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Generalized Sobolev–Morrey estimates for hypoelliptic operators on homogeneous groups

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Abstract
Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous group, Q is the homogeneous dimension of $\mathbb{G}$, $X_0, X_1, \ldots, X_m$ be left invariant real vector fields on $\mathbb{G}$ and satisfy Hörmander's rank condition on $\mathbb{R}^N$. Assume that $X_1, \ldots, X_m$ $(m \le N-1)$ are homogeneous of degree one and X_0 is homogeneous of degree two with respect to the family of dilations $\left(\delta_\lambda\right)_{\lambda>0}$. Consider the following hypoelliptic operator with drift on $\mathbb{G}$

$$\mathcal{L} = \sum_{i,j=1}^{m} a_{ij} X_i X_j + a_0 X_0,$$

where (a_{ij}) is a $m \times m$ constant matrix satisfying the elliptic condition in $\mathbb{R}^m$ and $a_0 \neq 0$. In this paper, for this class of operators, we obtain the generalized Sobolev–Morrey estimates by establishing boundedness of a large class of sublinear operators T_α, $\alpha \in [0, Q)$ generated by Calderón–Zygmund operators $(\alpha = 0)$ and generated by fractional integral operator $(\alpha > 0)$ on generalized Morrey spaces and proving interpolation results on generalized Sobolev–Morrey spaces on $\mathbb{G}$. The sublinear operators under consideration contain integral operators of harmonic analysis such as Hardy–Littlewood and fractional maximal operators, Calderón–Zygmund operators, fractional integral operators on homogeneous groups, etc.

Keywords Hypoelliptic operators with drift · Homogeneous group · Fractional integral operator · Singular integral operators · Generalized Morrey space · Generalized Sobolev–Morrey estimates

Mathematics Subject Classification Primary 35B65 · 35H10 · 35R03 · 42B20 · 42B35 · 43A15 · 43A80

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1 Introduction and the main results

Let $\mathbb{G}$ be a homogeneous group on $\mathbb{R}^N$ and $X_0, X_1, \dots, X_m$ $(m < N)$ be left invariant real vector fields on $\mathbb{G}$. Assume that $X_1, \dots, X_m$ are homogeneous of degree one and X_0 is homogeneous of degree two satisfying Hörmander's condition

$$rank\, L(X_0, X_1, \dots, X_m)(x) = N, \quad x \in \mathbb{G},$$

where $L(X_0, X_1, \dots, X_m)$ denotes the Lie algebra generated by $X_0, X_1, \dots, X_m$. In this paper we are interested in the following hypoelliptic operator with drift

$$\mathcal{L} = \sum_{i,j=1}^{m} a_{ij} X_i X_j + a_0 X_0, \tag{1.1}$$

where $a_0 \neq 0$, $(a_{ij})_{i,j=1}^m$ is a constant coefficients matrix satisfying that for some $\mu > 0$,

$$\mu^{-1}|\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j \le \mu|\xi|^2, \quad \xi \in \mathbb{R}^m.$$

Since Hörmander's classic work [32] for the operators sum of squares was published, the regularity of hypoelliptic operators structured on Hörmander's vector fields has attracted extensive attention [3,8,9,35]. The relative of properties of weak generalized solutions to elliptic equations constructed by Hörmander's vector fields was studied in [5,6]. Folland [21] proved that any Hörmander type operator like (1.1) has a homogeneous fundamental solution. For the further properties of the fundamental solutions, see Bramanti and Brandolini [7]. The authors of [7,31,34,44] considered a priori estimates for the operator $\mathcal{L}$. The operator $\mathcal{L}$ contains many particular cases. When $X_0 = \sum_{i,j=1}^{n} b_{ij} x_i \partial_{x_j} - \partial_t$, $X_i = \partial_{x_i}$, $i = 1, 2, \dots, m$, $\mathcal{L}$ is a Kolmogorov–Fokker–Planck ultraparabolic operator of the kind

$$\mathcal{L}_1 u = \sum_{i,j=1}^{m} a_{ij}\partial^2_{x_i x_j} u + \sum_{i,j=1}^{n} b_{ij} x_i \partial_{x_j} u - \partial_t u,$$

where $(x,t) \in \mathbb{R}^{n+1}$, $(a_{ij})_{i,j=1}^m$ is a positive definite matrix, $(b_{ij})_{i,j=1}^n$ is a constant coefficients matrix with a suitable upper triangular structure. It is clear that $\mathcal{L}_1$ is a heat operator, when $m = n$, $(b_{ij})_{i,j=1}^n = (0)_{i,j=1}^n$. For more details see [36,37]. The operator $\mathcal{L}_1$ arises in many research fields, for instance, stochastic processes and kinetic models [13,14,16], mathematical finance theory [2,36,45] etc. Since $\mathcal{L}_1$ owns a homogeneous fundamental solution with good properties, many authors still pay attention to it up to now [10,46,48]. In addition, other examples of (1.1) can see in [7,22].

Morrey spaces and their properties play an important role in the study of local behavior of solutions to elliptic partial differential equations, refer to [40,47]. In [1,15] the authors showed the boundedness in Morrey spaces for some important operators in harmonic analysis such as Hardy–Littlewood operators, Calderón–Zygmund singular integral operators and fractional integral operators. Moreover, various Morrey spaces are defined in the process of study. In [24,39,43] the authors introduced and studied the boundedness of the classical operators in generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [25,26,29,50]) and etc.

In this paper motivated by these articles, we will establish the boundedness of sublinear integral operators on generalized Morrey spaces in the framework of homogeneous groups. The sublinear operators under consideration contain integral operators of harmonic analysis such as Hardy–Littlewood and fractional maximal operators, Calderón–Zygmund operators, potential operators on homogeneous groups, etc. Homogeneous groups include the Euclidean space, the Heisenberg group, the Carnot group, see [4,12,22,52]. Furthermore, applications to generalized Sobolev–Morrey estimates for hypoelliptic operators with drift on homogeneous groups are given. Also, generalized Morrey estimates for the sublinear operators generated by fractional integral operators on the homogeneous group and an application are obtained. Recall that the local Morrey-type space was introduced and proved the boundedness in this spaces of the fractional integral operator and singular integral operators defined on homogeneous Lie groups by author in [24], see also [27,30].

Let us state the following three main results of the paper.

Theorem 1.1 (Generalized Sobolev–Morrey estimate). *Let $1 < p < \infty$ and $\varphi \in \Omega_p$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi(x,s)\, s^{\frac{Q}{p}}}{t^{\frac{Q}{p}}} \frac{dt}{t} \le C\varphi(x,r), \tag{1.2}$$

where C does not depend on x and r. Let also $u \in S^2_{p,\varphi}(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$. Then there exists a constant $C > 0$ such that

$$\|u\|_{S^2_{p,\varphi}(\mathbb{G})} \le C\Big(\|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}\Big), \tag{1.3}$$

where

$$\|u\|_{S^2_{p,\varphi}(\mathbb{G})} = \|u\|_{M_{p,\varphi}(\mathbb{G})} + \sum_{i=1}^m \|X_i u\|_{M_{p,\varphi}(\mathbb{G})} + \sum_{i,j=1}^m \|X_i X_j u\|_{M_{p,\varphi}(\mathbb{G})} + \|X_0 u\|_{M_{p,\varphi}(\mathbb{G})}.$$

Remark 1.1 Denote by $\mathcal{G}_p$ the set of all decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $r \in (0,\infty) \mapsto r^{\frac{Q}{p}}\varphi(r) \in (0,\infty)$ is almost increasing, here Q is the homogeneous dimension of $\mathbb{G}$. Then for $\varphi \in \mathcal{G}_p$ the condition (1.2) stays the following form

$$\int_r^\infty \varphi(t) \frac{dt}{t} \le C\,\varphi(r), \tag{1.4}$$

where C does not depend on r. For the nontriviality of generalized Morrey spaces $M_{p,\varphi}(\mathbb{G})$ we assumed in Theorem 1.1 and in the sequel that $\varphi \in \Omega_p$, see Lemma 2.2 and Remark 2.3.

Note that the condition (1.2) is weaker than (1.4). Indeed, if (1.4) holds, then

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi(s)\, s^{\frac{Q}{p}}}{t^{\frac{Q}{p}}} \frac{dt}{t} \le \int_r^\infty \varphi(t) \frac{dt}{t}.$$

The following example shows that there exist functions satisfying (1.2) but not (1.4).

