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ON LATTICES OF MONOTONIC MAPS AND GENERALIZED TOPOLOGIES

Ghasem Mirhosseinkhani[1]

[1]Department of Mathematics, Hormozgan University, Bandarabbas, Iran
e-mail: gh.mirhosseini@yahoo.com

Abstract. We study some properties between the lattice of all monotonic maps and the lattice of all generalized topologies on a nonempty set. We present a covariant and contravariant Galois connection between them. We also define the direct sum of two monotonic maps and characterize the direct sum; and give an interesting lower and upper bound for enlarging and restricting maps.

Keywords: generalized topology, monotonic map, lattice.

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1. Introduction and preliminaries

For the last one decade or so, the researchers are concerned with the investigations of generalized topological spaces. A'. Csa'sza'r [3, 4], using monotonic maps from the family of all subsets of a nonempty set X to itself and established some generalized topologies on X. The lattice of all generalized topologies on a nonempty set was studied in [2]. In this paper we discuss some properties between the lattice of all monotonic maps and the lattice of all generalized topologies on a nonempty set.

Let X be a set and denote $\Gamma(X)$ the collection of all monotonic maps from the power set $\rho(X)$ into itself (i.e. $A \subseteq B$ implies $\gamma A \subseteq \gamma B$ for $\gamma \in \Gamma(X)$, where we write γA for $\gamma(A)$). According to [3], a set $A \subseteq X$ is said to be γ-open iff $A \subseteq \gamma A$ and the collection μ_γ of all γ-open sets is a generalized topology (briefly GT) in the sense of [4], i.e. $\varnothing \in \mu_\gamma$ and any union of elements of μ_γ belongs to μ_γ. Similarly, for every $\gamma \in \Gamma(X)$ the collection $\overline{\mu}_\gamma = \{A \mid \gamma(X-A) \subseteq X-A\}$ is a GT on X. Conversely, according to [5], if μ is a GT on X and $A \subset X$, then $i_\mu A = \bigcup\{M \in \mu \mid M \subseteq A\}$ is a mapping $i_\mu : \rho(X) \to \rho(X)$ such that it is monotone, idempotent and restricting, where $\gamma \in \Gamma(X)$ is said to be idempotent iff $\gamma\gamma A = \gamma A$ for $A \subseteq X$, restricting iff $\gamma A \subseteq A$ for $A \subseteq X$. Similarly, if $c_\mu A = \bigcap\{N \mid A \subseteq N, X-N \in \mu\}$, then c_μ is again monotone and idempotent but enlarging, where $\gamma \in \Gamma(X)$ is said to be enlarging iff $A \subseteq \gamma A$ for $A \subseteq X$. Moreover, i_μ and c_μ are conjugate, i.e. $A \subseteq X$ implies $c_\mu A = X - i_\mu(X-A)$. We denote by $\Gamma_{ie}(X)$ and $\Gamma_{ir}(X)$ the collection of all idempotent enlarging maps, and the collection of all idempotent restricting maps in $\Gamma(X)$, respectively.

Let $\mathbf{g}(X)$ be the collection of all generalized topologies on X. According to [2], $(\mathbf{g}(X),\vee,\wedge,1,0)$ is a bounded lattice, neither distributive nor complemented, where its join and meet in $\mathbf{g}(X)$ are defined by $\mu\vee\lambda=\{A\cup B \mid A\in\mu, B\in\lambda\}$ and $\mu\wedge\lambda=\mu\cap\lambda$; and $1=\rho(X), 0=\{\varnothing\}$. Let $\gamma,\delta\in\Gamma(X)$. We say γ is weaker than of δ if $\gamma\le\delta$, i.e. $\gamma A\subseteq\delta A$ for $A\subseteq X$. It is well known that $(\Gamma(X),\vee,\wedge,1,0)$ is a complete lattice, where the join and meet of $\gamma,\delta\in\Gamma(X)$ are defined by $\gamma\vee\delta(A)=\gamma A\cup\delta A$ and $\gamma\wedge\delta(A)=\gamma A\cap\delta A$ for $A\subseteq X$. The least and the greatest elements 0,1 are maps $\gamma_\varnothing$ and γ_X, respectively, where we denote by γ_M the map $\gamma A=M$ for $A\subseteq X$, see [7].

