Proceedings of IAM, V.1, N.2, 2012, pp.203-217

SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS BY METHOD OF STATISTICAL MODELING

Sh. Formanov[1], A. Rasulov[1], G. Raimova[1]

[1]Institute of Mathematics and Information Technologies, National Academy of Sciences of Uzbekistan, Tashkent, Uzbekistan
e-mail: shakirformanov@yandex.ru

Abstract. In this paper the initial-boundary value problem for a parabolic equation with a polynomial non-linearity relative to the unknown function is considered. The representation of the solution this problem in that work is given in the form of a mathematical expectation. Further, in this paper some results of the above mentioned work would be used to derive probabilistic representation for our problem.

Keywords: initial-boundary value problem, parabolic equation, statistical modeling method.

AMS Subject Classification: 65K10.

1. Introduction

In this paper we will consider the initial-boundary value problem for a parabolic equation with a polynomial non-linearity relative to the unknown function $u(x,t)$

$$\frac{\partial u(x,t)}{\partial t} - a\Delta u(x,t) + cu(x,t) = f(u),$$

where $f(u) = \sum_{n=0}^{\infty} a_n(x,t)u^n(x,t)$. Theorems on the existence of solution for nonlinear equation are given in [3], [7]. The method of statistical modeling for solving the initial-boundary value problem for a linear equation are considered in [1] and also the case, when the right-hand side is a finite series, was given in [2], [9]. The representation of the solution this problem in that work is given in the form of a mathematical expectation, which are determined based on trajectories of branching processes. Further, in this paper some results of the above mentioned work would be used to derive probabilistic representation for our problem.

In the present paper under the assumption of the existence of the solution, an unbiased estimator is built using trajectories of a branching process. We will use a theorem of mean value to write out a special integral equation, that equates the value of function $u(x,t)$ with its integral over a spheroid and balloid with center at the point (x,t). A probabilistic representation of the solution to the problem in the form of mathematical expectation of some random variables is obtained. This

probabilistic representation uses branching process whose trajectories are used in the construction unbiased estimator for the solution. The derived unbiased estimator has finite variance, and is built up from trajectories of branching processes with a finite average number of branches.

2. Description of the problem

Let D be a bounded domain in R^m with boundary ∂D and $\Omega = D \times [0,T]$ is a cylinder in R^{m+1}. The following relations define functions $y_0(x) \in C(\overline{D})$, $y(x,t) \in C(\partial D \times [0,T])$, $f(x,t) \in C(\overline{\Omega})$, $a_n(x,t) \in C(\overline{\Omega})$ $(n = 0,1,2,\ldots)$ and the coefficients is $a > 0$, $c > 0$.

Let's consider the initial-boundary value problem for the following parabolic equation for $(x,t) \in \Omega$:

$$\frac{\partial u(x,t)}{\partial t} - a\Delta u(x,t) + cu(x,t) = \sum_{n=0}^{\infty} a_n(x,t)u^n(x,t), \tag{1}$$

with the initial and boundary conditions:

$$\begin{cases} u(x,t) = y(x,t), & x \in \partial D, \quad t \in [0,T], \\ u(x,0) = y_0(x), & x \in D. \end{cases} \tag{2}$$

Assume functions $a_n(x,t)$, $y_0(x)$, $y(x,t)$ and coefficients a, c are such that there exists a unique continuous solution of this nonlinear problem $u(x,t) \in C(\overline{D} \times [0,T]) \cap C^{2,1}(\overline{D} \times [0,T])$ ([3], p.201, [7], p.556). Relative to the functions $a_n(x,t)$, $(n = 0,1,2,\ldots)$ we will make the following assumptions:

$$\sup_{(x,t)\in\Omega} |a_n(x,t)| \le \overline{a}_n,$$

and

$$\sum_{k=0}^{\infty} \overline{a}_k k < \infty. \tag{3}$$

The purpose of this paper is to derive unbiased Monte Carlo estimators for solving the problem (1)-(2) with finite variance at some arbitrary point $(x,t) \in \Omega$.