Example 1.1 For $\beta \in (0, \frac{Q}{p})$ consider the weight function

$$\varphi(r) = r^{\beta - \frac{Q}{p}} \left|\sin\left(\max\left\{1, \frac{\pi}{r}\right\}\right)\right|.$$

If $r \in (0,\pi)$ then $\operatorname{ess\,inf}_{r<\zeta<\infty} \varphi(\zeta)\zeta^{\frac{Q}{p}} = 0$ while for $r \in (\pi,\infty)$, $\operatorname{ess\,inf}_{r<\zeta<\infty} \varphi(\zeta)\zeta^{\frac{Q}{p}} = r^\beta \sin 1$. Then

$$\int_r^\infty \frac{\operatorname*{ess\,inf}\limits_{s<\zeta<\infty} \varphi(\zeta)\zeta^{\frac{Q}{p}}}{s^{\frac{Q}{p}+1}} ds = \begin{cases} 0, & r \in (0,\pi) \\ r^{\beta-\frac{Q}{p}} \sin 1, & r \in (\pi,\infty) \end{cases} \le C\,\varphi(r).$$

The function φ does not satisfy the condition (1.4).

Corollary 1.1 *Let $1 < p < \infty$ and $\varphi \in \mathcal{G}_p$ satisfy the condition (1.4). Let also $u \in S^2_{p,\varphi}(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$. Then the inequality (1.3) is valid.*

If in Theorem 1.1 take $\varphi(r) = r^{\frac{\lambda-Q}{p}}$ with $0 < \lambda < Q$, then $M_{p,\varphi}(\mathbb{G}) = L_{p,\lambda}(\mathbb{G})$ is the classical Morrey space and we get the following corollary, which were proved in [33].

Corollary 1.2 [33] *Let $1 < p < \infty$, and $0 < k < 1$. Let also $u \in S^2_{p,\lambda}(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{S^2_{p,\lambda}(\mathbb{G})} \le C\Big(\|\mathcal{L}u\|_{L_{p,\lambda}(\mathbb{G})} + \|u\|_{L_{p,\lambda}(\mathbb{G})}\Big),$$

where

$$\|u\|_{S^2_{p,\lambda}(\mathbb{G})} = \|u\|_{L_{p,\lambda}(\mathbb{G})} + \sum_{i=1}^{m} \|X_i u\|_{L_{p,\lambda}(\mathbb{G})} + \sum_{i,j=1}^{m} \|X_i X_j u\|_{L_{p,\lambda}(\mathbb{G})} + \|X_0 u\|_{L_{p,\lambda}(\mathbb{G})}.$$

Theorem 1.2 (Higher order generalized Sobolev–Morrey estimate). *Let $1 < p < \infty$, $\varphi \in \Omega_p$ satisfy the condition (1.2) and k is a positive integer. Let also $u \in S^2_{p,\varphi}(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{S^{2k+2}_{p,\varphi}(\mathbb{G})} \le C\Big(\|\mathcal{L}u\|_{S^{2k}_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}\Big), \tag{1.5}$$

where $\|u\|_{S^{2k}_{p,\varphi}(\mathbb{G})} = \sum_{h=0}^{2k} \|D^h u\|_{M_{p,\varphi}(\mathbb{G})}$,

$$\|D^h u\|_{M_{p,\varphi}(\mathbb{G})} = \sum \|X_{ji} \dots X_{jl} u\|_{M_{p,\varphi}(\mathbb{G})},$$

where $X_{ji} \dots X_{jl}$ is homogeneous of degree h (let us note that X_0 is homogeneous of degree two while the remaining $X_1, \dots, X_m$ are homogeneous of degree one).

Corollary 1.3 *Let $1 < p < \infty$, k is a positive integer and $\varphi \in \mathcal{G}_p$ satisfy the condition (1.4). Let also $u \in S^2_{p,\varphi}(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$. Then the inequality (1.5) is valid.*

Corollary 1.4 [33] *let $1 < p < \infty$, $0 < k < 1$ and k is a positive integer. Let also $u \in S^2_{p,\lambda}(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{S^{2k+2}_{p,\lambda}(\mathbb{G})} \le C\Big(\|\mathcal{L}u\|_{S^{2k}_{p,\lambda}(\mathbb{G})} + \|u\|_{L_{p,\lambda}(\mathbb{G})}\Big).$$

To inspect two theorems, we first prove the boundedness of sublinear operators generated by Calderón–Zygmund operators T_0 in generalized Morrey space on $\mathbb{G}$ by applying the representation formulas of functions. These formulas depend on the fundamental solution of $\mathcal{L}$. Next generalized Sobolev–Morrey interpolations on the first order derivatives and higher order derivatives of vector fields are derived. Then based on these results, we obtain generalized Sobolev–Morrey estimates for $\mathcal{L}$. Instead, we shall apply representation formulas of higher order derivatives [7] to prove interpolations desired.

Theorem 1.3 (Generalized Morrey estimate). *Let $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, and $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{1.6}$$

where C does not depend on x and r. Then there exists a constant $C > 0$ such that for every $\mathcal{L}u \in M_{p,\varphi_1}(\mathbb{G})$, we have

$$\|X_i u\|_{M_{q,\varphi_2}(\mathbb{G})} \leq C\|\mathcal{L}u\|_{M_{p,\varphi_1}(\mathbb{G})}, \quad i = 1, 2, \ldots, m.$$

If in Theorem 1.3 take $\varphi_1(r) = \varphi(r) \in \mathcal{G}_p$, $\varphi_2(r) = r\varphi(r)$, then we get the following new corollary.

Corollary 1.5 *Let $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, and $\varphi \in \mathcal{G}_p$ satisfy the condition*

$$\int_r^\infty \varphi(t)\,dt \leq Cr\,\varphi(r), \tag{1.7}$$

where C does not depend on r. Then there exists a constant $C > 0$ such that for every $\mathcal{L}u \in M_{p,\varphi}(\mathbb{G})$, we have

$$\|X_i u\|_{M_{q,r\varphi(r)}(\mathbb{G})} \leq C\,\|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}, \quad i = 1, 2, \ldots, m.$$

Corollary 1.6 [33] *If $1 < p < \infty$, $1/q = 1/p - 1/Q$, and $0 < \lambda < p/q$, there exists a constant $c > 0$ such that for every $\mathcal{L}u \in L_{p,\lambda}(\mathbb{G})$, we have*

$$\|X_i u\|_{L_{q,\lambda q/p}(\mathbb{G})} \leq C\,\|\mathcal{L}u\|_{L_{p,\lambda}(\mathbb{G})}, \quad i = 1, 2, \ldots, m.$$

The proof uses the extension of generalized Morrey estimates for the sublinear operators generated by fractional integral operators T_α, $0 < \alpha < Q$ in the Euclidean space to the homogeneous group and application to $\mathcal{L}$.

Sobolev–Morrey spaces arose in the study of elliptic differential equations. Campanato considered Sobolev–Morrey spaces in [11]. More is investigated on Sobolev–Morrey spaces [19,20,33,44,48,49]. The embedding relation can be found in [41,42].

It is mentioned that since the second and higher order derivatives of vector fields are determined by Calderón–Zygmund operators rather than the fractional integral operators, we cannot use the method here to generalize estimates in Theorem 1.3 to the generalized Sobolev–Morrey estimates for $\mathcal{L}$.

The plan of the paper is the following. In Sect. 2, we introduce some knowledge of the homogeneous group $\mathbb{G}$, the fundamental solution for $\mathcal{L}$ and the generalized Morrey spaces. Section 3 is devoted to the proof of boundedness for sublinear operators generated by Calderón–Zygmund operators T_0 in generalized Morrey spaces. Generalized Morrey estimates for sublinear operators generated by fractional integral operators T_α, $0 < \alpha < Q$ are given. In Sect. 4 the generalized Sobolev–Morrey interpolation inequalities on $\mathbb{G}$ are shown. The main results are proved in Sect. 5.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

We now recall some basic notions concerning homogeneous Lie groups. We refer to the monograph [4] for a detailed treatment of the subject.

