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A MODULE WHOSE PRIMARY-LIKE SPECTRUM HAS THE ZARISKI-LIKE TOPOLOGY

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Abstract. Let R be a commutative ring with identity. The purpose of this paper is to introduce and study a new class of modules over R called top-like modules. Every top-like module possesses a primary-like spectrum with the Zariski-like topology. This class contains the family of multiplication R-modules properly. We show that a finitely generated R-module M is a top-like R-module iff M is a top R-module iff M is a multiplication R-module.

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1. Introduction

Throughout this pāper all rings are commutative with identity and modules are unitary. Let M be an R-module and $N -$ a submodule. The colon ideal of M into N is the ideal $(N:M) = \{r \in R: rM \subseteq N\}$ of R. A proper subbmodule P of M is called a prime submodule or p-prime submodule of M if for $p = (P:M)$, whenever $rx \in P$ for $r \in R$ and $x \in M$, we have $r \in p$ or $x \in P$ [13]. The intersection of all prime submodules of M containing N, denoted by $rad(N)$, is called prime radical (or simply, radical) of N [15]. The radical of an ideal I will be denoted by $\sqrt{I}$. The prime spectrum of , denoted by $Spec(M)$ is the set of all prime submodules of M. If $Spec(M) = \emptyset$, then M is called primeless [14] . For $p \in Spec(R)$, we denote $Spec_p(M)$as the set of all p-prime submodules of M [13]. Put $V(N) = \{P \in Spec(M): P \supseteq N\}$ } and $\zeta(M) = \{V(N): N$ is a submodule of $M\}$. Then there exists a topology τ, called quasi Zariski topology on $Spec(M)$, having $\zeta(M)$ as the set of closed subsets of $Spec(M)$ if and only if $\zeta(M)$ is closed under the finite union. In this case, M is called a top R-module [14]. We say that a submodule N of satisfies the primeful property if for every prime ideal p containing $(N:M)$ there exists $P \in V(N)$ such that $(P:M) = p$. Also M is called primeful if $M = 0$ or the zero submodule of M satisfying the primeful property [9]. If N is a submodule, then $(rad(N):M) = \sqrt{(N:M)}$. A proper submodule Q of M is called a primary-like submodule whenever $rx \in Q$ for $r \in R$ and $x \in M$, we have $r \in (Q:M)$ or

$x \in rad(Q)$ [8]. If Q is a primary-like submodule of M satisfying the primeful property, then $(Q{:}M)$ is a primary ideal of R [8, Lemma 2.1]. In this case, Q is called a p-primary-like submodule of M, where $p = \sqrt{(Q{:}M)}$. The primary-like spectrum of M, denoted by $Spec_L(M)$, is the set of all primary-like submodules of M satisfying the primeful property [8]. Also we set $\mathcal{X}_p = \left\{Q \in Spec_L(M){:}\sqrt{(Q{:}M)} = p\right\}$.

Recently, modules whose spectrums having various types of Zariski topologies have been received a good deal of attention (see for example [1, 3, 11, 14, 16]). Hereafter, we study the algebraic properties of a new class of modules which are equipped with a new Zariski topology, called Zariski-like topology, defined as follows. Let N be a submodule of an R-module M. We set $\upsilon(N) = \{Q \in Spec_L(M){:} N \subseteq rad(N)\}$. Some elementary facts about υ have been in the following lemma.

Lemma 1. Let M be an R-module. Let N, N' and $\{N_i{:}\, i \in I\}$ be submodules of M. Then the following statements hold.

(1) $\upsilon(M) = \emptyset$.

(2) $\upsilon(0) = Spec_L(M)$.

(3) $\cap_{i\in I}\, \upsilon(N_i) = \upsilon(\sum_{i\in I} N_i)$.

(4) $\upsilon(N) \cup \upsilon(N') \subseteq \upsilon(N \cap N')$.

(5) $\upsilon(N) = \upsilon\big(rad(N)\big)$.

Put $\eta(M) = \{\upsilon(N){:} N$ is a submodule of $M\}$. From (1), (2), (3) and (4) in Lemma 1, we can easily that there exists a topology, $\mathcal{T}$ say, on $Scpe_L\,(M)$. A module M is called a top-like module if $\eta(M)$ induces the topology $\mathcal{T}$. In Section 2, we study a class of R-modules whose primary-like spectrum is empty, called modules with empty primary-like spectrum or for short WEPS modules. We show that primeless R-modules are WEPS and the converse is true if R is a zero-dimensional ring (Lemma 2). In particular, every torsion divisible R-module is WEPS, however WEPS modules are neither torsion nor divisible in general (Example 1). In Section 3, for a module$M$ over an Artinian ring R we show that:

$$M \text{ is locally cyclic} \Longrightarrow M \text{ is top-like} \Longrightarrow M_p \text{ is a top-like } R_p\text{-module.}$$

Moreover, if M is finitely generated, then these conditions are equivalent (Theorem 1). An R-module M is called a multiplication module if for every submodule N of M, there exits an ideal I of R such that $N = IM$. In this case, we can take $I = (N{:}M)$ [5]. An R-module M is called weak multiplication if each prime submodule P of M has the form IM for some deal I of R [4]. Since the zero submodule of $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is the only prime submodule of $\mathbb{Q}$, then $\mathbb{Q}$ is a weak multiplication module, which is not multiplication. In Theorem 4 of Section 4, it is shown that every multiplication module is top-like. In particular, if M is a finitely generated module, then M is top-like $\Leftrightarrow$ M is top $\Leftrightarrow$ M is multiplication.

