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THE UNILATERAL QUADRATIC MATRIX EQUATION AND PROBLEM OF UPDATING OF PARAMETERS OF MODEL

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Abstract. The algorithm of construction of solutions of the unilateral quadratic matrix equation in case of complex eigenvalues of the corresponding matrix pencil is offered. As application, the problem of updating parameters of model by experimentally estimation of eigenvalues of this system is considered. Efficiency of the offered procedure of updating is shown on the example of system with two degree of freedoms.

Keywords: unilateral quadratic matrix equation, matrix sign function, parameters updating.

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1. Introduction

It is known, that various engineering problems are connected with the theory of oscillations. Here it is necessary to note the theory of strongly damped systems [4], in which central place occupy the problems of determination of matrix roots of the equations

$$A_2X^2 + A_1X + A_0 = 0. \tag{1}$$

In [2] matrix equation (1) is called as the unilateral quadratic matrix equation (UQME). Here the wide range of problems of control in which it is necessary to find the solution of UQME is noted. It is natural, that in different problems those or other solutions of (1) can be of interest. Thereupon, as well as in [4], we will compare with (1) the matrix pencil

$$L(\lambda) = A_2\lambda^2 + A_1\lambda + A_0. \tag{2}$$

Let a size of the matrices in (1) is equal to $n \times n$, $A_2 = I$, hereinafter I - an identity matrix of a corresponding size. Root (X_1) of the equation (1) allows factorizing the pencil (2):

$$L(\lambda) = (I\lambda - \widehat{X}_1)(I\lambda - X_1). \tag{3}$$

As it is noted in [4], generally speaking, the matrix $\widehat{X}_1 = -A_1 - X_1$ will not be solution of the equation (1). However, if $A_0 = A_0^T > 0$, $A_1 = A_1^T > 0$ together with the matrix X_1 solution of the equation (1) will be also the matrix

$$X_2 = -A_1 - X_1^T. \tag{4}$$

Hereinafter, the superscript means a transposition. It is essential, that in the depending on the eigenvalues of the pencil (2) it is possible to consider various problems of factorization, i.e. representation the pencil (2) in the form of (3). So, in [6], it is assuming, that the eigenvalues of the pencil (2) ordered by modulo , satisfy to the relation

$$|\lambda_n| < \rho < |\lambda_{n+1}|, \tag{5}$$

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where ρ -is some number. In [6], by this assumption, it is considered the algorithms of construction of solutions of (1) (X_+, X_-), such the eigenvalues X_+ coincide with $\lambda_1, \ldots, \lambda_n$, and the eigenvalues X_- coincide with $\lambda_{n+1}, \ldots, \lambda_{2n}$. Thus, it is possible to tell, that in the considered above case strongly damped systems, the problem of determination of X_+, X_- corresponds to the problem of factorization of the polynomial (2) concerning to a circle of radius ρ. However, if among eigenvalues of the pencil (2) there are complex eigenvalues with a small or zero real part, generally speaking, can not exist solutions of (1) with the properties noted above. For example, let in the equation (1) $A_2 = I,\ A_1 = 0,\ A_0 = diag\left\{\omega_1^2, \omega_2^2\right\},\ \omega_2^2 > \omega_1^2$. In this case $\lambda_{1,2} = \pm i\omega_1,\ \lambda_{3,4} = \pm i\omega_2$, i.e. the condition (5) is fulfilled. However, the equation (1) has no root X_1 which eigenvalues would coincide with $\lambda_{1,2}$. Really, if there was such root, the equality

$$X_1^2 = -A_0$$

would be fulfilled. But that is not possible, since both eigenvalues of matrix X_1 are equal to $-\omega_1^2$ while eigenvalues of matrix A_0 are equal to ω_1^2, ω_2^2. Thus, in a considered example it is expedient to choose as a solution of (1) the matrix

$$X_+ = diag\left\{i\omega_1, i\omega_2\right\},$$

which eigenvalues lay in upper half plane $(Im\lambda) > 0)$, or matrix $X_- = -X_+$ which eigenvalues lay in lower half-plane $(Im(\lambda) < 0)$. In other words, it is expedient to consider a problem of a factorization (2) concerning the real axis. Farther, assuming, that all eigenvalues of the pencil (2) are complex, the algorithm of construction of such solutions of (1) will be considered. This algorithm is based on procedure of computing of the matrix sign function [1].