2. Galois connections

Recall that a (covariant) Galois connection between preordered classes S and T is a pair (f,g) of order-preserving maps $f:S\to T$ and $g:T\to S$ with the property that for all $s\in S$ and $t\in T$, $g(t)\le s$ iff $t\le f(s)$. The latter condition is equivalent to: $g\circ f(s)\le s$ for all $s\in S$ and $t\le f\circ g(t)$ for all $t\in T$, see [1, 6]. If (f,g) is a Galois connection, then f preserves meets and g preserves joins. Moreover, $f\circ g$ and $g\circ f$ are idempotent, $g=g\circ f\circ g$ and $f=f\circ g\circ f$. Dually, a contravariant Galois connection is a pair (f,g) of order-reversing maps $f:S\to T$ and $g:T\to S$ between preordered classes provided that for all $s\in S$ and $t\in T$, $s\le g(t)$ iff $t\le f(s)$, or equivalently $s\le g\circ f(s)$ for all $s\in S$ and $t\le f\circ g(t)$ for all $t\in T$. Similarly, if (f,g) is a contravariant Galois connection, then $f\circ g$ and $g\circ f$ are idempotent, $g=g\circ f\circ g$ and $f=f\circ g\circ f$.

In this section we give a covariant and contravariant Galois connection between the lattices $\mathbf{g}(X)$ and $\Gamma(X)$.

Lemma 1. Let $\varphi:\mathbf{g}(X)\to\Gamma(X)$ and $\psi:\Gamma(X)\to\mathbf{g}(X)$ be defined by $\varphi(\mu)=i_\mu$ and $\psi(\gamma)=\mu_\gamma$. Then $\psi\circ\varphi=id_{\mathbf{g}(X)}$, $\varphi\circ\psi\le id_{\Gamma(X)}$ and φ,ψ are order-preserving, where id is the identity map.

Proof. Let $\mu\in\mathbf{g}(X)$. Then we have $\psi\circ\varphi(\mu)=\mu_{i_\mu}=\{A \mid A\subseteq i_\mu(A)\}=\{A \mid A\in\mu\}=\mu$. Thus $\psi\circ\varphi=id_{\mathbf{g}(X)}$. Suppose that $\gamma\in\Gamma(X)$. Then we have $\varphi\circ\psi(\gamma)=i_{\mu_\gamma}$. If $A\subseteq X$, then $i_{\mu_\gamma}(A)=\bigcup\{B \mid B\subseteq A, B\in\mu_\gamma\}=$ $=\bigcup\{B \mid B\subseteq A, B\subseteq\gamma B\}\subseteq\gamma A$. Therefore $\varphi\circ\psi(\gamma)\le\gamma$ and hence $\varphi\circ\psi\le id_{\Gamma(X)}$. Now let $\mu\le\lambda$ in $\mathbf{g}(X)$. If $A\subseteq X$, then $i_\mu(A)=\bigcup\{B\in\mu \mid A\subseteq B\}\subseteq$

$\subseteq \bigcup\{B \in \lambda \mid B \subseteq A\} = i_\lambda(A)$. Therefore $i_\mu \le i_\lambda$ and hence $\varphi(\mu) \le \varphi(\lambda)$ which shows that φ is order-preserving . If $\delta \le \gamma$ in $\Gamma(X)$ and $A \in \mu_\delta$, then $A \subseteq \delta A \subseteq \gamma A$ and hence $A \in \mu_\gamma$. Therefore $\psi(\delta) \le \psi(\gamma)$ which shows that ψ is order-preserving.

Lemma 2. Let $\varphi' : \mathbf{g}(X) \to \Gamma(X)$ and $\psi' : \Gamma(X) \to \mathbf{g}(X)$ be defined by $\varphi'(\mu) = c_\mu$ and $\psi'(\gamma) = \overline{\mu}_\gamma$. Then $\psi' \circ \varphi' = id_{\mathbf{g}(X)}$, $\varphi' \circ \psi' \ge id_{\Gamma(X)}$ and φ', ψ' are order-reversing.