Also it is proved that if M is a weak multiplication module over a PID such that for every $Q \in Spec_L(M)$, $\sqrt{(Q:M)} \neq 0$, then M is top-like (Theorem 5). By Example 5, we see that the converse dose not necessarily true.

2. Modules with empty primary-like spectrum

Hereafter we denote $Spec_L(M)$ by $\mathcal{X}$. Recall that an R-module M is said to be with empty primary-like spectrum or for short WEPS if $\mathcal{X} = \emptyset$. Note that we are not excluding the trivial case where $\mathcal{X}$ is empty; WEPS modules are top-like modules. Clearly, zero module is WEPS and primeless. As nontrivial example the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is WEPS (see [14, P. 81] and Lemma 2.1). Also for the $\mathbb{Z}$ -module $\mathbb{Q}$ of rational numbers $Spec(\mathbb{Q}) = 0$, i.e. $\mathbb{Q}$ is not primeless, however $\mathbb{Q}$ is WEPS since the submodule 0 of $\mathbb{Q}$ dose not satisfy the primeful property.

Lemma 2. Let M be an R-module. Consider the following statements.

(1) $pM = M$ for every $p \in V(Ann(M))$;
(2) M is primeless;
(3) M is WEPS;
(4) $mM = M$ for every maximal ideal m of R.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$. Moreover, if R is a zero-dimensional ring, then $(4) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2)$ Suppose on the contrary that P is a prime submodule of . Thus $(P:M)M = M$ by (1) and so $P = M$, a contradiction. $(2) \Longrightarrow (3)$ Suppose $Q \in \mathcal{X}$. Since Q satisfies the primful property, there exists a prime submodule Q of M containing Q which is a contradiction.

$(3) \Longrightarrow (4)$ Assume the contrary, $mM \neq M$ for some maximal ideal m of R. Thus $(mM:M) \neq R$. Hence $(mM:M) = m$ and so by [14, Corollary 1.2], mM is a prime submodule of . It is easily seen that $\in \mathcal{X}$, i.e. M is not WEPS.

$(4) \Longrightarrow (1)$ is clear.

Lemma 3. If M is a WEPS R-module, then M is not finitely generated and multiplication.

Proof. If M is either finitely generated or multiplication, M has a maximal submodule $\mathcal{m}$. It is evident that $\mathcal{m}$ satisfies the primeful property and $\mathcal{m} \in \mathcal{X}$. Thus M is not WEPS.

In [14, Lemma 1.3 (1)], it has been shown that any torsion divisible module over a domain R is primeless and so by Lemma 2.1 is WEPS. In general, a WEPS module is not torsion. It is clear that the $\mathbb{Z}$-module $\mathbb{Q}$ is WEPS which is also torsion-free. In the following a WEPS module is given which is not divisible.

Example 1. Let $R = K[x, y]$, the domain of polynomials over a field K. Let $m = Rx + Ry$. Then the R/m-module R/m is an injective hull of R/m since R/m is a field ([18, p. 50 Example]). Thus $R/m \cong E(R/m)$, a divisible R/m-module ([18, Remark before Proposition 2.22 and Proposition 2.6]). On the other hand since $E(R/m)$ is an essential extension of R/m, it is easily seen that $E(R/m)$ is a torsion R-module. Thus by [14, Lemma 1.3 (1)] $R/m \cong E(R/m)$is a primeless R-module and so by Lemma 1 it is a WEPS R-module. However, R/m is not a divisible R-module because for $x \in m = Rx + Ry$ and $1 + m \in R/m$ there is no $f + m \in R/m$ such that $x(f + m) = 1 + m$. Equivalently, there is no $g \in m$ such that $xf + g = 1$.

In the following we give conditions under which a WEPS module is divisible.

Proposition 1. Let R be a one-dimensional Noetherian domain and M be a module over R. If M is a WEPS module, then M is divisible.

Proof. Suppose that M is a WEPS R-module. By Lemma 2, $M = mM$ for every maximal ideal m of R. Assume $0 \neq r \in R$. Since R is one-dimensional domain, the ring R/Rr is zero- dimensional Noetherian and so is Artinian. So $m_1, \ldots, m_n \subseteq Rr$ for some positive integer n and maximal ideals m_i $(1 \leq i \leq n)$ of R. Hence $M = m_1 M = m_1 m_2 M = m_1 \ldots m_n M \subseteq rM \subseteq M$ and so $M = rM =$ rM. Thus M is divisible.