3. Integral representation of the solution

The basis for building the derived unbiased estimators will be the formula for the "parabolic mean" used in solving the heat equation. Various equations for the mean of parabolic equations were considered in [5], [6]. With the help of the fundamental solution, and Green's formula, we can transform this differential equations into an integral equations. In doing so we apply the results of lemma 3.1 ([10], p.106). We will obtain a special mean equation which equates the value of

the function $u(x,t)$ with its integral over a "balloid" and its surface with center at the point (x,t).

As is known, the fundamental solution, $Z(x,t;y,\tau)$, of the heat equation $u_t - a\Delta u = 0$ is

$$Z(x,t;y,\tau) = \left(4\pi a(t-\tau)\right)^{-m/2} \exp\left(-\frac{|x-y|^2}{4a(t-\tau)}\right).$$

Let $Z_r(x,t;y,\tau) = Z(x,t;y,\tau) - (4\pi ar)^{-m/2}$. With the help of this fundamental solution, we define the family of domains $Q_r(x,t)$, which depend on the parameter $r>0$ and the point $(x,t)\in R^{m+1}$, as

$$Q_r(x,t) = \left\{(y,\tau): Z(x,t;y,\tau) > (4\pi ar)^{-m/2}, \tau < t\right\}.$$

Let $(x,t)\in\Omega$, $r = r(x,t) = \min\left\{\frac{R^2(x)e}{2am}, t\right\}$, where $R(x)$ is the distance from x to the boundary of the domain ∂D: $R(x) = \inf\left\{|x-x'|, x\in\partial D, x'\in\overline{D}\right\}$. In this case $Q_r(x,t)\in\overline{\Omega}$. Applying the results of lemma 3.1 [10] for equation (1), we obtain the following integral representation:

$$u(x,t) = a\int\limits_{\partial Q_r(x,t)}\int\left(1-\frac{t-\tau}{r}\right)\left(-\frac{\partial Z(x,t;y,\tau)}{\partial n_y}\right)u(y,\tau)ds\,d\tau + \tag{4}$$

$$+\frac{1}{r}\int\limits_{Q_r(x,t)}\int Z_r(x,t;y,\tau)u(y,\tau)dy\,d\tau + F_r(x,t),$$

where

$$F_r(x,t) = \frac{1}{r}\int\limits_{Q_r(x,t)}\int (r-(t-\tau))Z_r(x,t;y,\tau)\,f(u(y,\tau))dy\,d\tau,$$

$$f(u(x,t)) = \sum_{n=0}^{\infty} a_n(x,t)u^n(x,t).$$

To obtain a probabilistic representation, we will consider these expressions in detail.

4. Probabilistic representation of the solution

To derive the probabilistic representation of the solution to our problem we will represent each term in (4) as the mathematical expectation of some random variable. Let's consider each term in the integral representation separately

$$I_1(x,t) = a\int\limits_{\partial Q_r(x,t)} \int \left(1 - \frac{t-\tau}{r}\right)\left(-\frac{\partial Z(x,t;y,\tau)}{\partial n_y}\right) u(y,\tau)\,ds\,d\tau.$$

We will make a transformation similar to transformations in the following way. We will parameterize the surface $\partial Q_r(x,t)$ with the help of the parameter $\rho \in (0,\infty)$ and unit vectors $w \in S_1(0)$ in the following manner. We will first transform to the polar coordinates, $y = x + R(t-\tau)$. In doing this, we make a change of variables $\tau_1 = t - \tau$, then $\rho = \frac{m}{2}\ln\frac{r}{\tau_1}$, and as a result we have

$$I_1 = (1-q_m)\int\limits_0^\infty q_1(\rho)\,d\rho \int\limits_{S_1(0)} q_2(w)\,u(y(\rho,w),\tau(\rho))\,ds = (1-q_m)Eu(y(\xi,\omega),\tau(\xi)),$$