Proof. Let $\mu \in \mathbf{g}(X)$. Then we have

$$\psi' \circ \varphi'(\mu) = \overline{\mu}_{c_\mu} = \{A \mid c_\mu(X - A) = X - A\} = \{A \mid A = i_\mu(A)\} = \mu .$$

Thus $\psi' \circ \varphi' = id_{\mathbf{g}(X)}$. Suppose that $\gamma \in \Gamma(X)$. Then we have $\varphi' \circ \psi'(\gamma) = c_{\overline{\mu}_\gamma}$. If $A \subseteq X$, then $c_{\overline{\mu}_\gamma}(A) = \bigcap\{B \mid A \subseteq B, X - B \in \overline{\mu}_\gamma\} = \bigcap\{B \mid A \subseteq B, \gamma B \subseteq B\} \supseteq \gamma A$. Therefore $\varphi' \circ \psi'(\gamma) \ge \gamma$ and hence $\varphi' \circ \psi' \ge id_{\Gamma(X)}$. Now let $\mu \le \lambda$ in $\mathbf{g}(X)$. If $A \subseteq X$, then we have

$$c_\mu(A) = \bigcap\{B \mid A \subseteq B, X - B \in \mu\} \supseteq \bigcap\{B \mid A \subseteq B, X - B \in \lambda\} = c_\lambda(A) .$$

Therefore $c_\mu \ge c_\lambda$ and hence $\varphi'(\mu) \ge \varphi'(\lambda)$ which shows that φ' is order-reversing. If $\delta \le \gamma$ in $\Gamma(X)$ and $A \in \overline{\mu}_\gamma$, then $\delta(X - A) \subseteq \gamma(X - A) \subseteq X - A$ and hence $A \in \overline{\mu}_\delta$. Therefore $\psi'(\delta) \ge \psi'(\gamma)$ which shows that ψ' is order-reversing.

Remark 1. Notice that since $\psi \circ \varphi = id_{\mathbf{g}(X)}$ and $\psi' \circ \varphi' = id_{\mathbf{g}(X)}$, so φ and φ' are injective, ψ and ψ' are surjective. Also $Im(\varphi) = \Gamma_{ir}(X)$ and $Im(\varphi') = \Gamma_{ie}(X)$. Therefore $\varphi : \mathbf{g}(X) \to \Gamma_{ir}(X)$ is an order-preserving isomorphism and $\varphi' : \mathbf{g}(X) \to \Gamma_{ie}(X)$ is an order-reversing isomorphism, and hence the lattices $\mathbf{g}(X)$, $\Gamma_{ir}(X)$ and $\Gamma_{ie}(X)^{op}$ are isomorphic, where $\Gamma_{ie}(X)^{op}$ is the dual lattice of $\Gamma_{ie}(X)$.

Now by the previous Lemmas we have the following Theorem.

Theorem 1. (1) The pair (ψ, φ) defined in Lemma 1, is a Galois connection between the lattices $\mathbf{g}(X)$ and $\Gamma(X)$; (2) The pair (ψ', φ') defined in Lemma 2, is a contravariant Galois connection between the lattices $\mathbf{g}(X)$ and $\Gamma(X)$.

By the properties of Galois connections we have the following Theorem.

Theorem 2. (1) ψ preserves meets and φ preserves joins; (2) $\varphi \circ \psi$ and $\varphi' \circ \psi'$ are order-preserving and idempotent maps on $\Gamma(X)$.

Corollary 1. Let $\{\gamma_k \mid k \in I\}$ be a family of monotonic maps on X and $\{\mu_k \mid k \in J\}$ be a family of GTs on X . Then

$$\mu_{(\bigcap_{k\in I}\gamma_k)} = \bigcap_{k\in I} \mu_{\gamma_k},$$

$$\mu_{(\bigcap_{k\in J}\mu_k)} = \bigcap_{k\in J} i_{\mu_k},$$

Proof. By Theorem 1, the result follows.

Recall that a closure operator on a preordered class S is an idempotent and order-preserving map $f : S \to S$ such that $id_S \le f$, and a interior (or kernel) operator on S is an idempotent and order-preserving map $f : S \to S$ such that $id_S \ge f$, see [6]. Therefore since $\varphi \circ \psi \le id_{\Gamma(X)}$ and $\varphi' \circ \psi' \ge id_{\Gamma(X)}$, by Theorem 2, we have the following Corollary:

Corollary 2. $\varphi' \circ \psi'$ is a closure operator and $\varphi \circ \psi$ is an interior operator on $\Gamma(X)$.

3. Lower and upper bounds

Clearly, a monotonic map γ on $\rho(X)$ is enlarging iff $id_X \le \gamma$ and it is restricting iff $\gamma \le id_X$. In this section we give an interesting lower and upper bound for every enlarging and every restricting map in $\Gamma(X)$, which are unique with respect to the idempotent property.