Let N be a submodule of M. In [8, Corollary 3.5] we showed that every primary-like submodule of M/N satisfying the primeful property has the form Q/N, where $Q \in \mathcal{X}$ and $N \subseteq Q$. Thus any homomorphic image of a WEPS module is WEPS. In particular, if $M = \oplus_{i \in I} M_i$ is a WEPS module, then for every $i \in I$, M_i is WEPS. The converse holds, if R is a zero-dimensinal ring (see [14, Proposition 1 .7] and Lemma 2.1). Also if for every $i \in I$, M_i is a primeless module, then $M = \oplus_{i \in I} M_i$ is a WEPS module (see [14, Proposition 1.7] and Lemma 2.1). In the following we investigate the similar assertion for direct product of modules.

Lemma 4. Let M be an R-module. If Q is a primary-like submodule of M and N a submodule of M such that $rad(Q) \cap rad(N) = rad(Q \cap N)$, then $N \subseteq Q$ or $Q \cap N$ is a primary-like submodule of N.

Proof. Let $N \nsubseteq Q$ and for $n \in N$, $rn \in Q \cap N$ such that $r \notin (Q \cap N: N)$. It implies that $rn \in Q$ and $r \notin (Q: M)$. Since Q is a primary-like submodule of M, we have $n \in rad(Q) \cap N$, and so by our assumption $n \in rad(Q \cap N)$. Thus $Q \cap N$ is a primary-like submodule of N.

Proposition 2. Let M_i $(i \in I)$ be R-modules and $M = \sqcap_{i \in I} M_i$. Let $rad(Q) \cap rad(N) = rad(Q \cap N)$ for every primary-like submodule Q of M_i and every submodule N of M_i. Then M is a WEPS module if and only if M_i is WEPS for every $i \in I$.

Proof. Suppose M_i is WEPS and M is not WEPS. Assume $Q \in \mathcal{X}$. Then $Q \cap M_i = M_i$ for each $i \in I$, by Lemma 4. Hence $M_i \subseteq Q$ for each $i \in I$. Thus $M \subseteq Q$, a contradiction. The converse follows from the fact that every homomorphic image of a WEPS module is WEPS.

Proposition 3. Let M be an R-module such that for every primary-like submodule Q of M and every submodule N of M we have $rad(Q) \cap rad(N) = rad(Q \cap N)$. Then if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules such that M' and M'' are both WEPS, then M is WEPS.

Proof. Suppose that $Q \in \mathcal{X}$. Then $Q \cap f(M') = f(M')$, by Lemma 4 and so $f(M') \subseteq Q$. Hence $Q/f(M')$ is a primary-like submodule of $M/f(M')$ satisfying the primeful property by [8, Corollary 3.5], which is a contradiction since $M/f(M') \cong M''$ and M'' is WEPS. Thus M is WEPS.

For the converse of Proposition 2.3, the homomorphic image of a WEPS module is WEPS. But the submodules of a WEPS module is not necessarily WEPS, even if $rad(Q) \cap rad(N) = rad(Q \cap N)$ for every primary-like submodule Q of M and every submodule N of M. The $\mathbb{Z}$-module $\mathbb{Q}$ is WWPS, while the $\mathbb{Z}$-module $\mathbb{Z}$ is not WEPS. In fact $Spec_L(\mathbb{Z}) = \{0\} \cup \{p^n\mathbb{Z}: n \in \mathbb{N}\}$. Also since $Spec(\mathbb{Q}) = 0$, then for every primary-like submodule Q of $\mathbb{Q}$ and every submodule N of $\mathbb{Q}$ either $rad(Q) \cap rad(N) = rad(Q \cap N) = 0$ or $rad(Q) \cap rad(N) = rad(Q \cap N) = \mathbb{Q}$.

3. Top-like modules

A submodule N of an R-module M is called semiprime. If N is an intersection of prime submodules. We say that a submodule $Q \in \mathcal{X}$ is phenomenal if whenever N and L are semiprime submodules of M with $N \cap L \subseteq rad(Q)$, then $N \subseteq rad(Q)$ or $N \subseteq rad(Q)$.

Theorem 1. Let M be an R-module. Consider the following statements.

(1) Every $Q \in \mathcal{X}$ is phenomenal;

(2) $\upsilon(N) \cup \upsilon(L) = \upsilon(N \cap L)$ for any submodules N and L of M;

(3) M is a top-like module.

Then(1)⟹(2)⟹(3) Furthermore, if every prime submodule of M satisfies the primeful property, then (3)⟹(1).

Proof. (1) ⟹ (2) Let N and L be semiprime submodules of M. Clearly $\upsilon(N) \cup \upsilon(L) \subseteq \upsilon(N \cap L)$. Let $Q \in \upsilon(N \cap L)$. Then $N \cap L \subseteq rad(Q)$ and hence $N \subseteq rad(Q)$ or $L \subseteq rad(Q)$. Thus $\upsilon(N \cap L) \subseteq \upsilon(N) \cup \upsilon(L)$.

(2) ⟹ (3) Assume N and L be submodules of . If $\upsilon(N)$ is empty, then $\upsilon(N) \cup \upsilon(L) = \upsilon(N \cap L)$. Suppose that $\upsilon(N)$ and $\upsilon(L)$ are both non-empty. Then $\upsilon(N) \cup \upsilon(L) = \upsilon((radN)) \cup \upsilon(rad(L)) = \upsilon(rad(N) \cap rad(L))$, by Lemma 1.