Here

$$q_m = (1+2/m)^{-(1+m/2)},$$

$$q_1(\rho) = \rho^{m/2}\exp(-\rho)\left(1-\exp\left(-\frac{2\rho}{m}\right)\right)\left((1-q_m)\Gamma\left(1+\frac{m}{2}\right)\right)^{-1},$$

$$q_2(w) = \frac{1}{\sigma_m} = \Gamma\left(\frac{m}{2}\right)\left(2\pi^{m/2}\right)^{-1},$$

$y(\rho,\omega)$ and $\tau(\rho)$ are determined by the following formulas

$$y(\rho,\omega) = x + \sqrt{4r\rho a\exp\left(-\frac{2\rho}{m}\right)}\,\omega, \tag{5}$$

$$\tau(\rho) = t - r\exp\left(-\frac{2\rho}{m}\right), \tag{6}$$

and the random variable ξ, is distributed with the probability density $q_1(\rho)$, ω is a random point on the surface, $S_1(0)$, and has density $q_2(w)$, $S_1(0)$ is the unit sphere, ds is the surface element, σ_m is the area of unit sphere, and $\Gamma(.)$ is the Gamma function.

Let's consider following terms

$$I_2(x,t) = \frac{1}{r}\int\limits_{Q_r(x,t)} \int Z_r(x,t;y,\tau)\,u(y,\tau)\,dy\,d\tau,$$

$$I_3(x,t) = \frac{1}{r}\int\limits_{Q_r(x,t)} \int (r-(t-\tau))\,Z_r(x,t;y,\tau)\,f(u(y,\tau))\,dy\,d\tau.$$

We transform to polar of coordinates $y = x + \rho w$, and in doing so make a few changes of variables such as

$$\tau_1 = t - \tau; \quad \rho = \sqrt{2m\tau_1 a \ln\left(\frac{w}{\tau_1}\right)}; \quad \tau_1 = \lambda w;$$

$$w = r\nu^{2/m}; \quad \lambda = \exp\left(-\frac{z}{1+m/2}\right).$$

Thus we obtain

$$I_2(x,t) = q_m E\left(1 - rc_1\left(1 - \nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right)\right)u\left(y_1(\xi_1,\nu,\omega),\tau_1(\xi_1,\nu)\right),$$

$$I_3(x,t) = rq_m \int_0^1 q_3(\nu)\int_0^\infty q_4(z)\int_{S_1(0)} q_2(w)\left(1 - \nu^{2/m}\exp\left(\frac{-2z}{m+2}\right)\right)\times$$

$$\times\, f\left(u\left(y_1(z,\nu,w),\tau_1(z,\nu)\right)\right)ds\,dz\,d\nu =$$

$$= rq_m E\left\{\left(1 - \nu^{2/m}\exp\left(\frac{-2\xi_1}{m+2}\right)\right)f\left(u(y_1(\xi_1,\nu,\omega),\tau_1(\xi_1,\nu))\right)\right\},$$

where $q_3(\nu) = \dfrac{(1-\nu)\nu^{2/m-1}}{B(2,2/m)}$ is the probability density function of the Beta distribution with parameters $(2,2/m)$; $B(.,.)$ - Beta function; $q_4(z) = \dfrac{\exp(-z)z^{m/2-1}}{\Gamma(m/2)}$ is the density of the Gamma distribution with parameter $(m/2)$ and ω is an isotropic vector. Then

$$y_1(\xi_1,\nu,\omega) = x + \left(\frac{4m}{m+2}ra\xi_1\nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right)^{1/2}\omega, \tag{7}$$

$$\tau_1(\xi_1,\nu) = t - r\nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right). \tag{8}$$

Here ξ_1 is a random variable with density $q_4(z)$, and ν is a random variable with density $q_3(\nu)$. Thus, we have:

$$u(x,t) = (1-q_m)Eu\left(y(\xi,\omega),\tau(\xi)\right) + q_m \times$$

$$\times E\left(1 - rc\left(1 - \nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right)\right)u\left(y_1(\xi_1,\nu,\omega),\tau_1(\xi_1,\nu)\right) +$$

$$+ q_m rE\left(1 - \nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right)f\left(u\left(y_1(\xi_1,\nu,\omega),\tau_1(\xi_1,\nu)\right)\right), \tag{9}$$

where $y(\xi,\omega)$, $\tau(\xi)$ are determined by formulas (5)-(6), and $y_1(\xi_1,\nu,\omega)$, $\tau(\xi_1,\nu)$ are determined by formulas (7)-(8).

