PSEUDO-SPHERICAL 2-DEGENERATE CURVES IN MINKOWSKI SPACE-TIME

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Abstract. In this paper, we characterized 2-degenerate curves lying on pseudo-sphere $S_1^3(r)$ and pseudohyperbolic space $H_0^3(r)$ in the Minkowski spacetime $R_4^1$.

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1. Introduction

The geometry of null hypersurfaces in space-times has played an important role in the development of general relativity, as well as in mathematics and physics of gravitation. It is necessary to understand the causal structure of space-times, black holes, asymptotically flat systems and gravitational waves. For details see [5] and references therein.

The null curves in Lorentzian space have been studied by several authors (see [1, 2, 3, 7]). In a null hypersurface, there are many other curves distinct from the null ones. They are $s$-degenerate curves as those ones whose derivative of order $s$ is a null vector provided that $s > 1$ and all derivatives of order less than $s$ are spacelike. Thus classical null curves are 1-degenerate curves.

On the other hand, many studies on the Lorentzian spherical spacelike, timelike and null curves have been done by many authors. For example in [10], the authors have characterized the Lorentzian spherical spacelike curves in the Minkowski 3-space $R_3^1$. Lorentzian spherical timelike and null curves in the same space have been characterized in [12]. Later, in [4] the authors studied the spacelike, timelike and null curves lying on the pseudohyperbolic space $H_0^3$ in the Minkowski space-time.

The articles concerning the $s$-degenerate curves are rather few. In [8] the authors introduced $s$-degenerate curves in Lorentzian space forms and obtained a reference along an $s$-degenerate curve in $n$-dimensional Lorentzian space with the minimum number of curvatures. Therefore in this paper, we studied the pseudo-spherical 2-degenerate curves in Minkowski space-time $R_4^1$. We firstly aimed to show that the Cartan reference of an $s$-degenerate curves in $R_4^1$. Next we defined the 2-degenerate helices in $R_4^1$ and gave some necessary sufficient conditions for a 2-degenerate curve to lies on the pseudo-sphere. Moreover, we have seen that...
there are no 2-degenerate geodesics and 2-degenerate cubic curves which lie on
\(H^3_0(r)\).

2. Cartan frames for s-degenerate curves

The goal of this section is to find Frenet frames for \(s\)-degenerate curves in
Minkowski space-time. Before to do that we need a technical result.

Let \(E\) be a real vector space with a symmetric bilinear mapping\(g : ExE \to \mathbb{R}\). We say that \(g\) is degenerate on \(E\) if there exists a vector \(\xi \neq 0\) in
\(E\) such that

\[ g(\xi, v) = 0 \quad \text{for all} \quad v \in E, \]

otherwise, \(g\) is said to be non-degenerate. The radical (also called the null space) of \(E\), with respect to \(g\) is the subspace \(\text{Rad}(E)\) of \(E\) defined by

\[ \text{Rad}(E) = \{ \xi \in E \mid g(\xi, v) = 0, v \in E \}. \]

The dimension of \(\text{Rad}(E)\) is called the nullity degree of \(g\) (or \(E\)) and is denoted
by \(r_E\).

If \(F\) is a subspace of \(E\), then we can consider \(g_F\) the symmetric bilinear
mapping on \(FxF\) obtained by restricting \(g\) and define \(r_F\) as the nullity degree of
\(F\) (or \(g_F\)). For simplicity, we will use \(<,>\) instead of \(g\) or \(g_F\).

A vector \(v\) is said to be timelike, lightlike or spacelike provided that
\(<v, v><0, \quad <v, v>=0\) (and \(v \neq 0\)), or \(<v, v>>0\), respectively. The vector
\(v=0\) is assumed to be spacelike. A unit vector is a vector \(u\) such that
\(<u, u>=\pm1\). Two vectors \(u\) and \(v\) are said to be orthogonal, written \(u \perp v\), if
\(<u, v>=0\), [8].