(3) ⟹ (1) Let $Q \in \mathcal{X}$ and let N and L be semiprime submodules of M such that $N \cap L \subseteq rad(Q)$. By hypothesis, there exists a submodule K of M such that $\upsilon(N) \cup \upsilon(L) = \upsilon(K)$. Since $N = \bigcap_{i \in I} P_i$, for some collection of prime submodules P_i $(i \in I)$, for each $i \in I$, $P_i \in \upsilon(N) \subseteq \upsilon(K)$ and so $K \subseteq P_i$. Thus $K \subseteq \bigcap_{i \in I} P_i = N$. Similarly $K \subseteq L$. Thus $K \subseteq N \cap L$. Hence we have $\upsilon(N) \cup \upsilon(L) \subseteq \upsilon(N \cap L) \subseteq \upsilon(K) \subseteq \upsilon(N) \cup \upsilon(L)$. It follows that $\upsilon(N) \cup \upsilon(L) = \upsilon(N \cap L)$. Now from $Q \in \upsilon(N \cap L)$. we have $Q \in \upsilon(N)$ or $Q \in \upsilon(N)$, i.e. $N \subseteq rad(Q)$ or $L \subseteq rad(Q)$.

Corollary 1. Let M be an R-module. Then the conditions (1), (2) and (3) in Theorem 3.1 are equivalent in each of the following cases.

(1) M is a finitely generated module.

(2) M is a primeful module and every prime submodule of M has the form pM for some prime ideal $p \in V\big(Ann(M)\big)$.

Proof. (1) follows from [9, Theorem 2.2] and Theorem 1.

(2) follows from [9, Proposition 4.5] and Theorem 3.1.

Corollary 2. Any homomorphic image of a top-like R-module is top-like. In particular every cyclic module is top-like.

Proof. Let N be a submodule of a top-like R-module M. Then the primary-like submodules of M/N which satisfy the primeful property are precisely the submodules Q/N, where Q is a primary-like submodule of M satisfying the primeful property with $N \subseteq Q$ [8, Corollary 3.5]. Similarly every prime (semiprime) submodule of M/N has the form K/N, where K is a prime (semiprime) submodule of M containing N [14, Lemma 1.1]. Hence we have $rad(Q/N) = rad(Q)/N$. Thus by Theorem 1 M/N is a top-like R-module. In particular if M' is a homomorphic image of a top-like module M under a surjective homomorphism φ, then $M \cong M/Ker\varphi$ and so by the above argument M' is top-like.

Corollary 3. Let $S \subseteq R$ be rings and M be an R-module such that the S-module M is a top-like module. Then the R-module M is a top-like module.

Proof. Let Q be a primary-like R-submodule of M satisfying the primeful property. It is clear that Q is a primary-like S-submodule of M satisfying the primeful property. Let N and L be semiprime R-submodule of M such that $N \cap L \subseteq rad(Q)$. Hence $N \subseteq rad(Q)$ or $L \subseteq rad(Q)$, since N and L are also semiprime S-submodule of M. Thus Q is phenomenal and so by Theorem 1, M is a top-like R-module.

Lemma 5. Let R be a field. Then there exists a phenomenal submodule $Q \in \mathcal{X}$ if and only if M is a one-dimensional vector space over R.

Proof. Suppose M is one-dimensional. Since every proper submodule of M is a prime submodule, $P = 0$ is the only prime submodule of M . So it is easily seen that $P \in \mathcal{X}$ and P is a phenomenal submodule. Conversely, suppose on the contrary that M is not one-dimensional and $Q \in \mathcal{X}$ is phenomenal. Since Q is a proper submodule and so is a prime submodule, $rad(Q) = Q$ and $dim_R M \neq 0$. So $dim_R M \geq 2$. Assume $dim_R M = 2$. Thus there exist non-zero elements $m_1, m_2 \in M$ such that $Rm_1 \cap Rm_2 = 0$. Since Q is phenomenal, $Q \neq 0$. Assume that $m \in M \setminus Q$ and $0 \neq q \in Q$. So Rm and $R(m+q)$ are subspaces of M with $Rm \cap R(m+q) = 0 \subseteq Q$ but $Rm \nsubseteq Q$ and $R(m+q) \nsubseteq Q$, a contradiction.

Theorem 2. Let m be a maximal ideal of R and M a top-like R-module. Then M/mM is a cyclic R-module.

Proof. Suppose that $M \neq mM$. In this case M/mM is a non-zero vector space over the field R/m and every proper subspace is primary-like and satisfies the primeful property. Since M is a top-like module, M/mM is a top-like R/m-module by Corollary 2. Hence M/mM contains a phenomenal submodule by Theorem 2. Now Lemma 5 shows that M/mM is one-dimensional over R/m, i.e. M/mM is a cyclic R-module.

In Corollary 2, we showed that top-like modules are closed under quotient. Now we use from Theorem 2 to show that a submodule of a top-like module is not necessarily top-like.

