \) therefore there exists a point \( w \in X \) s.t. \( Sw = z \) if \( Fw \neq z \).

Using (3), \( d(Fw, z, t) = d(G\alpha, Fw, t) \)
\[
\leq \phi[\min\{d(Sw, T\alpha, t), d(Fw, Sw, t), d(G\alpha, T\alpha, t), d(Fw, T\alpha, t), d(Sw, G\alpha, t)\}]
\leq \phi[\min\{d(z, z, t), d(Fw, z, t), d(z, z, t), d(Fw, z, t), d(z, z, t) \leq \phi[d(Fw, z, t)] \text{,}
\]

this gives \( d(Fw, z, t) \leq \phi[d(Fw, z, t)] \).

But \( \phi \) is a contractive modulus therefore \( \phi[d(Fz, z, t)] < d(Fz, z, t) \) this is a contradiction.

So \( Fw = z = Sw \), hence \( w \) is a co-occurrence point of \( F \) and \( S \). Since \( (F, S) \) is weakly compatible therefore \( FSw = SFw \Rightarrow Fz = Sz \).

Now if \( Fz \neq z \) then by using (3) we can get,
\[
d(Fz, z, t) = d(Fz, G\alpha, t)
\leq \phi[\min\{d(Sz, T\alpha, t), d(Fz, Sz, t), d(G\alpha, T\alpha, t), d(Fz, T\alpha, t), d(Sz, G\alpha, t)\}]
\leq \phi[\min\{d(Sz, z, t), d(Fz, Sz, t), d(z, z, t), d(z, z, t), d(Sz, z, t)\}].
\]
Since $Fz = Sz$, therefore $d(Fz, z, t) \leq \phi[d(Fz, z, t)]$. Also $\phi$ is a contractive modulus. Thus $\phi[d(Fz, z, t)] < d(Fz, z, t)$. This is a contradiction. Hence $Fz = Sz = z$. Now if $Gz \neq z$ then by using (3), we get $d(z, Gz, t) = d(Fz, Gz, t)$
\[ \leq \phi[\min\{d(Sz, Tz, t), d(Fz, Sz, t), d(Gz, Tz, t), d(Fz, Tz, t), d(Sz, Gz, t)\}] \]
\[ \leq \phi[\min\{d(z, Tz, t), d(z, z, t), d(Gz, Tz, t), d(z, Tz, t), d(z, Gz, t)\}] \]
And $Gz = Tz \Rightarrow d(z, Gz, t) \leq \phi[d(z, Gz, t)]$ and $\phi$ is a contractive modulus, therefore $\phi[d(z, Gz, t)] < d(z, Gz, t)$, which is a contradiction. So $Gz = z = Tz$, hence we have $Gz = Tz = Fz = Sz = z$.

Hence $F, S, T, G$ have a common fixed point in $X$.

Now we prove uniqueness.
Let there be another point say $w$ s.t. $w \neq z$, then by (3)
\[ d(Fz, Gw, t) \leq \phi[\min\{d(Sz, Tw, t), d(Fz, Sz, t), d(Gw, Tw, t), d(Fz, Tw, t), d(Sz, Gw, t)\}] \]
\[ d(z, w, t) \leq \phi[\min\{d(z, w, t), d(z, z, t), d(w, w, t), d(z, w, t), d(z, w, t)\}] \]
\[ \Rightarrow d(z, w, t) \leq \phi[d(z, w, t)] \]
Since $\phi$ is a contractive modulus, we get $\Rightarrow \phi[d(z, w, t)] < d(z, w, t)$, which is a contradiction.

Therefore fixed points are unique. This proves the Theorem 2.1.

**Corollary 1.** Let $F, G, S$ and $T$ be four self mappings of a 2-metric space $(X, d)$ satisfying the following conditions:
1. The pairs $(F, S)$ and $(G, T)$ are weakly compatible.
2. $\lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Gy_n = \lim_{n \to \infty} Ty_n = z$ for some $z$ in $X$.
3. $d(Fx, Gy, t) \leq \phi[\min\{d(Sx, Ty, t), d(Fx, Sx, t), d(Gy, Ty, t), d(Fx, Ty, t), d(Sx, Gy, t)\}]$,

where $\phi$ is a contractive modulus. Then the maps $F, G, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Using condition (2), since $\lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Gy_n = \lim_{n \to \infty} Ty_n = z$ for some $z$ in $X$ since $\lim_{n \to \infty} Ty_n = z$ then there exists a point $\alpha \in X$ s.t. $z = T\alpha$, refers this to the proof of theorem 3.1, we have corollary 1.

**References**


2-ölçülü metrik fəzalarda tərpənməz nöqtə haqqında vahid ümumi teorem

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XÜLASƏ

Bu işin məqsədi 2 ölçülü metrik fəzalarda zəif uyuşan inikas anlayışından istifadə etmək, 4 ayrı misal üçün tərpənməz nöqtə haqqında bəzi teoremlərin isbat edilməsidir. Bu nəzətələr Iseki və digar bəzi müəlliflərin nəticələrini genişləndirir və əməliyyatlı və əməliyyatlıdır.

Açar sözlər: tərpənməz nöqtə, 2 ölçülü metrik fəzalar, zəif uyğunluq, sıxılan modullar

Единая общая теорема о неподвижной точке в 2-мерном метрическом пространстве

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РЕЗЮМЕ

Целью данной работы является доказать некоторые общие теоремы о неподвижной точке для четырех самостоятельных примеров в 2-мерном метрическом пространстве с использованием понятие слабо совместимых отображений. Наши результаты расширяют и обобщают результаты Iseki и ряда других авторов.

Ключевые слова: неподвижная точка, 2-мерное метрическое пространство, слабо совместимость, сжимающие модули.