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PSEUDO-SPHERICAL 2-DEGENERATE CURVES IN MINKOWSKI SPACE-TIME

Handan Öztekin [1]

[1]Department of Mathematics, Firat University, Elazig, Turkey
e-mail: handanoztekin@gmail.com

Abstract. In this paper, we characterized 2-degenerate curves lying on pseudo-sphere $S_1^3(r)$ and pseudohyperbolic space $H_0^3(r)$ in the Minkowski spacetime $\mathbb{R}_1^4$.

Keywords: Null curve, s-degenerate curve, pseudo-sphere, pseudohyperbolic space.

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1. Introduction

The geometry of null hypersurfaces in space-times has played an important role in the development of general relativity, as well as in mathematics and physics of gravitation. It is necessary to understand the causal structure of space-times, black holes, asymptotically flat systems and gravitational waves. For details see [5] and references therein.

The null curves in Lorentzian space have been studied by several authors (see [1,2, 3,7]). In a null hypersurface, there are many other curves distinct from the null ones. They are s-degenerate curves as those ones whose derivative of order s is a null vector provided that $s > 1$ and all derivatives of order less than s are spacelike. Thus classical null curves are $1-$degenerate curves.

On the other hand, many studies on the Lorentzian spherical spacelike, timelike and null curves have been done by many authors. For example in [10], the authors have characterized the Lorentzian spherical spacelike curves in the Minkowski 3-space $\mathbb{R}_1^3$. Lorentzian spherical timelike and null curves in the same space have been characterized in [12]. Later, in [4] the authors studied the spacelike, timelike and null curves lying on the pseudohyperbolic space H_0^3 in the Minkowski space-time.

The articles concerning the s-degenerate curves are rather few. In [8] the authors introduced s-degenerate curves in Lorentzian space forms and obtained a reference along an s-degenerate curve in n-dimensional Lorentzian space with the minimum number of curvatures. Therefore in this paper, we studied the pseudo-spherical 2-degenerate curves in Minkowski space-time $\mathbb{R}_1^4$. We firstly aimed to show that the Cartan reference of an s-degenerate curves in $\mathbb{R}_1^4$. Next we defined the 2-degenerate helices in $\mathbb{R}_1^4$ and gave some necessary sufficient conditions for a 2-degenerate curve to lies on the pseudo-sphere. Moreover, we have seen that there are no 2-degenerate geodesics and 2-degenerate cubic curves which lie on $H_0^3(r)$.

2. Cartan frames for s-degenerate curves

The goal of this section is to find Frenet frames for s-degenerate curves in Minkowski space-time. Before to do that we need a technical result.

Let E be a real vector space with a symmetric bilinear mapping $g : ExE \to \mathbb{R}$. We say that g is degenerate on E if there exists a vector $\xi \neq 0$ in E such that

$$g(\xi, v) = 0 \quad for\, all\; v \in E,$$

otherwise, g is said to be non-degenerate. The radical (also called the null space) of E, with respect to g is the subspace $Rad(E)$ of E defined by

$$RadE = \{\xi \in E \,|\, g(\xi, v) = 0, v \in E\}.$$

The dimension of $Rad(E)$ is called the nullity degree of g (or E) and is denoted by r_E.

If F is a subspace of E, then we can consider g_F the symmetric bilinear mapping on FxF obtained by restricting g and define r_F as the nullity degree of F (or g_F). For simplicity, we will use $<,>$ instead of g or g_F.

A vector v is said to be timelike, lightlike or spacelike provided that $< v, v >< 0$, $< v, v >= 0$ (and $v \neq 0$), or $< v, v >> 0$, respectively. The vector $v = 0$ is assumed to be spacelike. A unit vector is a vector u such that $< u, u >= \pm 1$. Two vectors u and v are said to be orthogonal, written $u \perp v$, if $< u, v >= 0$, [8].

Let $\gamma : I \to \mathbb{R}_1^n$ be differentiable curve . For any vector field V along γ, V' be the covariant derivative of V along γ. Write

$$E_i(t) = Span\{\gamma'(t), \gamma''(t), ..., \gamma^{(i)}(t)\},$$

where $t \in I$ and $i = 1,2,...,n$. Let d be the number defined by $d = \max\{i : \dim E_i(t) = i, \;\; for \;\; all \;\; t \;\; \}$.

Definition 1. With the above notations, the curve $\gamma : I \to \mathbb{R}_1^n$ is said to be an $s-$ degenerate (or $s-$lightlike) curve if for all $1 \leq i \leq d$, $dimRad(E_i(t))$ is constant for all t, and there exists, $0 < s \leq d$, such that $Rad(E_s) \neq \{0\}$ and $Rad(E_s) = \{0\}$ for all $j < s$, [2].