Let $r = r(x,t) = \min\left\{\frac{eR^2(x)}{2ma}, t, \frac{1}{c}\right\}$, then $\overline{Q_r(x,t)} \in \Omega$ and at a given point (x,t), the functions

$$p_1(x,t;y,\tau) = \frac{\left(1-\frac{t-\tau}{r}\right)}{1-q_m}\left(-\frac{\partial Z(x,t;y,\tau)}{\partial n_y}\right) I_{\partial Q_r(x,t)}(y,\tau),$$

$$p_2(x,t;y,\tau) = \frac{\big(1-(r-(t-\tau))c\big)Z_r(x,t;y,\tau)}{rq_m(1-rq_{1m}c)} I_{Q_r(x,t)}(y,\tau),$$

$$p_3(x,t;y,\tau) = \frac{Z_r(x,t;y,\tau)}{rq_m} I_{Q_r(x,t)}(y,\tau),$$

are the densities of distributions in $\overline{Q}_r(x,t)$. Here $q_{1m} = 1 - \frac{1}{2}\left(\frac{m+2}{m+4}\right)^{1+m/2}$. Let (y_2,τ_2) be a random point of the balloid $Q_r(x,t)$, at the fixed (x,t), where (y_2,τ_2) is distributed with a density $p_2(x,t;y,\tau)$, then in expression (9) the second term can be substituted with

$$\frac{1}{r}\int\limits_{Q_r(x,t)}\int \big(1-(r-(t-\tau))c\big)Z_r(x,t;y,\tau)u(y,\tau)dyd\tau = q_m(1-rcq_{1m})Eu(y_2,\tau_2).$$

Thus, the following probabilistic representation of the solution to the problem is valid.

Theorem 1. For finding the solution of the problem (1)-(2) the following probabilistic representation is valid:

$$u(x,t) = (1-q_m)Eu\big(y(\xi,\omega),\tau(\xi)\big) + q_m(1-rcq_{1m})Eu(y_2,\tau_2) +$$

$$+ q_m rE\left(\left(1-\nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right) f\big(u(y_1(\xi_1,\nu,\omega),\tau_1(\xi_1,\nu))\big)\right). \qquad (10)$$

We use the following notation

$$(y_1,\tau_1) = \big(y(\xi,\omega),\tau(\xi)\big); \quad (y_3,\tau_3) = y_1(\xi_1,\nu,\omega),\tau_1(\xi_1,\nu);$$

$$\alpha_1 = 1-q_m; \quad \alpha_2 = q_m(1-rcq_{1m}); \quad \alpha_3 = rcq_m q_{1m}.$$

Here $\alpha_1+\alpha_2+\alpha_3 = 1$.

After substituting this notations, the expression (10) will take the form

$$u(x,t) = \alpha_1 Eu(y_1,\tau_1) + \alpha_2 Eu(y_2,\tau_2) +$$

$$+ \alpha_3 E\left(\left(1-\nu^{2/m}\exp\left(-\frac{2\xi}{m+2}\right)\right)\frac{f\big(u(y_3,\tau_3)\big)}{cq_{1m}}\right). \qquad (11)$$

So, theorem 1 gives us a representation of the solution to the problem (1)-(2) in the form of a mathematical expectation of some random variables. Thus makes it

possible to finding the solution by using a simulation of these random variables on the trajectories of branching random processes.

When applying the methods of Monte Carlo for the solution of the derived equations, the convergence of the iterative method was assumed. We can show, under what conditions the nonlinear integral operator, $F(u)$, in expression (11), which determined below, is a contraction operator and has a fixed point. (see [10], p.156).