Given a pair of smooth mappings

$$[(x, y) \to x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N; \quad [x \mapsto x^{-1}] : \mathbb{R}^N \mapsto \mathbb{R}^N,$$

the space $\mathbb{R}^N$ with these mappings forms a group, in which the identity is the origin. If there exist $0 < w_1 \leq w_2 \leq \cdots \leq w_N$, such that the dilations

$$\delta_\lambda : (x_1, \ldots, x_N) \mapsto (\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N), \quad \lambda > 0$$

are group automorphisms, then the space $\mathbb{R}^N$ with this structure is called a homogeneous group, denoted by $\mathbb{G}$.

Definition 2.1 A homogeneous norm $\|\cdot\|$ on $\mathbb{G}$ is defined in the following way: if for any $x \in \mathbb{G}$, $x \neq 0$, it holds

$$\|x\| = \rho \Leftrightarrow |\delta_{1/\rho} x| = 1,$$

where $|\cdot|$ denotes the Euclidean norm; also, let $\|0\| = 0$.

It is not difficult to verify that the homogeneous norm satisfies

1. $\|\delta_\lambda x\| = \lambda \|x\|$ for every $x \in \mathbb{G}$, $\lambda > 0$;
2. there exists $c_0 \equiv c(\mathbb{G}) \geq 1$, such that for every $x, y \in \mathbb{G}$,

$$\|x^{-1}\| \leq c_0 \, \|x\| \quad \text{and} \quad \|x \circ y\| \leq c_0 \, (\|x\| + \|y\|). \tag{2.1}$$

In view of the above properties, it is natural to define the quasi distance d:

$$d(x, y) = \|x \circ y^{-1}\|.$$

The ball with respect to d is defined by $B(x, r) \equiv B_r(x) = \{y \in \mathbb{G} : d(x, y) < r\}$. Note that $B(0, r) = \delta_r B(0, 1)$, therefore

$$|B(x, r)| = r^Q |B(0, 1)|, \quad x \in \mathbb{G}, r > 0, \tag{2.2}$$

where

$$Q = w_1 + \cdots + w_N.$$

We will call that Q is the homogeneous dimension of $\mathbb{G}$ and always require $Q > 4$ in the sequel. By (2.2) the doubling condition on $\mathbb{G}$ holds, that is

$$|B(x, 2r)| \leq c|B(x, r)|, \quad x \in \mathbb{G}, \ r > 0,$$

where c is some positive constant, and so $(\mathbb{G}, dx, d)$ is a space of homogeneous type.

Let B be a ball on $\mathbb{G}$ and λB $(\lambda > 0)$ denote the ball with the same center as B whose radius is λ times that of B.

Definition 2.2 Differential operators Y on $\mathbb{G}$ are said to be homogeneous of degree β $(\beta > 0)$, if for every test function φ,

$$Y(\varphi(\delta_\lambda x)) = \lambda^\beta (Y\varphi)(\delta_\lambda x), \quad \lambda > 0, x \in \mathbb{G};$$

a function f is called homogeneous of degree α, if

$$f((\delta_\lambda x)) = \lambda^\alpha f(x), \quad \lambda > 0, x \in \mathbb{G}.$$

Clearly, if Y is a homogeneous differential operator of degree β and f is a homogeneous function of degree α, then Yf is homogeneous of degree $\alpha - \beta$.

Lemma 2.1 (See [7]) *Let $\mathcal{L}$ be a left invariant homogeneous differential operator of degree 2 on $\mathbb{G}$, then there is a unique fundamental solution $\Gamma(\cdot)$ such that for every test function u and every $x \in \mathbb{G}$,*

(a) $\Gamma(\cdot) \in C^\infty(\mathbb{G}\backslash\{0\})$;
(b) $\Gamma(\cdot)$ *is homogeneous of degree* $2-Q$;
(c) $u(x) = (\mathcal{L}u * \Gamma)(x) = \int_{\mathbb{G}} \Gamma(x \circ y^{-1})\mathcal{L}u(y)dy$;
(d) $X_i u(x) = \int_{\mathbb{G}} X_i\Gamma(x \circ y^{-1})\mathcal{L}u(y)dy$.
Moreover, for $i, j = 1, \ldots, m$, *there exist constants* $c_{i,j}$ *such that*

$$X_i X_j u(x) = V.P. \int_{\mathbb{G}} X_i X_j \Gamma(x \circ y^{-1})\mathcal{L}u(y)dy + c_{ij}\mathcal{L}u(x).$$

Remark 2.1 If we set $\Gamma_i = X_i\Gamma$, $\Gamma_{i,j} = X_i X_j \Gamma$, then it is obvious that Γ_i is homogeneous of degree $1-Q$ and Γ_{ij} is homogeneous of degree $-Q$.

Several important integral operators are needed:

Definition 2.3 For any $f \in L_1^{\text{loc}}(\mathbb{G})$, the Hardy–Littlewood maximal operator on $\mathbb{G}$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \quad \text{a.e.} \quad x \in \mathbb{G}.$$

Definition 2.4 For any $f \in L_1^{\text{loc}}(\mathbb{G})$, we say that T is a Calderón–Zygmund operator on $\mathbb{G}$ if

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{G}: \|x \circ y^{-1}\| > \varepsilon\}} K(x \circ y^{-1}) f(y)dy = V.P. \int_G K(x \circ y^{-1}) f(y)dy,$$

where K satisfies

$$|K(x)| \le \frac{c}{\|x\|^Q}; \quad |\nabla K(x)| \le \frac{c}{\|x\|^{Q+1}}, \quad x \ne 0.$$

Definition 2.5 For any $f \in L_1^{\text{loc}}(\mathbb{G})$, the fractional maximal operator M_α and the fractional integral operator I_α on $\mathbb{G}$ are defined by

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{Q}} \int_{B(x,r)} |f(y)|dy, \quad 0 \le \alpha < Q,$$
$$I_\alpha f(x) = \int_{\mathbb{G}} \frac{f(y)}{\|x \circ y^{-1}\|^{Q-\alpha}} dy, \quad 0 < \alpha < Q,$$

respectively.

If $\alpha = 0$, then $M = M_0$ is the Hardy–Littlewood maximal operator.

Suppose that T_α, $\alpha \in [0, Q)$ represents a linear or a sublinear operator, which satisfies, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$, the inequality

$$|T_\alpha f(x)| \le c_1 \int_{\mathbb{G}} \frac{|f(y)|}{\|x \circ y^{-1}\|^{Q-\alpha}} dy, \tag{2.3}$$

where c_1 is independent of f and x.

We point out that the condition (2.3) with $\alpha = 0$ was first introduced by Soria and Weiss in [51] in the case $\mathbb{G} = \mathbb{R}^n$. Condition (2.3) is satisfied by many interesting operators in harmonic analysis, such as the Calderón–Zygmund operator, Carleson's maximal operator, Hardy–Littlewood maximal operators, fractional maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Riesz potentials, Ricci–Stein's oscillatory singular integrals, Bochner–Riesz means and so on (see [17,38,51] for details).

Note that, the maximal operator M, and the Calderón–Zygmund operator T satisfy the condition (2.3) with $\alpha = 0$, and the fractional maximal operator M_α, and the fractional integral operator I_α satisfy the condition (2.3) with $0 < \alpha < Q$.

Let $0 < \alpha < Q$, $1 \leq p < \frac{Q}{\alpha}$ and $f \in L_p(\mathbb{G})$. Then the integral $I_\alpha f(x)$ converges absolutely for almost every $x \in \mathbb{G}$, see [25, Theorem 3.2.1]. The Hardy–Littlewood–Sobolev result states that (see [22,24], [25, Theorem 3.2.1]) the operator I_α is bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ if and only if $1 < p < q < \infty$ and $\alpha = Q/p - Q/q$. Also I_α is bounded from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$ if and only if $1 < q < \infty$ and $\alpha = Q - Q/q$.

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(\mathbb{G})$ play an important role, see [23]. They were introduced by C. Morrey in 1938 [40]. The Morrey space in a Carnot group is defined as follows: for $1 \leq p \leq \infty$, $0 \leq \lambda \leq Q$, a function $f \in L_{p,\lambda}(\mathbb{G})$ if $f \in L_p^{\mathrm{loc}}(\mathbb{G})$ and

$$\|f\|_{L_{p,\lambda}} := \sup_{x\in\mathbb{G},\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

(If $\lambda = 0$, then $L_{p,0}(\mathbb{G}) = L_p(\mathbb{G})$; if $\lambda = Q$, then $L_{p,Q}(\mathbb{G}) = L_\infty(\mathbb{G})$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda}(\mathbb{G}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{G}$.)