As application of such solutions of (1), the iterative procedure of an improvement of estimation of parameters of weakly damped mechanical system is considered. In this procedure are using the results of estimation of eigenvalues which, in turn, can be received as a result of handling of transients in this system (see the example).

2. The unilateral quadratic matrix equation

Let's rewrite the equation (1) in the form

$$\begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} X = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix}. \tag{6}$$

In (6) and further, 0 -is a zero matrix of a corresponding size. Let's notice, that if matrix A_2 has inverse matrix, for determination of the solution of (1) it is possible to use the matrix sign function method [1]. In this case the relation (6) can be rewritten as:

$$\begin{bmatrix} I \\ X \end{bmatrix} X = H \begin{bmatrix} I \\ X \end{bmatrix}, H = \begin{bmatrix} 0 & I \\ -A_2^{-1}A_0 & -A_2^{-1}A_1 \end{bmatrix}. \tag{7}$$

Let's introduce concept of a matrix sign function [1, 3]. Let matrix Z has no eigenvalues on the imaginary axis. I.e. this representation takes place

$$Z = \tau \begin{bmatrix} v & 0 \\ 0 & \pi \end{bmatrix} \tau^{-1},$$

where the matrix v has eigenvalues only in the left half-plane, and the matrix π - only in the right half-plane. The sign function of matrix Z is defined as follows

$$sgnZ = \tau \begin{bmatrix} -I_v & 0 \\ 0 & I_\pi \end{bmatrix} \tau^{-1},$$

where the identity matrices I_v,I_π has sizes of blocks v and π accordingly. We will notice, that there is a simple procedure of evaluation of $sgnZ$, namely

$$sgnZ = \lim_{k\to\infty} it\, Z_k, Z_{k+1} = \frac{1}{2}\left(Z_k + Z_k^{-1}\right), Z_0 = Z. \tag{8}$$

Let's assume, that the matrix H in (7) has no real eigenvalues. In this case, the eigenvalues of the matrix H will be symmetrical arrangement concerning to the real axis. Let the eigenvalues of the matrix X_+ lay in the upper half plane. Then eigenvalues of the matrix iX_+ will lay in the left half-plane. Having multiplied left and right parts of (7) on i, we rewrite this relation in the form

$$\begin{bmatrix} I \\ X_+ \end{bmatrix} iX_+ = iH \begin{bmatrix} I \\ X_+ \end{bmatrix}. \tag{9}$$

Let's apply the procedure (8) to both sides of (9). We describ only the first step of this procedure. Having multiplied (9) at the left on $(iH)^{-1}$, and on the right on $(iX_+)^{-1}$ we get

$$(H)^{-1} \begin{bmatrix} I \\ X_+ \end{bmatrix} = \begin{bmatrix} I \\ X_+ \end{bmatrix} (iX_+)^{-1}. \tag{10}$$

Combining (9) and (10), and multiplying the result on $1/2$ we obtain

$$\begin{bmatrix} \mathrm{I} \\ \mathrm{X}_+ \end{bmatrix} \cdot \left(\frac{iX_+ + (iX_+)^{-1}}{2} \right) = \left(\frac{iH + (iH)^{-1}}{2} \right) \begin{bmatrix} I \\ X_+ \end{bmatrix}.$$

Continuing this process and taking into account, that $sgn(iX_+) = -I$, we have

$$-\begin{bmatrix} I \\ X_+ \end{bmatrix} = \mathrm{sgn}(iH) \cdot \begin{bmatrix} I \\ X_+ \end{bmatrix}. \tag{11}$$

The relation (11) can be rewritten as

$$(I + sgn(iH)) \cdot \begin{bmatrix} I \\ X_+ \end{bmatrix} = 0. \tag{12}$$

Similar calculations can be done with reference to other root of (1) (X_-). As eigenvalues of X_- lay in the lower half-plane, similar procedure allows one to get the following relation determining X_-

$$(I - sgn(iH)) \cdot \begin{bmatrix} I \\ X_- \end{bmatrix} = 0. \tag{13}$$

Thus, determination of solutions X_+, X_- of the equation (1) after evaluation of sign function of the matrix iH, is actually reduced to the procedure of solution of (1) to solution of system of linear equations (12), (13).