5. Constructing branching random processes

In accordance with representation (11), we will construct a random process in the space Ω. Let

$$r = r(x,t) = \min\left\{\frac{eR^2(x)}{2ma}, t, \frac{1}{c}\right\}. \tag{12}$$

We will determine a random branching process in Ω corresponding to the probabilistic representation in the following way. Let $M = \sum_{n=1}^{\infty} n\bar{a}_n$, and α be a real number so that $0 < \alpha < 1$. (Further conditions on α will be specified).

Suppose we have a particle initially at the point $(x_0,t_0) = (x,t)$. Let $n > 0$ and assume (x_n,t_n) be known. For a one step transition the particle moves with probability α_i to the point $(x_{n+1},t_{n+1}) = (y_i,\tau_i)$, which is distributed with probability density $p_i(y,\tau;x_n,t_n)$. If the particle moves to the point (y_3,τ_3), then here, with probability $\pi_n = \frac{\alpha}{M}\bar{a}_n$ $(n = 1,2,\cdots)$, the particle divides into n new particles. The probability of absorption at (y_3,τ_3) is equal to $\pi_0 = 1 - \sum_{n=1}^{\infty}\pi_n$. A new particles behaves independently from the others (similar to the original).

This process will terminate if all particles absorbed in $\overline{\Omega}$, or if all particles get to $\partial\Omega$. The parameter α allows us to regulate the number of branches: the smaller α the fewer of branches in the process.

The sequence of coordinates of the particle is determined in the following way

1. If the probability density of the point (x_{n+1},t_{n+1}) is equal to $p_1(x_n,t_n;y,\tau)$ at a fixed (x_n,t_n), then (x_{n+1},t_{n+1}) will be given by

$$\begin{aligned} x_{n+1} &= x_n + 2\left(r(x_n,t_n)\xi_n a\right)^{1/2}\exp\left(-\frac{\xi_n}{m}\right)\omega_n, \\ t_{n+1} &= t_n - r(x_n,t_n)\exp\left(-\frac{2\xi_n}{m}\right), \end{aligned} \tag{13}$$

where $\{\xi_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ are sequences of independent random variables with probability densities $q_1(\rho)$, and independent isotropic vectors respectively. $r(x_n,t_n)$ is determined by formula (12).

2. If the probability density of the point (x_{n+1},t_{n+1}) is equal to $p_2(x_n,t_n;y,\tau)$ at a fixed (x_n,t_n), then

$$x_{n+1} = x_n + 2\omega_n\left(\frac{m}{m+2}r(x_n,t_n)\xi_n'(\nu_n')^{2/m}a\exp\left(-\frac{2\xi_n'}{m+2}\right)\right)^{1/2},$$

$$t_{n+1} = t_n - r(x_n,t_n)(\nu_n')^{2/m}\exp\left(-\frac{2\xi_n'}{m+2}\right), \qquad (14)$$

where $\{\xi_n'\}_{n=0}^{\infty}$, and $\{\nu_n'\}_{n=0}^{\infty}$ are sequences of independent random variables, obtained using the following algorithm (the Neumann acceptance-rejection method):

The details of the rejection algorithm go as follows.

A. Simulate ξ as a Gamma distributed random variable with parameter $(m/2)$, γ is uniformly distributed in the unit interval $(0,1)$, ν a Beta distributed random variable with parameters $(2,2/m)$;

B. If $\gamma > 1 - cr\left(1-\nu^{2/m}\exp\left(-\frac{2\xi}{m+2}\right)\right)$, then we reject and return to A., otherwise $\nu' = \nu$, $\xi' = \xi$.

3. If the density of the point (x_{n+1},t_{n+1}) is equal to $p_3(x_n,t_n;y,\tau)$ at a fixed point (x_n,t_n), then

$$x_{n+1} = x_n + 2\omega_n\left(\frac{m}{m+2}r(x_n,t_n)\xi_n\nu_n^{2/m}a\exp\left(-\frac{2\xi_n}{m+2}\right)\right)^{1/2},$$

$$t_{n+1} = t_n - r(x_n,t_n)\nu_n^{2/m}\exp\left(-\frac{2\xi_n}{m+2}\right), \qquad (15)$$

where $\{\xi_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ are sequences of independent Gamma distributed random variables with parameter $(m/2)$ and Beta distributed random variables with parameters $(2,2/m)$ respectively. $\{\omega_n\}_{n=0}^{\infty}$ are independent isotropic vectors.