Example 2. Let $M = \mathbb{Q} \oplus \mathbb{Z}_p$, where $\mathbb{Z}_p$ be the cyclic group of order M. Then $Spec(M) = \{\mathbb{Q}\oplus 0,\ 0\oplus\mathbb{Z}_p\}$ by [14, Example 2.6].Clearly if N is a submodule of M such that $N \nsubseteq \mathbb{Q}\oplus 0$ or $N \nsubseteq 0\oplus\mathbb{Z}_p$, then N dose not satisfy the primeful property. Also if $N \subseteq 0\oplus\mathbb{Z}_p$, then $(N{:}M) = 0$ and so N dose not satisfy the primeful property. Consider the only remaining case $N \subseteq \mathbb{Q}\oplus 0$. In this case, if $(N{:}M) = p\mathbb{Z}$, then $N = \mathbb{Q}\oplus 0$ and so $\mathbb{Q} \oplus 0 \in \mathcal{X}$. If $(N{:}M) = 0$, then N does not satisfy the primeful property. The finial case is $0 \subset (N{:}M) \subset p\mathbb{Z}$. In this case if N is a primary-like submodule satisfying the primeful property, then $(N{:}M) = p^i\mathbb{Z}$ for some $i \geq 1$, since $(N{:}M)$ is a primary ideal of R. Assume $i = 1$ and $(0,b) \in M \setminus \mathbb{Q}\oplus 0$. Now we have $p(0,b) = (0,0)$, follows $p \in p^i\mathbb{Z}$ which is a contradiction. Therefore $\mathcal{X} = \{\mathbb{Q}\oplus 0\}$. Hence M is a top-like $\mathbb{Z}$ -module by Theorem 3.1. Put $= \mathbb{Z}\oplus\mathbb{Z}_p$ Hence by Theorem 2 N is not top-like, since $N/pN \cong \mathbb{Z}_p\oplus\mathbb{Z}_p$, a non-cyclic $\mathbb{Z}$-module.

Corollary 4. Let F be a free R-module. Then the following statements are equivalent.

(1) F is top-like;
(2) F is top;
(3) F is cyclic.

Proof. (1) $\Longrightarrow$ (3) Suppose that F is top-like. Hence F/mF is a cyclic R/m-module by Theorem 3.2. Thus F is cyclic.

(3) $\Longrightarrow$ (1) follows from Corollary 2.

(2) $\Leftrightarrow$ (3) holds by [14, Corollary 2.5].

Corollary 5. Let R be a semi-local ring and M be an R-module. Then the following statements are equivalent.

(1) M is top-like;
(2) M is top;
(3) M is cyclic.

Proof. (1) $\Longrightarrow$ (3) Assume that M is a top-like module. Since R is semi-local, R is containing precisely finite distinct maximal ideals. Suppose $m_1, \dots, m_n$ denote the distinct maximal ideals of R, where n is a positive integer. By Theorem 3.2 M/m_iM is cyclic for each $1 \le i \le n$. Thus M is cyclic.

(3) $\Longrightarrow$ (1) follows from Corollary 2.

(2) $\Leftrightarrow$ (3) holds by [14, Corollary 2.5].

Let N be a submodule of an R-module M and $p \in S \in pecc(R)$. The saturation $S_p(N)$ of N with respect to p is the contraction of N_p in M [10]. In [14, P. 92] it has been shown that for every submodule N of M and for any R_p-submodule L of M_p, $(R_pN) \cap M = S_p(N)$ and $L = R_p(L \cap M)$.

Lemma 6. Let m be a maximal ideal of a ring R. If Q is an m-primary-like submodule of M satisfying the primeful property, then $rad(Q)$ is an m-prime submodule of M.

Proof. Since Q satisfies the primeful property, we have $(rad(Q){:}M) = \sqrt{(Q{:}M)} = m$. Suppose $rx \in rad(Q)$ with $r \in R \setminus m$ and $x \in M$. Then $x = tx + srx$, for some $s \in R$ and $t \in m$ and therefore $x \in rad(Q)$.

Proposition 4. Let R be an Artinian ring and M be a top-like R-module. Then M_p is a top-like R_p-module for every prime ideal p of R.

Proof. Let Q be a primary-like submodule of the R_p-module M_p satisfying the primeful property. Then it is easily verified that $Q \cap M$ is a primary-like submodule of M satisfying the primeful property. Now if N and L are semiprime submodules of M_p with $N \cap L \subseteq rad(Q)$, then $N \cap M$ and $L \cap M$

are semiprime submodules of M with $(N \cap M) \cap (L \cap M) \subseteq rad(Q) \cap M$. Since R is an Artinian ring, $rad(Q)$ is a prime submodule of M_p by Lemma 3.2. So $rad(Q) \cap M$ is a prime submodule of M. Since M is a top-like module, by [6, Theorem 2.16] M is a top module. Hence by [14, Lemma 2.1], $N \cap M \subseteq rad(Q) \cap M$ or $L \cap M \subseteq rad(Q) \cap M$. It follows that $N = R_p(N \cap M) \subseteq R_p(rad(Q) \cap M) = rad(Q)$ or $L \subseteq rad(Q)$. Thus Q is phenomenal and so M_p is a top-like R_p-module by Theorem 1.

Proposition 5. Let M_p be a top-like R_p-module and every prime submodule of $M_p/R_p(pM)$ satisfies the primeful property for every prime ideal p of R. Then $S_p\big(rad(Q)\big) = S_p(pM)$ or $S_p\big(rad(Q)\big) = M$ for every $Q \in \mathcal{X}_p$.