We also denote by $WL_{p,\lambda}(\mathbb{G})$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{G})$ for which

$$\|f\|_{WL_{p,\lambda}(\mathbb{G})} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{G})} = \sup_{x\in\mathbb{G},\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} = \sup_{t>0} t\, |\{y \in B(x,r) : |f(y)| > t\}|^{1/p}.$$

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.6 Let $1 \leq p < \infty$ and $\varphi(r)$ be a positive measurable function on $(0,\infty)$. The generalized Morrey space $M_{p,\varphi}(\mathbb{G})$ is defined of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{G})$ by the finite norm

$$\|f\|_{M_{p,\varphi}(\mathbb{G})} = \sup_{x\in\mathbb{G}, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(r)} \|f\|_{L_p(B(x,r))}.$$

Also the weak generalized Morrey space $WM_{p,\varphi}(\mathbb{G})$ is defined of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{G})$ by the finite norm

$$\|f\|_{WM_{p,\varphi}(\mathbb{G})} = \sup_{x\in\mathbb{G}, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(r)} \|f\|_{WL_p(B(x,r))}.$$

Remark 2.2 (1) If $\varphi(r) = r^{\frac{\lambda-Q}{p}}$ with $0 < \lambda < Q$, then $M_{p,\varphi}(\mathbb{G}) = L_{p,\lambda}(\mathbb{G})$ is the classical Morrey space and $WM_{p,\varphi}(\mathbb{G}) = WL_{p,\lambda}(\mathbb{G})$ is the weak Morrey space.

(2) If $\varphi(r) \equiv r^{-\frac{Q}{p}}$, then $M_{p,\varphi}(\mathbb{G}) = L_p(\mathbb{G})$ is the Lebesgue space and $WM_{p,\varphi}(\mathbb{G}) = WL_p(\mathbb{G})$ is the weak Lebesgue space.

Lemma 2.2 [18] *Let $\varphi(r)$ be a positive measurable function on* $(0,\infty)$.

(i) *If*

$$\sup_{t<r<\infty} \frac{r^{-\frac{Q}{p}}}{\varphi(r)} = \infty \quad \textit{for some } t > 0 \textit{ and for all } x \in \mathbb{G},$$

then $M_{p,\varphi}(\mathbb{G}) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau} \varphi(r)^{-1} = \infty \quad \textit{for some } \tau > 0 \textit{ and for all } x \in \mathbb{G},$$

then $M_{p,\varphi}(\mathbb{G}) = \Theta$.

Remark 2.3 [18] We denote by Ω_p the sets of all positive measurable functions φ on $\mathbb{G} \times (0, \infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{G}} \left\| \frac{r^{-\frac{Q}{p}}}{\varphi(r)} \right\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{x\in\mathbb{G}} \left\| \varphi(r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.2, we always assume that $\varphi \in \Omega_p$.

We use the following simplified notation later:

$$\|Du\|_{M_{p,\varphi}(\mathbb{G})} = \sum_{i=1}^{m} \|X_i u\|_{M_{p,\varphi}(\mathbb{G})},$$

$$\|D^2 u\|_{M_{p,\varphi}(\mathbb{G})} = \sum_{i=1}^{m} \|X_i X_j u\|_{M_{p,\varphi}(\mathbb{G})} + \|X_0 u\|_{M_{p,\varphi}(\mathbb{G})},$$

and generally,

$$\|D^k u\|_{M_{p,\varphi}(\mathbb{G})} = \sum \|X_{ji} \dots X_{jl} u\|_{M_{p,\varphi}(\mathbb{G})},$$

where $X_{ji} \dots X_{jl}$ is homogeneous of degree k (let us note that X_0 is homogeneous of degree two while the remaining $X_1, \dots, X_m$ are homogeneous of degree one).

Definition 2.7 For $p \in [1, \infty)$, a nonnegative integer k, the generalized Sobolev–Morrey space $S^k_{p,\varphi}(\mathbb{G})$ consists of all $M_{p,\varphi}(\mathbb{G})$ functions such that

$$\|u\|_{S^k_{p,\varphi}(\mathbb{G})} = \sum_{h=0}^{k} \|D^h u\|_{M_{p,\varphi}(\mathbb{G})}$$

is finite.

The space $S^k_{p,\varphi}(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$ consists of all functions $u \in S^k_p(\mathbb{G}) \cap S^{1,0}_p(\mathbb{G})$ with $D^h u \in M_{p,\varphi}(\mathbb{G})$, and is endowed by the same norm. Recall that $S^{1,0}_p(\mathbb{G})$ is the closure of $C^\infty_0(\mathbb{G})$ with respect to the norm in $S^1_p(\mathbb{G})$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\, w(s)\, ds, \quad 0 < t < \infty,$$

where w is a weight. The following theorem was proved in [28].

Theorem 2.1 [28] *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\,ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3 Sublinear operators on the spaces $M_{p,\varphi}(\mathbb{G})$

The following is true for the homogeneous group space [4,22]. Let us note that the homogeneous group is a special case of homogeneous spaces, so we can state

Lemma 3.1 [4,22] *Let $1 \le p < \infty$. Then the maximal operator M and Calderón–Zygmund operator T are bounded on $L_p(\mathbb{G})$ for $p>1$ and from $L_1(\mathbb{G})$ to $WL_1(\mathbb{G})$.*

Lemma 3.2 [4,22] *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then the fractional integral operator I_α is bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ for $p>1$ and from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$.*

The following theorem is valid.

Theorem 3.1 *Let $1 \le p < \infty$ and T_0 be a sublinear operator satisfying condition (2.3) with $\alpha = 0$ bounded on $L_p(\mathbb{G})$ for $p>1$, and bounded from $L_1(\mathbb{G})$ to $WL_1(\mathbb{G})$. Then, for $p>1$ the inequality*

$$\|T_0 f\|_{L_p(B)} \le C\, r^{\frac{Q}{p}} \int_{2c_0 r}^\infty \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{Q}{p}-1}\, dt$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{G})$, where $c_0 \ge 1$ is the constant from the triangle inequality (2.1) and C does not depend on f, x_0 and $r>0$.

Moreover, for $p=1$ the inequality

$$\|T_0 f\|_{WL_1(B)} \le C\, r^{Q} \int_{2c_0 r}^\infty \|f\|_{L_1(B(x_0,t))}\, t^{-Q-1}\, dt \tag{3.1}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L_1^{\rm loc}(\mathbb{G})$, where C does not depend on f, x_0 and $r>0$.

Proof Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{G}$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2c_0B = B(x_0, 2c_0 r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2c_0B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2c_0B)}}(y), \quad r>0,$$

and have

$$\|T_0 f\|_{L_p(B)} \le \|T_0 f_1\|_{L_p(B)} + \|T_0 f_2\|_{L_p(B)}.$$

Since $f_1 \in L_p(\mathbb{G})$, $T_0 f_1 \in L_p(\mathbb{G})$ and from the boundedness of T_0 in $L_p(\mathbb{G})$ (see Lemma 3.1) it follows that:

$$\|T_0 f_1\|_{L_p(B)} \le \|T_0 f_1\|_{L_p(\mathbb{G})} \le C\|f_1\|_{L_p(\mathbb{G})} = C\|f\|_{L_p(2c_0B)},$$

where constant $C>0$ is independent of f.