Remark 1. Note that $1-$degenerate curves are precisely the null (or lightlike) curves (see [2,3,7]).

Definition 2. A basis $\mathrm{B}=\{L_1, N_1,...,L_r, N_r, W_1,...,W_m\}$ of $\mathbb{R}^n_q$, with $2r \le 2q \le n$ and $m=n-2r$, is said to be pseudo-orthonormal if it satisfies the following equations:

$$<L_i, L_j>=<N_i, N_j>=0, \quad <L_i, N_i>=\varepsilon_i, \quad <L_i, N_j>=0, i\ne j,$$
$$<L_i, W_\alpha>=<N_i, W_\alpha>=0, \quad <W_\alpha, W_\beta>=\varepsilon_\alpha \delta_{\alpha\beta},$$

where $i, j\in\{1,2,...,r\}$, $\alpha, \beta \in\{1,2,...,m\}$, $\varepsilon_\alpha=-1$ if $1\le\alpha\le q-r$ and $\varepsilon_\alpha=1$ if $q-r+1\le\alpha\le m$ and $\delta_{\alpha\beta}$ is a symbol of kronecker.

Let $\gamma: I\to\mathbb{R}^n_1$, $n=m+2$, be an $s-$ degenerate unit curve, $s>1$. Then we have the folowing equations:

$$\begin{aligned}
&\gamma' = W_1, \quad W_1' = k_1 W_2, \quad W_i' = -k_{i-1}W_{i-1}+k_i W_{i+1}, \quad 2\le i\le s-2,\\
&W_{s-1}' = -k_{s-2}W_{s-2}+L, \quad L' = k_{s-1}W_s, \quad W_s' = \varepsilon k_s L-\varepsilon k_{s-1}N,\\
&N' = -\varepsilon W_{s-1}-k_s W_s+k_{s+1}W_{s+1},\\
&W_{s+1}' = -\varepsilon k_{s+1}L+k_{s+2}W_{s+2},\\
&W_j' = -k_j W_{j-1}+k_{j+1}W_{j+1}, \quad s+2\le j\le m-1,\\
&W_m' = -k_m W_{m-1},
\end{aligned} \tag{1}$$

for certain functions $\{k_1,...,k_m\}$, [8].

Theorem 1. Let $\gamma: I\to\mathbb{R}^n_1$, $n=m+2$, be an $s-$ degenerate unit curve, $s>1$, and suppose that $\{\gamma'(t), \gamma''(t),...,\gamma^{(n)}(t)\}$ spans $T_{\gamma(t)}\mathbb{R}^n_1$ for all t. Then there exists a unique Frenet frame satisfying Eq. (1), [8].

Definition 3. An $s-$ degenerate curve, $s>1$, satisfying the above conditions is said to be $s-$ degenerate Cartan curve. The reference and curvature functions given by (1) is called the Cartan reference and Cartan curvatures of γ, respectively.

Definition 4. An $s-$ degenerate helix in $\mathbb{R}^n_1$ is an $s-$ degenerate Cartan curve having constant Cartan curvatures.

3. Pseudo-Spherical 2-degenerate Curves in $\mathbb{R}^4_1$

This section deals with pseudo-spherical 2-degenerate curves, that is, 2-degenerate curves that completely lie on a pseudo-sphere of radius $r>0$ and of center A and denoted by

$$S^3_1(r)=\{X\in\mathbb{R}^4_1 : <X-A, X-A>=r^2\} \, [9].$$

Minkowski space-time $\mathbb{R}^4_1$ is a Euclidean space $\mathbb{R}^4$ provided with the standart flat metric given by

$$<,>=-dx_1^2+dx_2^2+dx_3^2+dx_4^2,$$

where (x_1,x_2,x_3,x_4) is a rectengular coordinate system in $\mathbb{R}_1^4$.

Let γ be a 2-degenerate curve in $\mathbb{R}_1^4$. Then the Cartan equations can be written as follows:

$$\begin{aligned}\gamma' &= W_1,\\ W_1' &= L,\\ L' &= k_1W_2,\\ W_2' &= \varepsilon k_2 L-\varepsilon k_1 N,\\ N' &= -\varepsilon W_1 - k_2W_2.\end{aligned} \tag{2}$$

To have a characterization for pseudo-spherical 2-degenerate curves we use the osculating pseudo-sphere defined as below.