Let's continue to consider a case when matrix A_2 is invertible. In this case it is possible to specify rather simple procedure of improvement of the solution (1) received by means of described above algorithm [6]. So, let it is known X_0 – some approached value of a solution of the equation (1).

The solution of the equations (1) we will search in the form:

$$X = X_0 + \varepsilon X_1, \tag{14}$$

where εX_1 – is the small correction (ε -is small parameter). Having substituted (14) into (1) and neglecting the terms of order ε^2, we have

$$(X_0 + A_2^{-1}A_1)\varepsilon X_1 + \varepsilon X_1 X_0 = -X_0^2 - A_2^{-1}A_1X_0 - A_2^{-1}A_0. \tag{15}$$

The relation (15) allows using for determination of εX_1 standard procedure lyap.m of MATLAB package.

3. Sensitivity of eigenvalues

Let's consider a possibility to use described above solutions of the equation (1) in the problem of determination of sensitivity of eigenvalues to changing of parameters of system [8] and in the problem of updating of parameters of model [9]. Let it is assignment the system (model) movement of which be described by the following Lagrange equation

$$M\ddot{q} + B\ddot{q} + Kq = 0. \tag{16}$$

In this equation $q \in R^n$ is vector of generalized co-ordinates, the matrix is considered fixed, and matrices B, K depend on parameters β_i,γ_j as follows

$$B = B_0 + \delta B, K = K_0 + \delta K, \delta B = \sum_{i=1}^{s} \beta_i B_i, \delta K = \sum_{j=1}^{p} \gamma_j K_j. \tag{17}$$

In these relations, the matrices $B_i, i = 0, ..., s$, $K_j, j = 0, ..., p$ are specified. It is necessary to find the dependence on parameters β_i, γ_j (which are assumed small) changes of eigenvalues of system (16).

In other words, let are known $\lambda_i (i = 1, ..., 2n)$ eigenvalues of the differential equation (16) as zero values of δB and δK. It is necessary to find the dependence of increments $\delta\lambda_i$ of these eigenvalues on parameters β_i, γ_j. We will show, that in this problem it is possible effectively to use solutions of the equation (1), assuming, that all eigenvalues λ_i are complex. So, let solutions (roots) X_+, X_- of the equations (1), in which $A_2 = M$, $A_1 = B_0$, $A_0 = K_0$ are known. As it has been noted above, join of n eigenvalues of each of matrices $X_\pm$, gives $2n$ eigenvalues of the equation (1). Thus, it is enough to consider, for example, this problem only for n eigenvalues λ_i of the matrix X_+ which we will designate further as X (eigenvalues of the matrix X_- will be complexly conjugate to the eigenvalues of the matrix X_+).

Let's consider relations [8] which define sensitivity of eigenvalues. As well as in [8], it is supposed, that among eigenvalues λ_i there are no multiple ones. We designate as $D_x\lambda_i$ a matrix of sensitivity of eigenvalue λ_i to the changing of elements X_{kl} of matrix X

$$D_x\lambda_i = \left[\frac{\partial \lambda_i}{\partial X_{kl}} \right].$$

These matrices, according to (15) [8] are defined by the following relation

$$\left[\begin{array}{ccc} D_x\lambda_1 & \ldots & D_x\lambda_n \end{array} \right]^B = \left(V^{-1} \otimes I\right) \left[\begin{array}{ccc} I\, X^T & \ldots & \left(\mathrm{X}^{\mathrm{T}}\right)^{n-1} \end{array} \right]^B, \tag{18}$$

$$V = \begin{bmatrix} 1 & \ldots & 1 \\ \lambda_1 & & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & & \lambda_n^{n-1} \end{bmatrix}.$$