Proof. Suppose that $Q \in \mathcal{X}_p$. So $pM \subseteq rad(Q)$. It follows that $R_p(pM) \subseteq R_p rad(Q) \subseteq M_p$. By Theorem 3.2, $M_p/R_p(pM)$ is cyclic so that $R_p rad(Q) = R_p(pM)$ or $R_p rad(Q) = M_p$. Now, let $R_p rad(Q) = R_p(pM)$. Then $S_p\big(rad(Q)\big) = R_p rad(Q) \cap M = R_p(pM) \cap M = S_p(pM)$. If $R_p rad(Q) = M_p$, then $S_p\big(rad(Q)\big) = R_p rad(Q) \cap M = R_p M_p \cap M = M$.

An R-module M is called locally cyclic if M_p is a cyclic module over the local ring R_p for every prime ideal p of R.

Theorem 3. Let R be an Artinian ring and M be an R-module. Consider the following statements.

(1) M is cyclic.
(2) M is locally cyclic.
(3) M is top-like.
(4) M_p is a top-like R_p-module for every prime ideal p of R.
(5) $rad(Q) = S_p(pM)$ for every $Q \in \mathcal{X}_p$.
(6) $M/rad(pM)$ is a cyclic module for every prime ideal p of R.
(7) M/pM is a cyclic module for every prime ideal p of R.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5) \Longrightarrow (6) \Longrightarrow (7)$. Furthermore, if M is finitely generated, then $(7) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2)$ is clear.

$(2) \Longrightarrow (3)$ Suppose $Q \in \mathcal{X}$ and $p = \sqrt{(Q{:}\,M)} = (rad(Q){:}\,M)$. Let N and L be semiprime submodules of M with $N \cap L \subseteq rad(Q)$. Then $R_pN \cap R_pL \subseteq R_p rad(Q)$. Note that if $R_p rad(Q) = M_p$, by Lemma 3.2 $rad(Q) = M$, a contradiction. Thus $R_p rad(Q) \neq M_p$. But $pM \subseteq rad(Q)$ gives $R_p(pM) \subseteq R_p rad(Q)$. Since M_p is cyclic, $R_p rad(Q) = pR_pM_p$. Hence $R_p rad(Q)$ is a unique maximal submodule of the R_p- module M_p. Thus $R_pN \subseteq R_p rad(Q)$. or $R_pL \subseteq R_p rad(Q)$. Suppose that $R_pN \subseteq R_p rad(Q)$. Then $N \subseteq R_pN \cap M \subseteq$

$R_p rad(Q) \cap M = S_p(rad(Q)) = rad(Q)$. It follows that Q is phenomenal and so M top-like module by Theorem 1.

(3) $\Longrightarrow$ (4) follows from Proposition 5 .

(4) $\Longrightarrow$ (5) Suppose that $Q \in \mathcal{X}_p$. So $pM \subseteq rad(Q)$. It follows that $R_p(pM) \subseteq R_p rad(Q) \subseteq M_p$. By [6 , Theorem 2.16] and Theorem 2 , $M_p/R_p(pM)$ is cyclic so that $R_p rad(Q) = R_p(pM)$ or $R_p rad(Q) = M_p$. Now, if $R_p rad(Q) = M_p$, then by Lemma 6 $rad(Q) = M$ which is a contradiction. Thus we have $rad(Q) = S_p(rad(Q)) = R_p rad(Q) \cap M = R_p(pM) \cap M = S_p(pM)$.

(5) $\Longrightarrow$ (6) Suppose $M = rad(pM)$. So $p \subseteq (pM: M) \subseteq (rad(pM): M) = R$ so that $p = (pM: M)$. It follows that pM is a prime submodule of M satisfying the primeful property. Assume that $x \in M \setminus rad(pM)$. It implies that $p \subseteq (Rx + rad(pM): M) \subseteq R$ and so $(Rx + rad(pM): M) = R$. Thus $M = Rx + rad(pM)$, i. e. $M/rad(pM)$ is cyclic.

(6) $\Longrightarrow$ (7) Assume that $M = pM$. Hence similar to the proof (5) $\Longrightarrow$ (6) pM is a p-prime submodule of M and so $rad(pM) = pM$. Thus M/pM is cyclic.

(7) $\Longrightarrow$ (1) follows from [14, Theorem 3.5] and [7, Corollary 2.9].

4. Multiplication modules and weak multiplication modules

In this section we investigate the relationship between some certain classes of modules, spe cially multiplication modules, and top-like modules.

Hereafter we denote $Spec(M)$ and $Spec_p(M)$ for every $p \in Spec(R)$ by X and X_p, respectively. The map $\psi: X \longrightarrow Spec(R/Ann(M))$ given by $P \longmapsto (P: M)/Ann(M)$ is called the natural map of X. M is said to be X-injective if either $X = \emptyset$ or $X \neq \emptyset$ and the natural map of X is injective.

Theorem 4. Let M be a finitely generated R-module. Then the following statements are equivalent.