Definition 5. Let γ be a 2-degenerate curve in $\mathbb{R}_1^4$. Then the pseudo-sphere having five-point contact with γ is called the osculating pseudo-sphere of γ [6].

Theorem 2. Let γ be a 2-degenerate curve in $\mathbb{R}_1^4$. Then the center point of the osculating pseudo-sphere at a point $\gamma(t)$ is

$$A(t)=\gamma(t)-\varepsilon\frac{k_2(t)}{k_1(t)}L(t)-\varepsilon N(t).$$

Proof. Let $\{L,N,W_1,W_2\}$ be the Cartan frame. Then for any t the position vector $A(t)-\gamma(t),$ can be written as a linear combination of the frame in the form

$$A(t)-\gamma(t)=m_1(t)L(t)+m_2(t)N(t)+m_3(t)W_1(t)+m_4(t)W_2(t),$$

where m_i, $1\le i\le 4$ are differentiable functions on $\mathbb{R}$. Next, consider the function

$$f(t)=<A(t)-\gamma(t),A(t)-\gamma(t)>-r^2,$$

where r is the radius of the osculating pseudo-sphere, thus the equations

$$f(t)=f'(t)=f''(t)=f^{(3)}(t)=f^{(4)}(t)=0,$$

are satisfied due to the definition of the osculating pseudo-sphere at $t,$ then a straightforward computation leads to

$$\begin{aligned}m_1(t)&=<A(t)-\gamma(t),N(t)>=-\varepsilon\frac{k_2(t)}{k_1(t)},\\ m_2(t)&=<A(t)-\gamma(t),L(t)>=-\varepsilon,\\ m_3(t)&=<A(t)-\gamma(t),W_1(t)>=0,\\ m_4(t)&=<A(t)-\gamma(t),W_2(t)>=0.\end{aligned}$$

Thus we find

$$A(t)-\gamma(t)=-\varepsilon\frac{k_2(t)}{k_1(t)}L(t)-\varepsilon N(t)$$

and

$$r^2 = \left| 2\varepsilon \frac{k_2}{k_1} \right|. \tag{3}$$

Definition 6. Let γ be a 2-degenerate curve and $S_1^3(r)$ be a pseudo-sphere in $\mathbb{R}_1^4$. If $\gamma \subset S_1^3(r)$, then γ is called a pseudo-spherical 2-degenerate curve.

Definition 7. A 2-degenerate curve γ with zero first Cartan curvature k_1 is called 2-degenerate geodesic curve.

Definition 8. A 2-degenerate curve γ with zero first Cartan curvature k_1 and second curvature k_2 is called $2-$degenerate cubic curve.

Definition 9. A 2-degenerate curve γ is called a general $2-$ degenerate helix if $\frac{k_2}{k_1} = const.$, where k_1 and k_2 are nonzero Cartan curvatures of γ.

Theorem 3. Let $\gamma \subset \mathbb{R}_1^4$ be a 2-degenerate curve. Then γ is a pseudo-spherical 2-degenerate curve if and only if $\frac{k_2}{k_1} = const.$, where k_1 and k_2 are nonzero Cartan curvatures of γ.

Proof. Suppose that γ is a pseudo-spherical 2-degenerate curve. Then the osculating pseudo-spheres at all points of the curve are exactly $S_1^3(r)$, and so r is constant. Therefore from (3), $\frac{k_2}{k_1} = const.$

Conversely, assume that $\frac{k_2}{k_1} = const.$ Then all of the osculating pseudo-spheres have the same radius. Moreover, if we consider the function

$$A(t) = \gamma(t) - \varepsilon \frac{k_2(t)}{k_1(t)} L(t) - \varepsilon N(t).$$

giving the central point of the osculating pseudo-sphere whose derivative is zero everywhere, so it is constant. Consequently, γ lies on $S_1^3(r)$, since the equation

$$< A(t) - \gamma(t), A(t) - \gamma(t) >= r^2$$

is valid for all $t \in I$.

Since $< A(t) - \gamma(t), A(t) - \gamma(t) >= \left| 2\varepsilon \frac{k_2(t)}{k_1(t)} \right|$, 2-degenerate curves with $\frac{k_2}{k_1} = const.$ lie on pseudo-sphere $S_1^3(r)$.

Corollary 1. If we consider definition 7, there is no 2-degenerate geodesic which lies on $S_1^3(r)$.

Corollary 2. If we consider definition 8, there is no 2-degenerate cubic curve which lies on $S_1^3(r)$.