Let $T_0 = (x,t,1), T_1, T_2, \ldots, T_i, \ldots$ is a trajectory of a branching process, where

$$T_i = (x_i^1, t_i^1, n_i^1; x_i^2, t_i^2, n_i^2; \ldots; x_i^{l_i}, t_i^{l_i}, n_i^{l_i})$$

is a point distribution at time step i. The case $l_i = 0$ corresponds to the absence of particles in $\overline{\Omega}$. Such a point distribution is called zero and is denoted by θ. A

point distribution T_i may be interpreted in the following way: for $i=1,2,\ldots$ at each point (x_i^j,t_i^j) there are n_i^j number of particles $(j=\overline{1,2,...l_i})$.

Lemma 1. A branching process $\{T_i\}$ with probability one terminated in the domain Ω, or converges to the point distribution

$$T=(x^1,t^1,n^1;x^2,t^2,n^2;\ldots;x^k,t^k,n^k),$$

where $(x^j,t^j)\in\partial\Omega$, $(j=1,2,\ldots,k)$.

Proof. The process can be considered as a branching diffusion process for the particles that are diffusing in the bounded domain $\overline{\Omega}$ with absorbing boundaries. If the average number of particles, which generate one particle for the one step is $K<1$, then the constructed process is absorbed with probability 1, and the condition $K<1$ is a necessary and sufficient condition for the average number of the branches of the process to be finite. Let's show that for the process, which was described above, $K<1$. In accordance with our suggestions, the series $M=\sum_{n=0}^{\infty}n\bar{a}_n<\infty$ converges. We will fix $0<\alpha<1$. Taking into consideration that $\pi_n=\frac{\alpha}{M}\bar{a}_n$, we find that

$$K=\alpha_1\int_{\overline{\Omega}}p_1(x,t;y,\tau)d\mu+\alpha_2\int_{\overline{\Omega}}p_2(x,t;y,\tau)d\mu+\alpha_3\int_{\overline{\Omega}}p_3(x,t;y,\tau)\sum_{n=1}^{\infty}(n\pi_n)d\mu=$$

$$=\alpha_1+\alpha_2+\alpha_3\sum_{n=1}^{\infty}n\pi_n=\alpha_1+\alpha_2+\alpha_3\frac{\alpha}{M}\sum_{n=1}^{\infty}n\bar{a}_n<1.$$

Thus, the current process terminates and the general number of particles, which took part in the process, is finite [4]. If the process does not terminate inside Ω, then starting at some time all the particles go to the boundary, $\partial\Omega$. There exists a $n=n_0$, so that starting at T_n that has the following form, $T=(x_n^1,t_n^1,1;x_n^2,t_n^2,1;\ldots;x_n^k,t_n^k,1)$, where k does not depend on n. Further, all of the k particles independently transfer from the points (x_n^i,t_n^i) to the point (x_{n+1}^i,t_{n+1}^i). The coordinates of these points are determined either by formula (13), or by formula (14). We will show that $(x_n^i,t_n^i)\to(x_\infty^i,t_\infty^i)\in\partial\Omega$ when $n\to\infty$, $i=1,2,\ldots,k$ almost surely. The sequences $\{t_n^i\}_{n=n_0}^{\infty}$ are decreases and $t_n^i\geq 0$, so, it has as $n\to\infty$ a limit $t_\infty^i=\lim_{n\to\infty}t_n^i$ exists. Let $\Im_n$ be a σ- algebra, which was generated with random variables $\xi_{n_0}^i$, $\xi_{n_0+1}^i$, …, $\xi_{n_0+n}^i$; $\left(\xi'_{n_0}\right)^i$, $\left(\xi'_{n_0+1}\right)^i$, …, $\left(\xi'_{n_0+n}\right)^i$; $\left(\nu'_{n_0}\right)^i$, $\left(\nu'_{n_0+1}\right)^i$, …, $\left(\nu'_{n_0+n}\right)^i$ and vectors $\omega_{n_0}^i$, $\omega_{n_0+1}^i$, …, $\omega_{n_0+n}^i$. It is