Let \(\gamma : I \to \mathbb{R}^n_1\) be differentiable curve. For any vector field \(V\) along \(\gamma\), \(\dot{V}\)
be the covariant derivative of \(V\) along \(\gamma\). Write

\[ E_i(t) = \text{Span}\{\dot{\gamma}(t), \ddot{\gamma}(t), \ldots, \gamma^{(i)}(t)\}, \]

where \(t \in I\) and \(i=1,2,\ldots,n\). Let \(d\) be the number defined by
\(d = \max\{i : \dim E_i(t) = i, \text{ for all } t \}\).

Definition 1. With the above notations, the curve \(\gamma : I \to \mathbb{R}^n_1\) is said to be an \(s\)–
degenerate (or \(s\)-lightlike) curve if for all \(1 \leq i \leq d\), \(\dim \text{Rad}(E_i(t))\) is constant for
all \(t\), and there exists, \(0 < s \leq d\), such that \(\text{Rad}(E_s) \neq \{0\}\) and \(\text{Rad}(E_j) = \{0\}\) for all
\(j < s\), [2].

Remark 1. Note that \(1\)-degenerate curves are precisely the null (or lightlike)
curves (see [2,3,7]).
**Definition 2.** A basis \( \mathcal{B} = \{L_1, N_1, ..., L_r, N_r, W_1, ..., W_m\} \) of \( \mathbb{R}^n_q \), with \( 2r \leq 2q \leq n \) and \( m = n - 2r \), is said to be pseudo-orthonormal if it satisfies the following equations:

\[
\begin{align*}
\langle L_i, L_j \rangle &= \langle N_i, N_j \rangle = 0, \quad \langle L_i, N_j \rangle = \varepsilon_{ij}, \quad \langle L_i, W_{\alpha} \rangle = 0, \quad \langle W_{\alpha}, W_{\beta} \rangle = \varepsilon_{\alpha \beta} \delta_{\alpha \beta},
\end{align*}
\]

where \( i, j \in \{1, 2, ..., r\} \), \( \alpha, \beta \in \{1, 2, ..., m\} \), \( \varepsilon_{\alpha} = -1 \) if \( 1 \leq \alpha \leq q - r \) and \( \varepsilon_{\alpha} = 1 \) if \( q - r + 1 \leq \alpha \leq m \) and \( \delta_{\alpha \beta} \) is a symbol of kronecker.

Let \( \gamma : I \rightarrow \mathbb{R}^n_q \), \( n = m + 2 \), be an \( s \)-degenerate unit curve, \( s > 1 \). Then we have the following equations:

\[
\begin{align*}
\gamma' &= W_1, \quad W_1' = k_1 W_2, \quad W_i' = -k_{i-1} W_{i-1} + k_i W_{i+1}, \quad 2 \leq i \leq s - 2, \\
W_{s-1}' &= -k_{s-2} W_{s-2} + L, \quad \dot{L}' = k_{s-1} W_s, \quad W_s' = \varepsilon_2 L - \varepsilon_1 N, \\
N' &= -\varepsilon W_{s-1} - k_s W_s + k_{s+1} W_{s+1}, \\
W_{s+1}' &= -\varepsilon k_{s+1} L + k_{s+2} W_{s+2}, \\
W_j' &= -k_j W_{j-1} + k_{j+1} W_{j+1}, \quad s + 2 \leq j \leq m - 1, \\
W_m' &= -k_m W_{m-1},
\end{align*}
\]

for certain functions \( \{k_1, ..., k_m\}, [8] \).

**Theorem 1.** Let \( \gamma : I \rightarrow \mathbb{R}^n_q \), \( n = m + 2 \), be an \( s \)-degenerate unit curve, \( s > 1 \), and suppose that \( \{\gamma'(t), \gamma''(t), ..., \gamma^{(s-1)}(t)\} \) spans \( T_{\gamma(t)} \mathbb{R}^n_q \) for all \( t \). Then there exists a unique Frenet frame satisfying Eq. (1), [8].