Corollary 3. A $2-$degenerate curve $\gamma \subset \mathbb{R}_1^4$ fully lies on a pseudo-sphere if and only if there exists a fixed point A such that for $t \in I$

$$< A(t) - \gamma(t), \gamma'(t) >= 0.$$

Theorem 4. Let γ be a $2-$degenerate curve in $\mathbb{R}_1^4$. Then γ lies on $S_1^3(r)$ if and only if γ is a general $2-$ degenerate helix.

Proof. Let us first suppose that γ lies on $S_1^3(r)$ with center A. By definition we have $< A-\gamma, A-\gamma >= r^2$. Differentiating the previous equation four times with respect to t by using Cartan equations (2), we get

$$\begin{aligned} &< A-\gamma, W_1 >= 0, \quad < A-\gamma, L >= -1 \\ &< A-\gamma, W_2 >= 0, \quad < A-\gamma, N >= -\frac{k_2}{k_1} \end{aligned} \tag{4}$$

and

$$A - \gamma = -\varepsilon \frac{k_2}{k_1} L - \varepsilon N.$$

Thus we have

$$< A-\gamma, A-\gamma >= 2\varepsilon \frac{k_2}{k_1} = r^2.$$

Moreover, differentiating the last equation (4) with respect to t, we find

$$(-\frac{k_2}{k_1})' = 0.$$

Thus

$$\frac{k_2}{k_1} = const., \tag{5}$$

which means that, γ is a general $2-$ degenerate helix.

Conversely, γ is a general $2-$ degenerate helix. Thus we have $\frac{k_2}{k_1} = const.$

Then we have

$$A(t) = \gamma - \varepsilon \frac{k_2}{k_1} L - \varepsilon N$$

and $A' = 0$, that is, $A = const.$ Thus we get

$$< A-\gamma, A-\gamma >= r^2,$$

so γ lies on $S_1^3(r)$.

If we consider pseudohyperbolic space with center $A \in \mathbb{R}_1^4$ and radius $r \in \mathbb{R}^+$ in Minkowski space-time $\mathbb{R}_1^4$

$$H_0^3(r) = \{X \in \mathbb{R}_1^4 \mid < X - A, X - A >= -r^2\},$$

then we have following theorems:

Theorem 5. Let γ be a $2-$degenerate curve in $\mathbb{R}_1^4$. Then the center point of the pseudohyperbolic space at a point $\gamma(t)$ is

$$A(t) = \gamma(t) - \varepsilon \frac{k_2(t)}{k_1(t)} L(t) - \varepsilon N(t).$$

Proof. It can be proved easily similar to Theorem 2.

Theorem 6. Let $\gamma \subset \mathbb{R}_1^4$ be a $2-$degenerate curve. Then γ lies on $H_0^3(r)$ if and only if k_1 is a nonzero constant and k_2 is aconstant at every point of the 2-degenerate curve γ.

Proof. It is similar to Theorem 2.

Corollary 4. There is no $2-$degenerate geodesic which lies on $H_0^3(r)$.

Corollary 5. There is no $2-$degenerate cubic curve which lies on $H_0^3(r)$.

Corollary 6. A $2-$degenerate curve $\gamma \subset \mathbb{R}_1^4$ fully lies on $H_0^3(r)$ if and only if there exists a fixed point A such that for $t \in I$

$$< A(t) - \gamma(t), \gamma'(t) >= 0.$$

Theorem 7. Let γ be a $2-$degenerate curve in $\mathbb{R}_1^4$. Then γ lies on $H_0^3(r)$ if and only if $\frac{k_2}{k_1} = const.$, that is γ is a general $2-$ degenerate helix.

Proof. It is similar to Theorem 4.

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Minkowski məkan-zaman fəzasında psevdosferik ikiqat cırlaşan əyrilər

Handan Öztekin

XÜLASƏ

Bu məqalədə biz Minkowski məkan-zaman fəzasının $S_1^3(r)$ psevdo sferasında və $H_0^3(r)$ psevdo hiperbolik fəzasında ikiqat cırlaşan əyrilərin xarakteristikası verilmişdir.

Açar sözlər: Null əyriləri, s-cırlaşan əyrilər, psevdo sfera, psevdo hiperbolik fəza.

Псевдосферические 2- вырожденные кривые в пространстве-времени Минковского

Ханьдан Озтекин

РЕЗЮМЕ

В этой статье мы характеризуем 2- вырожденные кривые, лежащие на псевдо-сфере $S_1^3(r)$ и псевдо-гиперболическом пространстве $H_0^3(r)$ пространства-времени Минковского.

Ключевые слова: кривая Нулль, s-вырожденная кривая, псевдо-сфера, псевдо-гиперболическое пространство.