In (18) " $\otimes$ " means the Kronecker tensor product (procedure kron.m of MATLAB package), matrix V is the Vandermonde matrix and for its definition it is possible to use procedure vander.m of MATLAB package. The superscript "B", as well as in [8] means the block transpose. So, in the case of matrix 2×2, the relation (18) has a form

$$\begin{bmatrix} D_x\lambda_1 \\ D_x\lambda_2 \end{bmatrix} = \left(V^{-1} \otimes I\right) \begin{bmatrix} I \\ X^T \end{bmatrix}.$$

According to (23), (24) [8], the following relations, which can be used for control of accuracy of the evaluation of matrices $D_x\lambda_i$, take place:

$$\sum_{i=1}^{n} D_X\lambda_i = I, \sum_{i=1}^{n} \lambda_i^j D_x\lambda_i = (X^T)^j.$$

Let's notice, that use in this problem of the solution of the equation (1) has allowed to halve a size of the Vandermonde matrix.

Thus, if it is known a variation of matrix X caused by a changing of this or that parameter (β_i or γ_j), then, is possible, using (18) to find a corresponding variation of eigenvalues λ_i. We determine the connection of variation δX of matrix X with parameters β_i, γ_j, considering the small value of matrices δX, δB, δK. According to (1), in linear approach, the following relation takes place

$$M\left(X\delta X + \delta X X\right) + B_0\delta X + \delta B X + \delta K = 0.$$

If $M^{=1}$ exist's, this relation can be rewritten in the form of Lyapunov equation

$$\left(X + M^{-1}B_0\right)\delta X + \delta X X = -M^{-1}\delta B X - M^{-1}\delta K. \tag{19}$$

Let's notice, that according to (17) it is possible to present δX as a linear combination of the solutions of the equations similar to (19):

$$\delta X = \sum_{\mathrm{i}=1}^{\mathrm{s}} \beta_i \delta X_2^{\mathrm{i}} + \sum_{j=1}^{p} \gamma_j \delta X_k^j. \tag{20}$$

Here δX_2^V and $\delta \mathrm{X}_{\mathrm{k}}^{\mathrm{j}}$ are solutions of the following equations similar to (19)

$$\left(X + M^{-1}B_0\right)\delta X_2^V + \delta \mathrm{X}_2^{\mathrm{V}} X = -M^{-1}B_i X, \tag{21}$$

$$\left(X + M^{-1}B_0\right)\delta X_k^j + \delta \mathrm{X}_{\mathrm{k}}^{\mathrm{j}} X = -M^{-1}K_j. \tag{22}$$

Relations (18), (20) – (22) will allow one to construct the dependence of $\delta\lambda$ on parameters β_i, γ_j. Let $\delta\lambda_r^i$ be variation of eigenvalue λ_r which is caused by changing of the parameter β_i, and $\delta\lambda_r^j$ is variation of eigenvalue λ_r which is caused by changing of the parameter γ_j. These variations are defined by following relations

$$\delta\lambda_r^i = \beta_i f_{ri}, f_{ri} = tr(\delta X_2^{\mathrm{i}}\left(D_x\lambda_r\right)^T), \tag{23}$$

$$\delta\lambda_r^i = \gamma_j \psi_{rj}, \psi_{rj} = tr(\delta X_k^j\left(D_x\lambda_r\right)^T). \tag{24}$$

Here "tr" – means a trace of matrix. In others words, the dependance on parameters of variation of eigenvalue $\delta\lambda_r$ is determined as follows

$$\delta\lambda_r = \sum_{i=1}^{s} \beta_i f_{ri} + \sum_{j=1}^{p} r_j \psi_{rj}. \tag{25}$$

Taking into consideration (25), designating $\delta\lambda = \left[\begin{array}{ccc} \delta\lambda_1 & \ldots & \delta\lambda_n \end{array}\right]^T$, $\beta = \left[\begin{array}{ccc} \beta_1 & \ldots & \beta_s \end{array}\right]^T$, $\gamma = \left[\begin{array}{ccc} \gamma_1 & \ldots & \gamma_p \end{array}\right]^T$, we get following relation

$$\delta\lambda = F\beta + \psi\gamma, F = \left[f_{ri}\right], \psi = \left[\psi_{rj}\right]. \tag{26}$$

Elements f_{ri}, ψ_{rj} of matrices F, ψ are defined by (23), (24).