The following Proposition is an immediate consequence of the properties of enlarging and restricting maps.

Proposition 1. Let $\delta, \gamma \in \Gamma(X)$ such that $\delta \le id_X \le \gamma$. then

(1) $\mu_\delta = \{A \mid \delta A = A\}$ and $\overline{\mu}_\delta = \rho(X)$.

(2) $\overline{\mu}_\gamma = \{A \mid \gamma(X - A) = X - A\}$ and $\mu_\gamma = \rho(X)$.

(3) $\mu_\delta = \overline{\mu}_\gamma$ iff $\gamma(X - A) = X - \delta A$ for each $A \in \mu_\delta$. Therefore if γ and δ are conjugate, then $\mu_\delta = \overline{\mu}_\gamma$.

Theorem 3. Let $\gamma \in \Gamma(X)$ such that $id_X \le \gamma$ then

(1) There are idempotent maps $\gamma', \gamma'' \in \Gamma(X)$ such that $id_X \le \gamma' \le \gamma \le \gamma''$ and $\gamma'\gamma A = \gamma A$ for $A \subseteq X$.

(2) If there is $\eta \in \Gamma(X)$ such that $\gamma' \le \eta \le \gamma$ and $\eta\gamma A = \gamma A$ for $A \subseteq X$, then $\eta = \gamma'$.

(3) If there is $\eta \in \Gamma(X)$ such that $\gamma \le \eta \le \gamma''$ and $\eta^2 = \eta$, then $\eta = \gamma''$.

(4) If γ is idempotent, then $\gamma = \gamma' = \gamma''$.

Proof. (1): We define γ' as following:

$$\gamma' A = \bigcap \{\gamma B \mid A \subseteq \gamma B\}, \quad (A \subseteq X).$$

It is clear that γ' is monotone and $id_X \le \gamma' \le \gamma$. Since $A \subseteq \gamma A$, so we have $\gamma'\gamma A = \gamma A$. To show that γ' is idempotent, if $A \subseteq X$, then $A \subseteq \gamma' A$ and hence $\gamma' A \subseteq \gamma'\gamma' A$. Conversely, let $x \in \gamma'\gamma' A$. Then $x \in \gamma B$ for every $B \subseteq X$ such that $\gamma' A \subseteq \gamma B$. Since $A \subseteq \gamma' A$, so $x \in \gamma B$ for every $B \subseteq X$ such that $A \subseteq \gamma B$ which shows that $x \in \gamma' A$. To find γ'', we define an ascending of monotonic maps, by putting $\gamma^1 = \gamma, \gamma^\alpha = \gamma \circ \gamma^{\alpha-1}$ and $\gamma^\beta = \bigvee_{\alpha<\beta} \gamma^\alpha$, for every successor ordinal α and for every limit ordinal β. Thus we obtain a increasing sequence of maps $\gamma^1, \gamma^2, \cdots$, the sequence stabilizes, so we have $\gamma^\sigma = \gamma^{\sigma+1}$ for some ordinal σ. Now we put $\gamma'' = \gamma^\sigma$, clearly, γ'' is idempotent and $\gamma \le \gamma''$.

(2): Let $\eta \in \Gamma(X)$ such that $\gamma' \le \eta \le \gamma$ and $\eta\gamma A = \gamma A$ for $A \subseteq X$. If $A \subseteq \gamma B$, then $\eta A \subseteq \eta\gamma B = \gamma B$. Thus by the definition of γ', $\eta A \subseteq \gamma' A$ and hence $\eta \le \gamma'$ which shows that $\eta = \gamma'$.

(3): Let $\eta \in \Gamma(X)$ such that $\gamma \le \eta \le \gamma''$ and $\eta^2 = \eta$. Then for every ordinal α we have $\gamma^\alpha \le \eta^\alpha = \eta$. Thus $\gamma'' = \gamma^\sigma \le \eta$ and hence $\eta = \gamma''$.

(4): By parts 2 and 3, the result follows.

Theorem 4. Let $\delta \in \Gamma(X)$ such that $\delta \le id_X$. then

(1) There are idempotent maps $\delta', \delta'' \in \Gamma(X)$ such that $\delta'' \le \delta \le \delta' \le id_X$ and $\delta'\delta A = \delta A$ for $A \subseteq X$.