It's clear that $x \in B$, $y \in {}^{\complement}(2c_0B)$ implies $\frac{1}{2c_0}\|x_0 \circ y^{-1}\| \le \|x \circ y^{-1}\| \le \frac{3c_0}{2}\|x_0 \circ y^{-1}\|$. We get

$$|T_0 f_2(x)| \lesssim \int_{\complement_{(2c_0B)}} \frac{|f(y)|}{\|x_0 \circ y^{-1}\|^Q} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2c_0B)}} \frac{|f(y)|}{\|x_0 \circ y^{-1}\|^Q} dy &\approx \int_{\complement_{(2c_0B)}} |f(y)| \int_{\|x_0\circ y^{-1}\|}^{\infty} \frac{dt}{t^{Q+1}} dy \\ &\approx \int_{2c_0r}^{\infty} \int_{2c_0r \le \|x_0\circ y^{-1}\| < t} |f(y)| dy \frac{dt}{t^{Q+1}} \\ &\lesssim \int_{2c_0r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{Q+1}}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}|T_0 f_2(x)| &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_1(B(x_0,t))}\, t^{-Q-1} dt \\ &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, \|1\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{Q+1}} \\ &\le \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{Q}{p}-1}\, dt. \end{aligned} \tag{3.2}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|T_0 f_2\|_{L_p(B)} \lesssim r^{\frac{Q}{p}} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{Q}{p}-1}\, dt \tag{3.3}$$

is valid. Thus

$$\|T_0 f\|_{L_p(B)} \lesssim \|f\|_{L_p(2c_0B)} + r^{\frac{Q}{p}} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{Q}{p}-1}\, dt.$$

On the other hand,

$$\begin{aligned}\|f\|_{L_p(2c_0B)} &\approx r^{\frac{Q}{p}} \|f\|_{L_p(2c_0B)} \int_{2c_0r}^{\infty} t^{-\frac{Q}{p}-1}\, dt \\ &\le r^{\frac{Q}{p}} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{Q}{p}-1}\, dt.\end{aligned} \tag{3.4}$$

Thus

$$\|T_0 f\|_{L_p(B)} \lesssim r^{\frac{Q}{p}} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{Q}{p}-1}\, dt.$$

Let $p = 1$. From the weak $(1,1)$ boundedness of T_0 (see Lemma 3.1) and (3.4) it follows that:

$$\begin{aligned}\|T_0 f_1\|_{WL_1(B)} \le \|T_0 f_1\|_{WL_1(\mathbb{G})} \lesssim \|f_1\|_{L_1(\mathbb{G})} &= \|f\|_{L_1(2c_0B)} \\ &\approx r^Q \|f\|_{L_1(2c_0B)} \int_{2c_0r}^{\infty} t^{-Q-1} dt\end{aligned}$$

$$\leq r^Q \int_{2c_0 r}^{\infty} \|f\|_{L_1(B(x_0,t))}\, t^{-Q-1} dt. \tag{3.5}$$

Then by (3.3) and (3.5) we get the inequality (3.1). □

Theorem 3.2 *Let $1 \leq p < \infty$ and $\varphi_1, \varphi_2 \in \Omega_p$ satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{Q}{p}}}{t^{\frac{Q}{p}}} \frac{dt}{t} \leq C\varphi_2(x,r), \tag{3.6}$$

where C does not depend on x and r. Let T_0 be a sublinear operator satisfying the condition (2.3) with $\alpha = 0$ bounded on $L_p(\mathbb{G})$ for $p > 1$, and bounded from $L_1(\mathbb{G})$ to $WL_1(\mathbb{G})$. Then the operator T_0 is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{p,\varphi_2}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{1,\varphi_2}(\mathbb{G})$.

Proof By condition (3.6) and Theorems 2.1, 3.1 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1} r^{-\frac{Q}{p}}$, $g(r) = \|f\|_{L_p(B(x,r))}$ and $w(r) = r^{-\frac{Q}{p}-1}$ we have for $p > 1$

$$\|T_0 f\|_{M_{p,\varphi_2}(\mathbb{G})} \lesssim \sup_{x\in\mathbb{G},\, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x,t))}\, t^{-\frac{Q}{p}-1}\, dt$$
$$= \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, r^{-\frac{Q}{p}}\, \|f\|_{L_p(B(x,r))} = \|f\|_{M_{p,\varphi_1}(\mathbb{G})}$$

and for $p = 1$

$$\|T_0 f\|_{WM_{1,\varphi_2}(\mathbb{G})} \lesssim \sup_{x\in\mathbb{G},\, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_1(B(x,t))}\, t^{-Q-1}\, dt$$
$$= \sup_{x\in\mathbb{G}, r>0} \varphi_1(x,r)^{-1}\, r^{-Q}\, \|f\|_{L_1(B(x,r))} = \|f\|_{M_{1,\varphi_1}(\mathbb{G})}.$$

□

Corollary 3.1 *Let $1 \leq p < \infty$ and $\varphi_1, \varphi_2 \in \Omega_p$ satisfy the condition (3.6). Then the maximal operator M and Calderón–Zygmund operator T are bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{p,\varphi_2}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{1,\varphi_2}(\mathbb{G})$.*

Corollary 3.2 *Let $1 \leq p < \infty$ and $\varphi \in \mathcal{G}_p$ satisfy the condition (1.4). Let T_0 be a sublinear operator satisfying condition (2.3) with $\alpha = 0$ bounded on $L_p(\mathbb{G})$ for $p > 1$, and bounded from $L_1(\mathbb{G})$ to $WL_1(\mathbb{G})$. Then the operator T_0 is bounded on $M_{p,\varphi}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi}(\mathbb{G})$ to $WM_{1,\varphi}(\mathbb{G})$.*

Corollary 3.3 *Let $1 \leq p < \infty$ and $\varphi \in \mathcal{G}_p$ satisfy the condition (1.4). Then the operators M, T are bounded on $M_{p,\varphi}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi}(\mathbb{G})$ to $WM_{1,\varphi}(\mathbb{G})$.*

Note that for $\varphi_1(x,r) = \varphi_2(x,r) \equiv |B(x,r)|^{\frac{\lambda-1}{p}}$, from Theorem 3.2 we get the following new result.

Corollary 3.4 *Let $1 \leq p < \infty$ and $0 < \lambda < 1$. Let T_0 be a sublinear operator satisfying condition (2.3) with $\alpha = 0$ bounded on $L_p(\mathbb{G})$ for $p > 1$, and bounded from $L_1(\mathbb{G})$ to $WL_1(\mathbb{G})$. Then the operator T_0 is bounded on $L_{p,\lambda}(\mathbb{G})$ for $p > 1$ and from $L_{1,\lambda}(\mathbb{G})$ to $WL_{1,\lambda}(\mathbb{G})$.*

The following corollary for the operators M and T was proved in [33].

Corollary 3.5 [33] *Let $1 \le p < \infty$ and $0 < \lambda < 1$. Then for $p > 1$, the operators M, T are bounded on $L_{p,\lambda}(\mathbb{G})$ and for $p = 1$, the operators M, T are bounded from $L_{1,\lambda}(\mathbb{G})$ to $WL_{1,\lambda}(\mathbb{G})$.*

Next we state one of our main results. First we present some estimates which are the main tools for proving our theorems, on the boundedness of the operators T_α with $\alpha \in (0, Q)$ on the generalized Morrey spaces.

Theorem 3.3 *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Let also T_α be a sublinear operator satisfying condition (2.3), bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ for $p > 1$, and bounded from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$ for $p = 1$.*

Then, for $1 < p < \frac{Q}{\alpha}$ the inequality

$$\|T_\alpha f\|_{L_q(B(x,r))} \le C\, r^{\frac{Q}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt$$

holds for any ball $B(x,r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{G})$, where C does not depend on f, x and $r > 0$.

Moreover, for $p = 1$ the inequality

$$\|T_\alpha f\|_{WL_q(B(x,r))} \le C\, r^{\frac{Q}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_1(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt, \tag{3.7}$$

holds for any ball $B(x,r)$ and for all $f \in L_1^{\mathrm{loc}}(\mathbb{G})$, where C does not depend on f, x and $r > 0$.

Proof Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. For arbitrary $x \in \mathbb{G}$, set $B = B(x,r)$, $2c_0 B \equiv B(x, 2c_0 r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2c_0 B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2c_0 B)}}(y), \quad r > 0,$$

and have

$$\|T_\alpha f\|_{L_q(B)} \le \|T_\alpha f_1\|_{L_q(B)} + \|T_\alpha f_2\|_{L_q(B)}.$$

Since $f_1 \in L_p(\mathbb{G})$, $T_\alpha f_1 \in L_q(\mathbb{G})$ and from the boundedness of T_α from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ (see Lemma 3.2) it follows that:

$$\|T_\alpha f_1\|_{L_q(B)} \le \|T_\alpha f_1\|_{L_q(\mathbb{G})} \le C\|f_1\|_{L_p(\mathbb{G})} = C\|f\|_{L_p(2c_0 B)},$$

where constant $C > 0$ is independent of f.