(1) M is multiplication;

(2) M is top-like;

(3) M is top;

(4) $|X_p| \leq 1$ for every $p \in Spec(R)$;

(5) If $V(P) = V(P')$ for $Spec(M)$, then $P = P'$;

(6) M is X-injective;

(7) For every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$;

(8) M_p is a top R_p-module for every prime ideal p of R;

(9) M/mM is cyclic for every maximal ideal m of R.

Proof. (1) $\Longrightarrow$ (2) Suppose N and L are two submodule of . Therefore $N = IM$ and $L = JM$ some ideals I and J of R. It is easy to verify that $\upsilon(IM) \cup \upsilon(JM) \subseteq \upsilon(IM \cap JM) \subseteq \upsilon(IJM)$. Let

$Q \in \upsilon(IJM)$. Then $IJM \subseteq rad(Q)$. So $IJ \subseteq (IJM: M) \subseteq (rad(Q): M)$. Hence $I \subseteq (rad(Q): M)$ or

$J \subseteq (rad(Q): M)$ and so $Q \in \upsilon(IM) \cup \upsilon(JM)$.

(2) $\Longrightarrow$ (3) follows from the fact that (X, τ) is a topological subspace of $(\mathcal{X}, \mathcal{T})$.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) is by [14, Theorem 3.5].

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows from [11, Proposition 3.2].

Corollary 6. If M is a finitely generated top-like module over an Artinian ring R, then M is cyclic.

The following example shows that every top-like module is not multiplication in general.

Example 3. Let $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the cyclic group of order p. Then $(M) = pM$. So M is a top-like $\mathbb{Z}$-module by Theorem 1. But M is not a multiplication $\mathbb{Z}$-module by [14, Example 3.7].

An R-module M is called distributive if the lattice of its submodules is distributive, i.e. $N \cap (L + K) = (N \cap L) + (N \cap K)$ or equivalently $N + (L \cap K) = (N + L) \cap (N + K)$ for all submodules N, L and K of M [5]. Some authors call such modules arithmetical modules. An R-module M is called a Bezout module if every finitely generated submodule is cyclic [19]. It is easy to see that every Bezout R-module is distributive [19, P. 307, Corollary 2].

Proposition 6. Let R be an Artinian ring and M be an R-module. Consider the following statements.

(1) M is distributive.

(2) M is Bezout.

(3) M is top-like.

Then (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3). Furthermore, if R is a local ring and M is finitely generated, then (3) $\Rightarrow$ (2).

Proof. (1) $\Leftrightarrow$ (2) follows from [5, Propositions 4, 7].

(2) $\Longrightarrow$ (3) Assume $Q \in \upsilon(N \cap L)$. Let $N \nsubseteq rad(Q), n \in N \setminus rad(Q)$ and $l \in L$. Then there exists $m \in M$ such that $Rn + Rl = Rm$. Thus there exist $r, s \in R$ such that $n = rm$ and $l = sm$. Therefore $sn \in rad(Q)$. Now by Lemma 3.2 $s \in (rad(Q): M)$. In particular, $sm \in rad(Q)$, whence $l \in rad(Q)$. This implies that M is a top-like module.

(3) $\Longrightarrow$ (2) Since M is finitely generated, by Theorem 4.1 M is a multiplication module. Hence M is cyclic by [7, Corollary 2.9]. Since R is an

Artinian local ring, every ideal of R is principal and so every submodule of M is cyclic.

Recall that an R-module M is called weak multiplication if each prime submodule P of M has the form IM for some ideal I of R [4]. In this case, we can take $I=(P:M)$.

Theorem 5 Let R be a PID and M a weak multiplication R-module. If for every $Q\in\mathcal{X}, \sqrt{(Q:M)}\neq 0$, then M is a top-like R-module.