(2) If there is $\eta \in \Gamma(X)$ such that $\delta \le \eta \le \delta'$ and $\eta\delta A = \delta A$ for $A \subseteq X$, then $\eta = \delta'$.

(3) If there is $\eta \in \Gamma(X)$ such that $\delta'' \le \eta \le \delta$ and $\eta^2 = \eta$, then $\eta = \delta''$.

(4) If δ is idempotent, then $\delta = \delta' = \delta''$.

Proof. (1): We define δ' as following:

$$\delta' A = \bigcup \{\delta B \mid \delta B \subseteq A\}, \quad (A \subseteq X).$$

It is clear that δ' is monotone and $\delta \le \delta' \le id_X$. Since $\delta A \subseteq A$, so we have $\delta'\delta A = \delta A$. To show that δ' is idempotent, if $A \subseteq X$, then $\delta' A \subseteq A$ and hence $\delta'\delta' A \subseteq \delta' A$. Conversely, let $x \in \delta' A$. Then $x \in \delta B$ for some $B \subseteq X$ such that $\delta B \subseteq A$. Since $\delta B = \delta'\delta B \subseteq \delta' A$, so $x \in \delta B$ for some $B \subseteq X$ such that $\delta B \subseteq \delta' A$ which shows that $x \in \delta'\delta' A$. To find δ'', we define an ascending of monotonic maps, by putting $\delta^1 = \delta, \delta^\alpha = \delta \circ \delta^{\alpha-1}$ and $\delta^\beta = \vee_{\alpha<\beta} \delta^\alpha$, for every successor ordinal α and for every limit ordinal β. Thus we obtain a decreasing sequence of maps $\delta^1, \delta^2, \cdots$, the sequence stabilizes, so we have $\delta^\sigma = \delta^{\sigma+1}$ for some ordinal σ. Now we put $\delta'' = \delta^\sigma$, clearly δ'' is idempotent and $\delta'' \le \delta$.

The proofs of the parts 2, 3 and 4 are similar to Theorem 3.

4 Complement and direct sum

Recall that $\mu^c \in \mathbf{g}(X)$ is said to be a complement of μ if $\mu \vee \mu^c = \rho(X)$ and $\mu \wedge \mu^c = \{\varnothing\}$. The complement of a GT μ on X is not unique in general and a characterization for the existence of complement was given in [2], that is, μ^c exists iff for every nonempty set $A \in \mu$, there is $x_0 \in A$ such that $\{x_0\} \in \mu$. Similarly, we say that $\gamma^c \in \Gamma(X)$ is a complement of γ if $\gamma \vee \gamma^c = \gamma_X$ and $\gamma \wedge \gamma^c = \gamma_\varnothing$.

The following Theorem gives a characterization for the existence of a complement of a monotonic map γ and shows that the complement is unique.

Theorem 5. Let $\gamma \in \Gamma(X)$. Then γ^c exists iff $\gamma = \gamma_M$ for some $M \subseteq X$. Moreover, $\gamma^c = \gamma_{X-M}$ and γ, γ^c are conjugate.

Proof. Let $\gamma \in \Gamma(X)$ such that γ^c exists. If $A \subseteq X$, then $\gamma A \cup \gamma^c A = X$ and $\gamma A \cap \gamma^c A = \varnothing$, and hence $\gamma^c A = X - \gamma A$. But we have $\gamma\varnothing \subseteq \gamma A$ and $\gamma^c \varnothing \subseteq \gamma^c A$. Therefore $\gamma A = \gamma\varnothing$. Now if put $\gamma\varnothing = M$, then $\gamma A = M$ for each subset A of X and hence $\gamma = \gamma_M$. Also we have $\gamma^c A = X - \gamma A = X - M$, so $\gamma^c = \gamma_{X-M}$. Conversely, Let $\gamma = \gamma_M$ for some $M \subseteq X$. We define $\gamma^c = \gamma_{X-M}$. It is clear that γ^c is the complement of γ and γ, γ^c are conjugate.

Corollary 2. If $\gamma \in \Gamma(X)$ such that γ^c exists, then μ_{γ^c} is a complement of μ_γ and $\overline{\mu}_{\gamma^c}$ is a complement of $\overline{\mu}_\gamma$.