**Definition 3.** An \( s \)-degenerate curve, \( s > 1 \), satisfying the above conditions is said to be \( s \)-degenerate Cartan curve. The reference and curvature functions given by (1) is called the Cartan reference and Cartan curvatures of \( \gamma \), respectively.

**Definition 4.** An \( s \)-degenerate helix in \( \mathbb{R}^n_q \) is an \( s \)-degenerate Cartan curve having constant Cartan curvatures.

3. **Pseudo-Spherical 2-degenerate Curves in \( \mathbb{R}^4_1 \)**

This section deals with pseudo-spherical 2-degenerate curves, that is, 2-degenerate curves that completely lie on a pseudo-sphere of radius \( r > 0 \) and of center \( A \) and denoted by

\[
\delta^3_1(r) = \{X \in \mathbb{R}^4_1 : \langle X - A, X - A \rangle = r^2\} [9].
\]

Minkowski space-time \( \mathbb{R}^4_1 \) is a Euclidean space \( \mathbb{R}^4 \) provided with the standard flat metric given by
\[ \langle x_i - y_i \rangle = -dx^2_1 + dx^2_2 + dx^2_3 + dx^4_2, \]

where \((x_1, x_2, x_3, x_4)\) is a rectangular coordinate system in \(R^4_1\).

Let \(\gamma\) be a 2-degenerate curve in \(R^4_1\). Then the Cartan equations can be written as follows:

\[
\gamma' = W_1, \\
W_1' = L, \\
L' = k_1W_2, \\
W_2' = \varepsilon k_2L - \varepsilon k_1N, \\
N' = -\varepsilon W_1 - k_2W_2.
\]

To have a characterization for pseudo-spherical 2-degenerate curves we use the osculating pseudo-sphere defined as below.

**Definition 5.** Let \(\gamma\) be a 2-degenerate curve in \(R^4_1\). Then the pseudo-sphere having five-point contact with \(\gamma\) is called the osculating pseudo-sphere of \(\gamma\) [6].

**Theorem 2.** Let \(\gamma\) be a 2-degenerate curve in \(R^4_1\). Then the center point of the osculating pseudo-sphere at a point \(\gamma(t)\) is

\[
A(t) = \gamma(t) - \varepsilon \frac{k_2(t)}{k_1(t)} L(t) - \varepsilon N(t).
\]

**Proof.** Let \(\{L, N, W_1, W_2\}\) be the Cartan frame. Then for any \(t\) the position vector \(A(t) - \gamma(t)\) can be written as a linear combination of the frame in the form

\[
A(t) - \gamma(t) = m_1(t)L(t) + m_2(t)N(t) + m_3(t)W_1(t) + m_4(t)W_2(t),
\]

where \(m_i, \ 1 \leq i \leq 4\) are differentiable functions on \(R\). Next, consider the function

\[
f(t) = A(t) - \gamma(t), A(t) - \gamma(t) > -r^2,
\]

where \(r\) is the radius of the osculating pseudo-sphere, thus the equations

\[
f(t) = f'(t) = f''(t) = f^{(3)}(t) = f^{(4)}(t) = 0,
\]

are satisfied due to the definition of the osculating pseudo-sphere at \(t\), then a straightforward computation leads to

\[
m_1(t) = \langle A(t) - \gamma(t), N(t) \rangle = -\varepsilon \frac{k_2(t)}{k_1(t)}, \]

\[
m_2(t) = \langle A(t) - \gamma(t), L(t) \rangle = -\varepsilon, \]

\[
m_3(t) = \langle A(t) - \gamma(t), W_1(t) \rangle = 0, \]

\[
m_4(t) = \langle A(t) - \gamma(t), W_2(t) \rangle = 0.
\]

Thus we find

\[
A(t) - \gamma(t) = -\varepsilon \frac{k_2(t)}{k_1(t)} L(t) - \varepsilon N(t)
\]
\[ r^2 = \left| 2\varepsilon \frac{k_2}{k_1} \right| \]  

(3)

**Definition 6.** Let \( \gamma \) be a 2-degenerate curve and \( S^3_1(r) \) be a pseudo-sphere in \( \mathbb{R}^4_1 \). If \( \gamma \subseteq S^3_1(r) \), then \( \gamma \) is called a pseudo-spherical 2-degenerate curve.