It is clear that $z \in B$, $y \in {}^{\complement}(2c_0 B)$ implies $\frac{1}{2c_0}\|x \circ y^{-1}\| \le \|z \circ y^{-1}\| \le \frac{3c_0}{2}\|x \circ y^{-1}\|$. We get

$$|T_\alpha f_2(z)| \lesssim \int_{\complement_{(2c_0 B)}} \frac{|f(y)|}{\|x \circ y^{-1}\|^{Q-\alpha}}\, dy.$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{\complement_{(2c_0 B)}} \frac{|f(y)|}{\|x \circ y^{-1}\|^{Q-\alpha}}\, dy &\approx \int_{\complement_{(2c_0 B)}} |f(y)| \Big(\int_{\|x\circ y^{-1}\|}^{\infty} \frac{dt}{t^{Q+1-\alpha}} \Big)\, dy \\
&\approx \int_{2c_0 r}^{\infty} \Big(\int_{2c_0 r \le \|x\circ y^{-1}\| < t} |f(y)|\, dy \Big) \frac{dt}{t^{Q+1-\alpha}} \\
&\le \int_{2c_0 r}^{\infty} \Big(\int_{B(x,t)} |f(y)|\, dy \Big) \frac{dt}{t^{Q+1-\alpha}}.
\end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned} |T_\alpha f_2(x)| &\lesssim \int_{2c_0 r}^{\infty} \|f\|_{L_1(B(x_0,t))}\, t^{\alpha-Q-1} dt \\ &\lesssim \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t))}\, \|1\|_{L_{p'}(B(x,t))}\, \frac{dt}{t^{Q+1-\alpha}} \\ &\lesssim \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt. \end{aligned} \tag{3.8}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|T_\alpha f_2\|_{L_q(B)} \lesssim r^{\frac{Q}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt \tag{3.9}$$

is valid. Thus

$$\|T_\alpha f\|_{L_q(B)} \lesssim \|f\|_{L_p(2c_0 B)} + r^{\frac{Q}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt.$$

On the other hand,

$$\begin{aligned} \|f\|_{L_p(2c_0 B)} &\approx r^{\frac{Q}{q}} \|f\|_{L_p(2c_0 B)} \int_{2c_0 r}^{\infty} t^{-\frac{Q}{q}-1}\, dt \\ &\lesssim r^{\frac{Q}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt. \end{aligned} \tag{3.10}$$

Thus

$$\|T_\alpha f\|_{L_q(B)} \lesssim r^{\frac{Q}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt.$$

Let $p=1$. From the weak $(1,q)$ boundedness of T_α (see Lemma 3.2) and (3.10) it follows that

$$\begin{aligned} \|T_\alpha f_1\|_{WL_q(B)} \le \|T f_1\|_{WL_q(\mathbb{G})} &\lesssim \|f_1\|_{L_1(\mathbb{G})} = \|f\|_{L_1(2c_0 B)} \\ &\approx r^{\frac{Q}{q}}\, \|f\|_{L_1(2c_0 B)} \int_{2c_0 r}^{\infty} t^{-\frac{Q}{q}-1}\, dt \\ &\lesssim r^{\frac{Q}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_1(B(x,t))}\, t^{-\frac{Q}{q}-1}\, dt. \end{aligned} \tag{3.11}$$

By (3.9) and (3.11) we get the inequality (3.7). □

Theorem 3.4 *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.12}$$

where C does not depend on x and r. Let T_α be a sublinear operator satisfying condition (2.3) with $\alpha \in (0,Q)$, bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ for $p>1$, and bounded from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$ for $p=1$. Then the operator T_α is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$ for $p>1$ and from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$ for $p=1$. Moreover, for $p>1$

$$\|T_\alpha f\|_{M_{q,\varphi_2}(\mathbb{G})} \lesssim \|f\|_{M_{p,\varphi_1}(\mathbb{G})},$$

and for $p=1$

$$\|T_\alpha f\|_{WM_{q,\varphi_2}(\mathbb{G})} \lesssim \|f\|_{M_{1,\varphi_1}(\mathbb{G})}.$$

Proof By condition (3.12) and Theorems 2.1, 3.3 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1} r^{-\frac{Q}{q}}$, $g(r) = \|f\|_{L_p(B(x,r))}$ and $w(r) = r^{-\frac{Q}{q}-1}$ we have for $p > 1$

$$\begin{aligned}\|T_\alpha f\|_{M_{q,\varphi_2}(\mathbb{G})} &\lesssim \sup_{x\in\mathbb{G},\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x,t))} |B(x,t)|^{-\frac{1}{q}} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{G}, r>0} \varphi_1(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))} = \|f\|_{M_{p,\varphi_1}(\mathbb{G})}\end{aligned}$$

and for $p = 1$

$$\begin{aligned}\|T_\alpha f\|_{WM_{q,\varphi_2}(\mathbb{G})} &\lesssim \sup_{x\in\mathbb{G},\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x,t))} |B(x,t)|^{-\frac{1}{q}} \frac{dt}{t} \\ &= \sup_{x\in\mathbb{G}, r>0} \varphi_1(x,r)^{-1} |B(x,r)|^{-1} \|f\|_{L_1(B(x,r))} = \|f\|_{M_{1,\varphi_1}(\mathbb{G})}.\end{aligned}$$

□

Corollary 3.6 [18] *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ satisfy condition (3.12). Then the fractional maximal operator M_α and the fractional integral operator I_α are bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$ for $p = 1$.*

If in Theorem 3.4 take $\varphi_1(r) = \varphi(r) \in \mathcal{G}_p$, $\varphi_2(r) = r^\alpha \varphi(r)$, then we get the following new corollary.

Corollary 3.7 *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and $\varphi \in \mathcal{G}_p$ satisfy the condition*

$$\int_r^\infty t^{\alpha-1} \varphi(t)\, dt \le C r^\alpha \varphi(r), \tag{3.13}$$

where C does not depend on r. Let T_α be a sublinear operator satisfying condition (2.3) with $\alpha \in (0, Q)$, bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ for $p > 1$, and bounded from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$ for $p = 1$. Then the operator T_α is bounded from $M_{p,\varphi}(\mathbb{G})$ to $M_{q,r^\alpha\varphi(r)}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,r^\alpha\varphi(r)}(\mathbb{G})$ for $p = 1$.

Corollary 3.8 *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and $\varphi \in \mathcal{G}_p$ satisfy the condition (3.13). Then the operators M_α and I_α are bounded from $M_{p,\varphi}(\mathbb{G})$ to $M_{q,r^\alpha\varphi(r)}(\mathbb{G})$ for $p > 1$ and from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,r^\alpha\varphi(r)}(\mathbb{G})$ for $p = 1$.*

For $\varphi_1(x,r) = \varphi_2(x,r) \equiv |B(x,r)|^{\frac{\lambda-1}{p}}$, from Theorem 3.4 we get the following new result.

Corollary 3.9 *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and $0 < \lambda < \frac{p}{q}$. Let also T_α be a sublinear operator satisfying condition (2.3) with $\alpha \in (0, Q)$ bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ for $p > 1$, and from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$. Then the operator T_α is bounded from $L_{p,\lambda}(\mathbb{G})$ to $L_{q,\lambda q/p}(\mathbb{G})$ for $p > 1$ and from $L_{1,\lambda}(\mathbb{G})$ to $WL_{q,\lambda q}(\mathbb{G})$ for $p = 1$.*

The following corollary for the operator I_α was proved in [33].

Corollary 3.10 *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and $0 < \lambda < \frac{p}{q}$. Then the operators M_α and I_α are bounded from $L_{p,\lambda}(\mathbb{G})$ to $L_{q,\lambda q/p}(\mathbb{G})$ for $p > 1$ and from $L_{1,\lambda}(\mathbb{G})$ to $WL_{q,\lambda q}(\mathbb{G})$ for $p = 1$.*

4 Interpolation inequalities

We recall a statement in [44]:

Lemma 4.1 *Let $K \in C(\mathbb{G}\backslash\{0\})$ be homogeneous of degree α $(\alpha \in \mathbb{R})$ with respect to the dilations $(\delta_\lambda)_{\lambda>0}$, then there exists a constant $c > 0$ such that*

$$|K(z)| \le c\|z\|^{\alpha},$$

where $c = \sup\limits_{\Sigma_N} |K(z)|$, Σ_N denotes the unit sphere of $\mathbb{G}$.

Observe that if the integral kernel $K(\cdot)$ is homogeneous of degrees $-Q$, then

$$Tf(x) = V.P. \int_{\mathbb{G}} K(x \circ y^{-1}) f(y)dy$$

is obviously a Calderón–Zygmund operator.