Proof. Since $\gamma = \gamma_M$ and $\gamma^c = \gamma_{X-M}$ for some $M \subseteq X$. Then we have $\mu_\gamma = \overline{\mu}_{\gamma^c} = \rho(M)$ and $\mu_{\gamma^c} = \overline{\mu}_\gamma = \rho(X - M)$. Thus by Theorem 2.6 in [2], the result holds.

Let $\mu, \lambda \in \mathbf{g}(X)$ such that every $A \in \rho(X)$ can be uniquely expressed as a union of a μ-open set and a λ-open set, then $\rho(X)$ is the direct sum of μ and λ and written $\rho(X) = \mu \oplus \lambda$. A characterization for the direct sum of two GTs on X was given in [2]. Similarly, Let $\gamma, \delta \in \Gamma(X)$ be idempotent maps such that $id_X = \gamma \vee \delta$ and $\gamma_\varnothing = \gamma \wedge \delta$. Then we say that id_X is the direct sum of γ and δ and we write $id_X = \gamma \oplus \delta$.

Theorem 6. If $id_X = \gamma \oplus \delta$, then $\rho(X) = \mu_\gamma \oplus \mu_\delta$.

Proof. Let $A \in \mu_\gamma \cap \mu_\delta$. Then we have $A \subseteq \gamma A \cap \delta A = \varnothing$ and hence $\mu_\gamma \cup \mu_\delta = \{\varnothing\}$. Since γ and δ are idempotent, so $\gamma A \in \mu_\gamma$ and $\delta A \in \mu_\delta$ for each subset A of X, and by assumption we have $A = \gamma A \cup \delta A$. Thus $\rho(X) = \mu_\gamma \vee \mu_\delta$. Now let $A = B \cup C$ such that $B \in \mu_\gamma$ and $C \in \mu_\delta$. Then $B \subseteq \gamma B \subseteq \gamma A$ and $C \subseteq \delta C \subseteq \delta A$ which shows that $B \cap C = \varnothing$. But we have $A = \gamma A \cup \delta A = B \cup C$, so $\gamma A = B$ and $\delta A = C$. Therefore every subset A of X can be uniquely expressed as a union of a μ_γ-open set and a μ_δ-open set and hence $\rho(X) = \mu_\gamma \oplus \mu_\delta$.

The following Example shows that the converse of Theorem 6 need not be true.

Example 1. Let $X = \{a,b,c\}$ and we define the monotonic maps γ and δ by $\gamma\varnothing = \gamma\{c\} = \varnothing$, $\gamma\{a\} = \gamma\{a,c\} = \{a\}$, $\gamma\{b\} = \gamma\{b,c\} = \{c\}$, $\gamma\{a,b\} = \gamma X = \{a,c\}$, and

$$\delta\varnothing = \varnothing,\ \delta\{a\} = \delta\{b\} = \delta\{a,b\} = \{b\},\ \delta\{c\} = \{c\},$$
$$\delta\{a,c\} = \delta\{b,c\} = \delta X = \{b,c\}.$$

Then we have $\mu_\gamma = \{\varnothing,\{a\}\} = \rho(\{a\})$ and $\mu_\delta = \{\varnothing,\{b\},\{c\},\{b,c\}\} = \rho(\{b,c\})$ and hence $\rho(X) = \mu_\gamma \oplus \mu_\delta$. But $id_X \neq \gamma \oplus \delta$, because $\gamma \wedge \delta(\{b,c\}) = \{c\}$ and hence $\gamma \wedge \delta \neq \gamma_\varnothing$.

The following Theorem characterizes the direct sum of two monotonic maps.

Theorem 7. Let $\gamma,\delta \in \Gamma(X)$. Then $id_X = \gamma \oplus \delta$ if and only if $\gamma = id_X \wedge \gamma_M$ and $\delta = id_X \wedge \gamma_{X-M}$ for some $M \subseteq X$.