Now give detail description of the procedure of use of this relation in the considered problem of updating of parameters of model, where vector $\delta\lambda$ is considered as a known, and vectors β, γ are subject to determination. We will notice, that elements $\delta\lambda_i$ of the vector $\delta\lambda$ are complex numbers determining a changing of the real and imaginary part of the corresponding complex eigenvalue (λ_i). Thereupon it is expedient to present vector $\delta\lambda$ in the form of the vector of size $2n$ (vector θ) the first n components of which coincide with the real parts of the vector $\delta\lambda$, and remained n component corresponds to the imaginary part. In turn, it will allow rewriting system (26) in the form of system $2n$ linear equations

$$\theta = Dz, D = [F \quad \psi], z = \left[\beta^T \quad \gamma^T\right]^T. \tag{27}$$

Thus, if in (17) the number of parameters of model, which are subject to updating, does not exceed $2n$ ($s + p \leq 2n$), the system (27) allows one to find the corresponding corrections. So, for example, if $s + p = 2n$ and matrix D^{-1} exists, the system (27) can be rewritten in the form

$$z = D^{-1}\theta. \tag{28}$$

As these linear relations ((26) - (28)) are correct for enough small values of θ and z, generally speaking, the procedure of updating of parameters β and γ should be iterative (see example). We will notice, that in differ from [9], in examined statement of the problem of updating of parameters of model, it is supposed to use the "experimental" information about eigenvalues of system (16) only (that allows generating vector θ). The presence of the similar information about eigenvectors of this system is not supposed.

4. Example

Let's consider the mechanical system consisting of two masses connected by springs and dampers, movement of which is described by the equation (16). The matrices, appearing in (17)are taken as

$$\begin{gathered} M = diag\{m_1, m_2\}, B_0 = \begin{bmatrix} b_1 & -b_1 \\ -b_1 & b_1 + b_2 \end{bmatrix}, K_0 = \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 + c_2 \end{bmatrix}, \\ B_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, K_1 = B_1, K_2 = B_2, \\ m_1 = 10, m_2 = 1, b_1 = 0, b_2 = 0, c_1 = 40, c_2 = 5, \\ \beta_1 = 0, \beta_2 = 5, \gamma_1 = 10, \gamma_2 = 5. \end{gathered} \tag{29}$$

In [5], the procedure is described for the identification of the system (16), (17), (29) by results of transient registration. By using of this procedure, the following estimations of eigenvalues of this system have been obtained (table 2 [5]):

$$\begin{gathered} \lambda_{12} = -2.3316 \pm 7.5532i, \\ \lambda_{34} = -0.1745 \pm 0.8765i, \end{gathered} \tag{30}$$

which will be used in the problem of updating parameters of model as the "experimental" obtained data. Thus, in the considered problem, the initial value of parameters is determined by matrices M, B_0, K_0 ($\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$). It is necessary by experimental data (by estimations (30)) to find the matrices B, K, i.e. to estimate values of the parameters $\beta_1, \beta_2, \gamma_1, \gamma_2$

$$\bar{b}_1 = b_1 + \beta_1,\ \bar{b}_2 = b_2 + \beta_2,\ \bar{c}_1 = c_1 + \gamma_1,\ \bar{c}_2 = c_2 + \gamma_2. \tag{31}$$

As it was already noted, the relation (28) is correct for enough small value of θ and z. Therefore, in the considered problem it expedient to use two iterative cycles of updating of parameters. Explicitly we will describe the first cycle, since the second is similar. So, applying algorithm of the item 2 to the equation (1), in which $A_2 = M,\ A_1 = B_0,\ A_0 = K_0$, according to (12), we get the following solution

$$X_+ = \begin{bmatrix} 0 + 1.1130\mathrm{i} & 0 - 0.5255\mathrm{i} \\ 0 - 5.2548\mathrm{i} & 0 + 6.4991\mathrm{i} \end{bmatrix}, \tag{32}$$

which is satisfied (1) with accuracy 10^{-14}. Thereupon there is no necessity to use the procedure of the improvement by (15).