Given two balls B_{r_1}, B_{r_2} and a function $\phi \in C_0^\infty(\mathbb{G})$, let us write $B_{r_1} \prec \phi \prec B_{r_2}$ to mean that $0 \le \phi(x) \le 1$, $\phi(x) \equiv 1$ on B_{r_1} and $\operatorname{supp}\phi \subseteq B_{r_2}$. Now we show several interpolation inequalities in generalized Sobolev–Morrey spaces on $\mathbb{G}$.

Lemma 4.2 *Let $1 < p < \infty$ and $\varphi \in \Omega_p$ satisfy the condition (1.2). Then there exists a constant $c > 0$ such that for any $\varepsilon > 0$ and any test function u, the following inequality holds*

$$\|Du\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \varepsilon \, \|D^2u\|_{M_{p,\varphi}(\mathbb{G})} + \frac{1}{\varepsilon} \, \|u\|_{M_{p,\varphi}(\mathbb{G})}.$$

Proof From Lemma 2.1, we have

$$X_i u(x) = \int_{\mathbb{G}} X_i \Gamma(x \circ y^{-1}) \mathcal{L}u(y)dy = \int_{\mathbb{G}} \Gamma_i(x \circ y^{-1}) \mathcal{L}u(y)dy.$$

Let ϕ be a cutoff function with $B_{1/2}(0) \prec \phi \prec B_1(0)$, and split Γ_i as

$$\Gamma_i = \phi\Gamma_i + (1-\phi)\Gamma_i = K_0 + K_\infty,$$

where K_0 and K_∞ are all homogeneous of degrees $1 - Q$, then

$$\begin{aligned} X_i u(x) = & \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} K_0(x \circ y^{-1}) \mathcal{L}u(y)dy \\ & + \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|\ge 1/2\}} K_\infty(x \circ y^{-1}) \mathcal{L}u(y)dy := I + II. \end{aligned} \tag{4.1}$$

In terms of Lemma 4.1 (see [33, pp. 1332]),

$$|I| \le \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} |K_0(x \circ y^{-1})||\mathcal{L}u(y)|dy \le CM\mathcal{L}u(x), \tag{4.2}$$

where C does not depend on x.

Using Corollary 3.3, we infer that

$$\|I\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|M\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^2u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{4.3}$$

In terms of Lemma 4.1 (see, [33, pp. 1332]),

$$|II| = \left| \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|\ge 1/2\}} \widetilde{K}_\infty(x^{-1} \circ y) \mathcal{L}u(y)dy \right| \le CMu(x), \tag{4.4}$$

where C does not depend on x.

It follows by Lemma 3.5 that

$$\|II\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|Mu\|_{M_{p,\varphi}(\mathbb{G})} \leq c\|u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{4.5}$$

Summing (4.3) and (4.5), we obtain

$$\|Du\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^2u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}.$$

A dilation argument leads to

$$\varepsilon\|Du\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \varepsilon^2\|D^2u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})},$$

and the proof of the lemma is concluded. □

In the case of Euclidean space, the interpolation result on higher order derivatives can be deduced by the induction. But in our context the interpolation lemma on higher order derivative of vector fields cannot be deduced simply from that on lower order derivative by the induction. Now we need to use the representation formula of higher order derivative on homogeneous groups to arrive at our aim.

Lemma 4.3 (See [7]) *Let $Q > 4$, for every integer $k \geq 2$ and any couple of left invariant differential monomials P^{2k-1} and P^{2k-2}, homogeneous of degrees $2k-1$ and $2k-2$, respectively, we can determine two kernels $K^{(1)}$, $K^{(2)} \in C^\infty(\mathbb{G}\backslash\{0\})$ which are homogeneous of degrees $1-Q$ and $2-Q$, respectively, such that for any test function u,*

$$P^{2k-1}u(x) = \Big((\mathcal{L}^k u) * K^{(1)}\Big)(x),$$
$$P^{2k-2}u(x) = \Big((\mathcal{L}^k u) * K^{(2)}\Big)(x),$$

where $\mathcal{L}^k = \underbrace{\mathcal{L}\mathcal{L}\ldots\mathcal{L}}_{k\ times}$.

Lemma 4.4 *Let $1 < p < \infty$ and $\varphi \in \Omega_p$ satisfy the condition (1.2). If $k \geq 2$ is an integer, then there exists a constant $c = c(Q,k) > 0$ such that for every $\varepsilon > 0$ and any test function u,*

$$\|D^{2k-1}u\|_{M_{p,\varphi}(\mathbb{G})} \leq \varepsilon\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} + \frac{c}{\varepsilon^{2k-1}}\|u\|_{M_{p,\varphi}(\mathbb{G})}.$$

Proof Suppose that ϕ is a cutoff function with $B_{1/2}(0) \prec \phi \prec B_1(0)$. By Lemma 4.3, we have

$$P^{2k-1}u(x) = \Big((\mathcal{L}^k u) * K^{(1)}\Big)(x).$$

Now let us split $K^{(1)}$ in the following way

$$K^{(1)} = \phi K^{(1)} + (1-\phi)K^{(1)} = K_0^{(1)} + K_\infty^{(1)},$$

where $K_0^{(1)}$ and $K_\infty^{(1)}$ are homogeneous of degrees $1-Q$. Thus

$$\begin{aligned} P^{2k-1}u(x) &= \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} K_0^{(1)}(x\circ y^{-1})\mathcal{L}^k u(y)dy \\ &\quad + \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|\geq 1/2\}} K_\infty^{(1)}(x\circ y^{-1})\mathcal{L}^k u(y)dy \\ &= I_1(x) + I_2(x). \end{aligned} \tag{4.6}$$

It is easy to see with (4.2) that

$$|I_1(x)| \le \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} |K_0^{(1)}(x\circ y^{-1})|\,|\mathcal{L}^k u(y)|dy \lesssim M\mathcal{L}^k u(x).$$

From Corollary 3.3

$$\|I_1(\cdot)\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|M\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{4.7}$$

We have by using Lemma 4.1 and the way in (4.4) (see [33, pp. 1333]),

$$|I_2(x)| \lesssim \sum_{i=0}^{\infty} \int_{\{y\in\mathbb{G}:2^{i-1}\le\|x\circ y^{-1}\|<2^i\}} \frac{|u(y)|dy}{\|x\circ y^{-1}\|^{Q+2k-1}} \lesssim Mu(x).$$

Applying Lemma 3.5,

$$\|I_2(\cdot)\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|Mu\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{4.8}$$

Combining (4.7) and (4.8), we have from (4.6) that

$$\|D^{2k-1}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}.$$

A dilation argument shows

$$\varepsilon^{2k-1}\|D^{2k-1}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \varepsilon^{2k}\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})},$$

and this ends the proof. □

Lemma 4.5 *Let $1 < p < \infty$ and $\varphi \in \Omega_p$ satisfy the condition (1.2). If $k \ge 2$ is an integer, there exists a constant $c = c(Q,k) > 0$ such that for every $\varepsilon > 0$ and any test function u,*

$$\|D^{2k-2}u\|_{M_{p,\varphi}(\mathbb{G})} \le \varepsilon^2\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} + \frac{c}{\varepsilon^{2k-2}}\|u\|_{M_{p,\varphi}(\mathbb{G})}.$$

Proof Let ϕ be a cutoff function with $B_{1/2}(0) \prec \phi \prec B_1(0)$. By Lemma 4.3, we see

$$P^{2k-2}u(x) = \left((\mathcal{L}^k u) * K^{(2)}\right)(x).$$

Split $K^{(2)}$ as

$$K^{(2)} = \phi K^{(1)} + (1-\phi)K^{(2)} = K_0^{(2)} + K_\infty^{(2)},$$

where $K_0^{(2)}$ and $K_\infty^{(2)}$ are homogeneous of degrees $2-Q$, then

$$\begin{aligned} P^{2k-2}u(x) &= \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} K_0^{(2)}(x\circ y^{-1})\mathcal{L}^k u(y)dy \\ &\quad + \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|\ge 1/2\}} K_\infty^{(2)}(x\circ y^{-1})\mathcal{L}^k u(y)dy \\ &= J_1(x) + J_2(x). \end{aligned}$$

Analogously to the proof of Lemma 4.4, it yields

$$\begin{aligned} &\|J_1(\cdot)\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|M\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})}, \\ &\|J_2(\cdot)\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|Mu\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|u\|_{M_{p,\varphi}(\mathbb{G})}. \end{aligned}$$

Therefore

$$\|D^{2k-2}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}.$$

A dilation argument deduces

$$\varepsilon^{2k-2}\|D^{2k-2}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \varepsilon^{2k}\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}.$$

This completes the proof. □

5 Proof of the main theorems

The following result is known, see [7,21].