**Definition 7.** A 2-degenerate curve \( \gamma \) with zero first Cartan curvature \( k_1 \) is called 2-degenerate geodesic curve.

**Definition 8.** A 2-degenerate curve \( \gamma \) with zero first Cartan curvature \( k_1 \) and second curvature \( k_2 \) is called 2-degenerate cubic curve.

**Definition 9.** A 2-degenerate curve \( \gamma \) is called a general 2-degenerate helix if

\[
\frac{k_2}{k_1} = \text{const.}, \quad \text{where } k_1 \text{ and } k_2 \text{ are nonzero Cartan curvatures of } \gamma.
\]

**Theorem 3.** Let \( \gamma \subseteq \mathbb{R}^4_1 \) be a 2-degenerate curve. Then \( \gamma \) is a pseudo-spherical 2-degenerate curve if and only if \( \frac{k_2}{k_1} = \text{const.} \), where \( k_1 \) and \( k_2 \) are nonzero Cartan curvatures of \( \gamma \).

**Proof.** Suppose that \( \gamma \) is a pseudo-spherical 2-degenerate curve. Then the osculating pseudo-spheres at all points of the curve are exactly \( S^3_1(r) \), and so \( r \) is constant. Therefore from (3), \( \frac{k_2}{k_1} = \text{const.} \).

Conversely, assume that \( \frac{k_2}{k_1} = \text{const.} \). Then all of the osculating pseudo-spheres have the same radius. Moreover, if we consider the function

\[
A(t) = \gamma(t) - \varepsilon \frac{k_2(t)}{k_1(t)} L(t) - \varepsilon N(t).
\]

giving the central point of the osculating pseudo-sphere whose derivative is zero everywhere, so it is constant. Consequently, \( \gamma \) lies on \( S^3_1(r) \), since the equation

\[
\langle A(t) - \gamma(t), A(t) - \gamma(t) \rangle = r^2
\]

is valid for all \( t \in I \).

Since

\[
\langle A(t) - \gamma(t), A(t) - \gamma(t) \rangle = \left| 2\varepsilon \frac{k_2(t)}{k_1(t)} \right|,
\]

2-degenerate curves with \( \frac{k_2}{k_1} = \text{const.} \) lie on pseudo-sphere \( S^3_1(r) \).

**Corollary 1.** If we consider definition 7, there is no 2-degenerate geodesic which lies on \( S^3_1(r) \).

**Corollary 2.** If we consider definition 8, there is no 2-degenerate cubic curve which lies on \( S^3_1(r) \).
Corollary 3. A 2-degenerate curve $\gamma \subset R_1^4$ fully lies on a pseudo-sphere if and only if there exists a fixed point $A$ such that for $t \in I$
\[ < A(t) - \gamma(t), \gamma'(t) >= 0. \]

Theorem 4. Let $\gamma$ be a 2-degenerate curve in $R_1^4$. Then $\gamma$ lies on $S_1^3(r)$ if and only if $\gamma$ is a general 2-degenerate helix.