Proof. Suppose $\gamma = id_X \wedge \gamma_M$ and $\delta = id_X \wedge \gamma_{X-M}$ for some $M \subseteq X$. Clearly, γ and δ are idempotent. If $A \subseteq X$, then $A = (A \cap M) \cup (A \cap (X - M)) = \gamma A \cup \delta A$ and $\gamma A \cap \delta A = \varnothing$, and hence $id_X = \gamma \oplus \delta$. Conversely, suppose that $id_X = \gamma \oplus \delta$. Then by Theorem 6, we have $\rho(X) = \mu_\gamma \oplus \mu_\delta$, and hence by Theorem 2.15 in [2], $\mu_\gamma = \rho(M)$ and $\mu_\delta = \rho(X - M)$ for some $M \subseteq X$. Thus $\{A \mid A \subseteq \gamma A\} = \{A \mid A \subseteq M\}$ and $\{A \mid A \subseteq \delta A\} = \{A \mid A \subseteq X - M\}$. Now if $A \subseteq X$, then by assumption we have $\gamma A \subseteq A$ and $\gamma A \in \mu_\gamma$, so $\gamma A \subseteq A \cap M = id_X \wedge \gamma_M(A)$. But $A \cap M \in \rho(M)$, so $A \cap M \subseteq \gamma(A \cap M) \subseteq \gamma A$. Therefore we have $\gamma = id_X \wedge \gamma_M$. Similarly, we have $\delta = id_X \wedge \gamma_{X-M}$.

Theorem 8. Let $\gamma,\delta \in \Gamma_{ir}(X)$. Then $id_X = \gamma \oplus \delta$ if and only if $\rho(X) = \mu_\gamma \oplus \mu_\delta$.

Proof. If $id_X = \gamma \oplus \delta$, then by Theorem 6, we have $\rho(X) = \mu_\gamma \oplus \mu_\delta$. Conversely, suppose that $\rho(X) = \mu_\gamma \oplus \mu_\delta$. Then by Theorem 2.15 in [2], we have $\mu_\gamma = \rho(M)$ and $\mu_\delta = \rho(X - M)$ for some $M \subseteq X$. Since γ is idempotent and restricting if $A \subseteq X$, then we have $\gamma A \subseteq A$ and $\gamma A \in \mu_\gamma$, so $\gamma A \subseteq A \cap M = id_X \wedge \gamma_M(A)$. But $A \cap M \in \rho(M)$, so $A \cap M \subseteq \gamma(A \cap M) \subseteq \gamma A$. Therefore we have $\gamma = id_X \wedge \gamma_M$. Similarly, we have $\delta = id_X \wedge \gamma_{X-M}$ and hence by Theorem 7, the result follows.

References

1. Adamek J. Herrlich H., Strecker G.E. Abstract and Concrete Categories, John Willey and Sons Inc., New York, 1990.
2. Baskaran R., Murugalingam M., Sivaraj D., Lattice of generalized maps, Acta Math. Hungar., vol.133, N.4, 2011, pp.365-371.
3. Csaszar A., Generalized open sets, 75, 1997, pp.65-87.
4. Csaszar A.,. Generalized topology, generalized continuity, Acta Math. Hungar., 96, 2002, pp.351-357.
5. Csaszar A., Generalized open sets in generalized topologies, Acta Math. Hungar., 106, 2005, pp.53-66.
6. Gratzer G., Lattice Theory: foundation, Springer, 2011, Basel AG.
7. Slapal J., On lattices of generalized topologies, Math. Slovaca, vol.38, N.1, 1988, pp.51-59.

Monoton inikas və ümumiləşmiş topoloqiya şəbəkələri haqqinda

Haşem Mirhasenxani

XÜLASƏ

İşdə monoton inikasın şəbəkələri və boş olmayan çoxluqlar üzərində bütün ümumiləşmiş topoloqiyalar arasındakı münasibətləri öyrənilir. Onlar arasında kovariant və kontravariant Kalois əlaqələri təsvir edilir. Həmçinin iki monoton inikasın düz cəmini təyin edirik və bu cəmin xarakteristikasını və iİnikasın genişlənməsi və məhdudluğunun yuxarı və aşağı sərhədləni verilir.

Açar sözlər: ümumiləşmiş topologiya, monoton inikas , şəbəkə.

О решетках монотонных отображений и обобщенных топологий

Гашем Мирхосейнхани

РЕЗЮМЕ

Мы изучаем некоторые отношения между решетками монотонных отображений и всех обобщенных топологии по непустых множеств. Мы представляем ковариантные и контравариантные Калоис связи между ними. Мы также определяем прямую сумму двух монотонных отображений и характеризируем эту сумму, даем интересные нижнюю и верхнюю границу для расширения и ограничения отображений.

Ключевые слова: обобщенная топология, монотоническое отображение, решетка.