Lemma 5.1 *For some integer h with $0 < h < Q$, assume that $K_h \in C^\infty(\mathbb{G}\backslash\{0\})$ is homogeneous of degree $h - Q$, f is an integrable function and T_h is defined by*

$$T_h f = f * K_h,$$

P^h is a left invariant homogeneous differential operator of degree h, then

$$P^h T_h f = V.P.(f * P^h K_h) + c f,$$

for some constant c depending on K_h and P^h.

Proof of Theorem 1.1. It holds from Lemma 2.1 that

$$X_i X_j u(x) = V.P.\int_{\mathbb{G}} \Gamma_{ij}(x \circ y^{-1})\mathcal{L}u(y)dy + c_{ij}\,\mathcal{L}u(x),$$

and using Corollary 3.3,

$$\begin{aligned}\|X_i X_j u\|_{M_{p,\varphi}(\mathbb{G})} &\lesssim \left\| \int_{\mathbb{G}} \Gamma_{ij}(\cdot \circ y^{-1})\mathcal{L}u(y)dy \right\|_{M_{p,\varphi}(\mathbb{G})} + \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} \\ &\lesssim \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}. \end{aligned} \tag{5.1}$$

Due to $a_0 X_0 u = \mathcal{L}u - \sum_{i,j=1}^{m} a_{ij} X_i X_j$, it follows that

$$\|X_0 u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{5.2}$$

Then by (5.1) and (5.2)

$$\|D^2 u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{5.3}$$

From Lemma 4.2, we have

$$\begin{aligned}\|Du\|_{M_{p,\varphi}(\mathbb{G})} &\lesssim \varepsilon\,\|D^2u\|_{M_{p,\varphi}(\mathbb{G})} + \frac{1}{\varepsilon}\,\|u\|_{M_{p,\varphi}(\mathbb{G})} \\ &\lesssim \varepsilon\,\|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} + \frac{1}{\varepsilon}\,\|u\|_{M_{p,\varphi}(\mathbb{G})}. \end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4)

$$\begin{aligned}\|u\|_{S^2_{p,\varphi}(\mathbb{G})} &= \|u\|_{M_{p,\varphi}(\mathbb{G})} + \|Du\|_{M_{p,\varphi}(\mathbb{G})} + \|D^2u\|_{M_{p,\varphi}(\mathbb{G})} \\ &\lesssim \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}, \end{aligned}$$

the proof is ended. □

Proof of Theorem 1.2. In order to prove the conclusion, we need to establish the following inequality: if k is a positive integer, there exists a constant $c > 0$ such that for every test function u,

$$\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} \le c\|D^{2k-2}\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{5.5}$$

When $k=1$, by (5.3),

$$\|D^2u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{5.6}$$

When $k \geq 2$, since X_0 cannot be expressed as the composition of two vector field with homogeneity of degree 1, it follows that D^k cannot be obtained from $D(D^{k-1})$ directly. But P^k can be written as X_0P^{2k-2} or X_iP^{2k-1} $(i=1,\dots,m)$, denoted by P^2P^{2k-2} and PP^{2k-1}, respectively. Furthermore, it holds from Lemma 4.3 that

$$P^{2k-1}u(x) = \Big((\mathcal{L}^k u) * K^{(1)}\Big)(x),$$
$$P^{2k-2}u(x) = \Big((\mathcal{L}^k u) * K^{(2)}\Big)(x),$$

where $K^{(1)}$, $K^{(2)}$ are homogeneous of degrees $1-Q$ and $2-Q$, respectively.

In the case of $P^{2k} = P^2P^{2k-2}$, we have from Lemma 5.1,

$$P^{2k}u(x) = V.P.\int_{\mathbb{G}} P^2K^{(2)}(x\circ y^{-1})(\mathcal{L}^k u)(y)dy + c_{ij}(\mathcal{L}^k u)(x),$$

where $P^2K^{(2)}$ is homogeneous of degree $-Q$. Applying Corollary 3.3,

$$\begin{aligned}\|P^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} &\lesssim \left\|\int_{\mathbb{G}} P^2K^{(2)}(\cdot\circ y^{-1})(\mathcal{L}^k u)(y)dy\right\|_{M_{p,\varphi}(\mathbb{G})} + \|\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})}\\ &\lesssim \|\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k-2}\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}.\end{aligned}$$

In the case of $P^{2k} = PP^{2k-1}$, we obtain by using Lemma 5.1,

$$P^{2k}u(x) = V.P.\int_{\mathbb{G}} PK^{(1)}(x\circ y^{-1})(\mathcal{L}^k u)(y)dy + c_{ij}(\mathcal{L}^k u)(x),$$

where $PK^{(1)}$ is homogeneous of degree $-Q$. By virtue of Corollary 3.3,

$$\begin{aligned}\|P^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} &\lesssim \left\|\int_{\mathbb{G}} PK^{(1)}(\cdot\circ y^{-1})(\mathcal{L}^k u)(y)dy\right\|_{M_{p,\varphi}(\mathbb{G})} + \|\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})}\\ &\lesssim \|\mathcal{L}^k u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k-2}\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}.\end{aligned}$$

As a consequence

$$\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k-2}\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})},$$

and (5.5) is proved.

Then

$$\|D^{2k+2}u\|_{M_{p,\varphi}(\mathbb{G})} \lesssim \|D^{2k}\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})}. \tag{5.7}$$

Lemma 4.4 implies that

$$\begin{aligned}\|D^{2k+1}u\|_{M_{p,\varphi}(\mathbb{G})} &\lesssim \varepsilon\,\|D^{2k+2}u\|_{M_{p,\varphi}(\mathbb{G})} + \frac{1}{\varepsilon^{2k+1}}\,\|u\|_{M_{p,\varphi}(\mathbb{G})}\\ &\lesssim \varepsilon\|D^{2k}\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} + \frac{1}{\varepsilon^{2k+1}}\,\|u\|_{M_{p,\varphi}(\mathbb{G})}.\end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8), we have

$$\begin{aligned}\|u\|_{S^{2k+2}_{p,\varphi}(\mathbb{G})} &= \|u\|_{M_{p,\varphi}(\mathbb{G})} + \|D^{2k+1}u\|_{M_{p,\varphi}(\mathbb{G})} + \|D^{2k+2}u\|_{M_{p,\varphi}(\mathbb{G})}\\ &\lesssim \|D^{2k}\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}\\ &= \|\mathcal{L}u\|_{S^{2k}_{p,\varphi}(\mathbb{G})} + \|u\|_{M_{p,\varphi}(\mathbb{G})}.\end{aligned}$$

Theorem 1.2 is proved.

Proof of Theorem 1.3. From Lemma 2.1, we get

$$X_i u(x) = \int_{\mathbb{G}} \Gamma_i(x \circ y^{-1})\, \mathcal{L}u(y)\, dy.$$

Since the function $\Gamma_i(\cdot)$ is homogeneous of degree $1 - Q$, it follows by Lemma 4.1 that

$$|X_i u(x)| \lesssim \int_{\mathbb{G}} \frac{|\mathcal{L}u(y)|}{\|x \circ y^{-1}\|^{Q-1}}\, dy,$$

and we finish the proof by applying Corollary 3.6 with $\alpha = 1$.

$$\begin{aligned}\|X_i u\|_{M_{q,\varphi_2}(\mathbb{G})} &\lesssim \Big\| \int_{\mathbb{G}} \frac{|\mathcal{L}u(y)|}{\|\cdot \circ y^{-1}\|^{Q-1}}\, dy \Big\|_{M_{q,\varphi_2}(\mathbb{G})} \\ &\lesssim \|\mathcal{L}u\|_{M_{p,\varphi_1}(\mathbb{G})}, \quad i = 1, 2, \ldots, m.\end{aligned}$$

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Compliance with ethical standards

Conflict of interest The author declare that there is no conflict of interest.

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