Proof. Suppose N and L be non-zero semiprime submodules of such that $N\cap L\subseteq rad(Q)$. We show that Q is phenomenal. So by Theorem 1 M is top-like. Since R is PID, $rad(Q)$ is prime by Lemma 6 . Hence $rad(Q)=pM$ for some prime ideal p of R because M is weak multiplication. Also let $\{p_i,\ i\in I\}$ and $\{q_j,\ j\in J\}$ be families of maximal ideals of R such that $N=\cap_{i\in I}p_iM$ and $L=\cap_{j\in J}q_jM$. If $(N:M)\nsubseteq(rad(Q):M)$ or $(L:M)\nsubseteq(rad(Q):M)$, then $N\subseteq rad(Q)$ or $L\subseteq rad(Q)$ b y [12, Lemma 2]. Hence we consider just the case that $(N:M)\subseteq(rad(Q):M)$ and $(L:M)\subseteq(rad(Q):M)$. Then we have $\cap_{i\in I}p_i\subseteq p$ and $\cap_{j\in J}q_j\subseteq p$. If I or J is a finite set, then the claim follows from the above arguments. So we assume that I and J are infinite sets. Now we show that if $\mathrm{N}\nsubseteq\mathrm{rad(Q)}$, then $\mathrm{L}\subseteq\mathrm{rad(Q)}$. Suppose $\mathrm{n}\in\mathrm{N}\setminus\mathrm{rad(Q)}$. Therefore by [15, Lemma 2.12], $(\mathrm{L:n})\subseteq(\mathrm{rad(Q):M})=(\mathrm{pM:M})=\mathrm{p}$. If $(\mathrm{L:n})=(0)$, then $\mathrm{n}+\mathrm{L}\notin\mathrm{T(M/L)}$ the torsion submodule of $\mathrm{M/L}$, so $\mathrm{T(M/L)}=\mathrm{M/L}$. Since M is a weak multiplication module, $\mathrm{M/L}$ is also a weak multiplication module. But every weak multiplication module over an integral domain is either torsion or torsion-free by [4, Proposition 2.4(iii)]. Hence $\mathrm{M/L}$ is a torsion-free R-module. On the other hand we have $(\mathrm{L:M})\subseteq(\mathrm{L:n})=(0)$. Thus $\mathrm{L}\in\mathrm{Spec_0(M)}$ by [14, Lemma 1.1]. Therefore $\mathrm{L}=(0)\mathrm{M}=\mathrm{M}$. Now we consider our claim in the case that $(\mathrm{L:n})=(0)$. In this case we set $\Omega=\{\mathrm{q_j:n}\notin\mathrm{q_jM}\}$. Since $\mathrm{N}\cap\mathrm{L}\subseteq\mathrm{pM}$ and $\mathrm{n}\in\mathrm{N}\setminus\mathrm{pM}$, we have $\mathrm{n}\notin\mathrm{L}$ so that $\Omega\neq\emptyset$. Since $(\mathrm{q_jM:n})=\mathrm{R}$ if $\mathrm{n}\in\mathrm{q_jM}$, then we have $(\mathrm{L:n})=\left(\cap_{\mathrm{j}\in\mathrm{J}}\mathrm{q_jM:n}\right)=\cap_{\mathrm{q_{i\in\Omega}}}(\mathrm{q_jM:n})$. But for every $\mathrm{q_j}\in\Omega,\left(\mathrm{q_jM:n}\right)=\mathrm{q_j}$. Hence we have $(0)\neq(\mathrm{L:n})=\cap_{\mathrm{q_{i\in\Omega}}}\mathrm{q_j}\subseteq\mathrm{p}$. Since R is a PID, $(\mathrm{L:n})=(\mathrm{r})$ by some $\mathrm{r}\in\mathrm{R}$ and r has only a finite number of prime factors. Hence there exist only a finite number of prime ideal containing $(\mathrm{L:n})$. Thus Ω is a finite set. It follows that there exists $q_j\in\Omega$ such that $q_j\subseteq p$. It implies that $L\subseteq pM=rad(Q)$.

The following example shows that the converse of Theorem 5 is not true in general.

Example 4. Let $M=\mathbb{Q}\oplus\mathbb{Z}_p$. Then M is a top-like $\mathbb{Z}$-module which is not a weak multiplication module.

For any element x of an R-module M, we denote $c(x)=\cap\{I:I$ is an ideal of R. Such that $x\in IM\}$. M is called a content R-module if for every $x\in M, x\in$

$c(x)M$. Every free module or, more generally every projective module is content module [17, P. 51]. Also every faithful multiplication module is a content module [7, Theorem 1.6].

Theorem 6. Let M be a content weak multiplication R-module. Then M is top-like.

Proof. Let N be a semiprime submodule of , and let $N = \cap_{i \in I} P_i$, where, P_i is p_i-prime submodule of M for each $i \in I$. Since M is weak multiplication, for each $i \in I$. Since M is a content module, we have $N = (N: M)M$. Now, assume L that is a submodule of M. If $rad(L) = M$, then $v(L) = v\big(rad(L)\big) = v(RM)$. If $rad(L) \neq M$, then $rad(L)$ is a semiprime submodule of M. Hence $v(L) = v\big(rad(L)\big) = v\big((rad(L): M)M\big)$. Thus M is top-like, by the proof (1) $\Longrightarrow$ (2) of Theorem 4.

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Əsas spektri Zariski tip topologiyaya malik modullar

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XÜLASƏ

Bu işin məqsədi vahidə malik kommutativ R halqası üzərində yeni sinif “top-like” adlanan modulların öyrənilməsidir. Hər bir “üst” modul Zariski-topologiyası olan əsas spektrə malikdir. Bu sinif R-modulların vurulmasından alınan ailəni özündə saxlayır. Biz götərəcəyik ki, M-in məhdud R-modulları yalnız o zaman “top-like” modullar olacaqlar ki, onlar R-modullarn vurulmasından ibarət olsunlar.

Açar sözlər: əsas tip alt modul, əsas xassələr, “üst” modul, Zariski tipli topologiya, vurma modulu, WEPS modul.

Модули, основной спектр которых имеет топологию типа Зариски

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РЕЗЮМЕ

Целью данной работы является ознакомление и изучение нового класса модулей над коммутативным кольцом с единицей R, называемых «top-like» модули. Каждый «top-like» модуль обладает основным спектром с топологией типа Зариски. Этот класс содержит семейство умножений R-модулей. Мы покажем, что порожденные конечные R-модули являются «top-like» R-модуль тогда и только тогда, когда M является умножением R-модулей.

Ключевые слова: подмодуль типа основного, основное свойство, top-like модуль, топология типа Зарискому, модуль умножения, WEPS модуль.