Eigenvalues of the matrix X_+, defined by (32), are the following

$$\lambda_1 = 0.6416i, \quad \lambda_2 = 6.9705i. \tag{33}$$

Using (18), we find the matrices $D_x\lambda_1$, $D_x\lambda_2$. Using these matrices it is possible to construct the matrix D, which is appearing in (28)

$$D = \begin{bmatrix} -0.0005 & -0.0372 & 0 & 0 \\ -0.5495 & -0.4628 & 0 & 0 \\ 0 & 0 & 0.0008 & 0.0580 \\ 0 & 0 & 0.0788 & 0.0664 \end{bmatrix}.$$

According to (30), (33) the vector θ in (28) has a form

$$\theta = [-0.1745 \quad -2.3316 \quad 0.2349 \quad 0.5827]^{\mathrm{T}}.$$

By these data, according to (28), the vector z and corresponding estimations of parameters $\bar{b}_1,\ \bar{b}_2,\ \bar{c}_1,\ \bar{c}_2$, which were used as initial data in the second iterative cycle, have been found.

Let's notice, that in the second cycle the matrices X_+. D and the vector θ are

$$X_+ = \begin{bmatrix} -0.0150 + 1.2526\mathrm{i} & -0.1336 - 0.5065\mathrm{i} \\ 1.6370 - 5.0651\mathrm{i} & -2.4911 + 6.6387\mathrm{i} \end{bmatrix},$$

$$D = \begin{bmatrix} -0.0008 & -0.0351 & -0.0010 & 0.0054 \\ -0.5492 & -0.4649 & 0.0010 & -0.0054 \\ -0.0010 & -0.0020 & 0.0011 & 0.0413 \\ -0.1745 & -0.1972 & 0.0774 & 0.0676 \end{bmatrix},$$

$$\theta = [-0.0162\ 0.0162\ 0.0457\ 0.4926]^{\mathrm{T}}.$$

Result for the evaluations are given in the table 1.

Table 1

	I	II	III	IV	V
$\bar{b}_1$	0	0.3005	-0.1081	0	0
$\bar{b}_2$	0	4.6816	5.1311	5.0042	5
$\bar{c}_1$	40	44.0283	49.7709	49.7096	50
$\bar{c}_2$	5	8.9937	9.9576	9.9379	10
$d\Delta$	3.4239	0.7004	0.0245	0.0245	0.0146

In the table 1 the following notation are accepted: $\bar{b}_1, \bar{b}_2, \bar{c}_1, \bar{c}_2$ – estimations of parameters (31) of system; $d\Lambda$ – norm of a difference of the vectors, one of which is the vector of "experimentally received eigenvalues (30), the second – the vector of eigenvalues of the system (16), parameters of which are given in the corresponding column of table. So, columns I - IV contain estimations of parameters of system: the column I corresponds to the initial approach ($B = B_0,\ K = K_0$); the column II contains the estimations of parameters received after the first iteration; in the column III the estimations received after the second iteration are given. Here it should be noted, that after the second iteration, as a result of use of the relation (28), the value of estimation $\bar{b}_1$ has turned out negative. Thereupon, in the second iteration of the procedure of solution, so-called NNLS problems [7] with reference to system (27) (procedure nnls.m of MATLAB package) have been used. These outcomes are given in the column IV. Exact values of the parameters of system are given in the column V. Thus, comparing the initial values of estimations of parametres of system (a column I), the values of estimations obtained as a result of two iterations (a column IV) and exact values of parametres of system (a column V), it is possible to state, about the high efficiency of the proposed procedure of the model parameters updating in the considered example.

5. Conclusion

The algorithm of construction of solutions of the unilateral quadratic matrix equation in case of complex eigenvalues of the corresponding matrix pencil is offered. As application, the problem of updating parameters of model by experimentally estimation of eigenvalues of this system is considered. Efficiency of the offered procedure of updating is shown on the example of system with two degree of freedoms.

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