Proof. Let us first suppose that $\gamma$ lies on $S_1^3(r)$ with center $A$. By definition we have $< A - \gamma, A - \gamma >= r^2$. Differentiating the previous equation four times with respect to $t$ by using Cartan equations (2), we get
\[
< A - \gamma, W_1 >= 0, \quad < A - \gamma, L >= -1
\]
\[
< A - \gamma, W_2 >= 0, \quad < A - \gamma, N >= -\frac{k_2}{k_1}
\]
and
\[
A - \gamma = -\varepsilon \frac{k_2}{k_1} L - \varepsilon N.
\]
Thus we have
\[
< A - \gamma, A - \gamma >= 2\varepsilon \frac{k_2}{k_1} = r^2.
\]
Moreover, differentiating the last equation (4) with respect to $t$, we find
\[
(-\frac{k_2}{k_1})' = 0.
\]
Thus
\[
\frac{k_2}{k_1} = \text{const.},
\]
which means that, $\gamma$ is a general 2-degenerate helix.

Conversely, $\gamma$ is a general 2-degenerate helix. Thus we have $\frac{k_2}{k_1} = \text{const.}$

Then we have
\[
A(t) = \gamma - \varepsilon \frac{k_2}{k_1} L - \varepsilon N
\]
and $A' = 0$, that is, $A = \text{const.}$ Thus we get
\[
< A - \gamma, A - \gamma >= r^2,
\]
so $\gamma$ lies on $S_1^3(r)$.

If we consider pseudohyperbolic space with center $A \in R_1^4$ and radius $r \in R^+$ in Minkowski space-time $R_1^4$,
\[
H_0^3(r) = \{ X \in R_1^4 | < X - A, X - A > = -r^2 \},
\]
then we have following theorems:
Theorem 5. Let $\gamma$ be a $2 -$ degenerate curve in $\mathbb{R}^4_1$. Then the center point of the pseudohyperbolic space at a point $\gamma(t)$ is

$$A(t) = \gamma(t) - \varepsilon \frac{k_2(t)}{k_1(t)} L(t) - \varepsilon N(t).$$

Proof. It can be proved easily similar to Theorem 2.

Theorem 6. Let $\gamma \subset \mathbb{R}^4_1$ be a $2 -$ degenerate curve. Then $\gamma$ lies on $H^3_0(r)$ if and only if $k_1$ is a nonzero constant and $k_2$ is a constant at every point of the $2 -$ degenerate curve $\gamma$.

Proof. It is similar to Theorem 2.

Corollary 4. There is no $2 -$ degenerate geodesic which lies on $H^3_0(r)$.

Corollary 5. There is no $2 -$ degenerate cubic curve which lies on $H^3_0(r)$.

Corollary 6. A $2 -$ degenerate curve $\gamma \subset \mathbb{R}^4_1$ fully lies on $H^3_0(r)$ if and only if there exists a fixed point $A$ such that for $t \in I$

$$< A(t) - \gamma(t), \gamma'(t) > = 0.$$

Theorem 7. Let $\gamma$ be a $2 -$ degenerate curve in $\mathbb{R}^4_1$. Then $\gamma$ lies on $H^3_0(r)$ if and only if $\frac{k_2}{k_1} = const.$, that is $\gamma$ is a general $2 -$ degenerate helix.

Proof. It is similar to Theorem 4.

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References


Minkowski məkan-zaman fəzasında psevdosferik ikiqat çırlaşan ayrılər

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XÜLASƏ

Bu məqalədə biz Minkowski məkan-zaman fəzasının $S^3_1(r)$ psevdosferasında və $H^3_0(r)$ psevdohiperbolik fəzasında ikiqat çırlaşan ayrılərin xarakteristikası verilmişdir.

Açar sözlər: Null ayrılər, s-çırlaşan ayrılər, psevdosfera, psevdohiperbolik faza.

Псевдосферические 2- вырожденные кривые в пространстве-времени Минковского

Ханьдан Озтекин

РЕЗЮМЕ

В этой статье мы характеризуем 2- вырожденные кривые, лежащие на псевдо-сфере $S^3_1(r)$ и псевдо-гиперболическом пространстве $H^3_0(r)$ пространства-времени Минковского.

Ключевые слова: кривая Нуль, s-вырожденная кривая, псевдо-сфера, псевдо-гиперболическое пространство.