obvious, that coordinates of the vector process $\{x_n^i\}_{n=n_0}^{\infty}$ form bounded martingales relative to $\{\Im_n\}_{n=1}^{\infty}$. This is why the coordinates, and the process $\{x_n^i\}_{n=n_0}^{\infty}$ itself converge with probability one. If $t_\infty^i = 0$, then it is obvious that $(x_\infty^i, t_\infty^i) \in \partial\Omega$. Let's suppose $t_\infty^i > 0$. As far as the process converges, then

$$E_{(x_{n_0}^i, t_{n_0}^i)} \left| x_{n+1}^i - x_n^i \right| = const \quad E_{(x_{n_0}^i, t_{n_0}^i)} \sqrt{r(x_n^i, t_n^i)} \to 0.$$

Applying Lebesgue's theorems about bounded convergence, we obtain

$$E_{(x_{n_0}^i, t_{n_0}^i)} \left(r\left(x_\infty^i, t_\infty^i\right)\right)^{1/2} = 0.$$

Thus $r(x_\infty^i, t_\infty^i) = 0$ almost surely. Then from the definition for $r(x,t)$ via formula (12), we obtain $R(x_\infty^i) = 0$, i.e. $(x_\infty^i, t_\infty^i) \in \partial\Omega$, and the lemma has been proven.

Further, we will give a recurrence formula for the average number of particles in the n-th generation. Let $I_A(y,\tau)$ be the indicator of set $A \in \Omega$. Let's define $K\left((x,t),A\right) = E_{(x,t)} T_1(A)$ as the average number of particles of the first generation, which are in the set A. Let $dK\left((x,t),(y,\tau)\right)$ means that differential is taken from the second argument. $K_1\left((x,t),A\right) = K\left((x,t),A\right)$ $K_0\left((x,t),A\right) = 1$, if $(x,t) \in A$ and $K_0\left((x,t),A\right) = 0$, if $(x,t) \notin A$. Then for the average number of particles in the $(n+1)$ -st generation, which will be in A, obeys the following recurrence:

$$K_{n+1}\left((x,t),A\right) = \int_\Omega K_n\left((y,\tau),A\right) dK\left((x,t),(y,\tau)\right) =$$

$$= \int_\Omega K\left((y,\tau),A\right) dK_n\left((x,t),(y,\tau)\right), \quad n = 1,2,\ldots.$$

The process $T_0, T_1, T_2, \ldots, T_n, \ldots$ has $K\left((x,t),\overline{\Omega}\right) = K$. Here $K\left((x,t),\overline{\Omega}\right)$ is the average number of particles in the first generation or the average number of particles which arise from one particle, $K_n\left((x,t),\overline{\Omega}\right)$ is the average number of particles in n-th generation. From the recurrence it follows, that if $K\left((x,t),\overline{\Omega}\right) < b$ $(b = const)$, then $K_n(x,D) < b^n$. For the process $T_0, T_1, T_2, \ldots, T_n, \ldots$, will be $K < 1$ and we thus have $K_n\left((x,t),\overline{\Omega}\right) < 1$.

In the next section we will construct an unbiased estimator of the solution of problem (1)-(2) using branching processes trajectories.

6. Construction of unbiased and ε -biased estimators of the solution

We will give the sequence of estimators $\{\zeta_k(x,t)\}_{k=0}^{\infty}$ with the following recurrence. Let $\zeta_0(x,t)=u(x,t)$, $\zeta_k(x,t)=\Psi(\zeta_{k-1}(x,t))$, where

$$\Psi(\zeta(x,t))=\begin{cases}\zeta(y,\tau), & \text{in the case 1.}\\ W_n(y,\tau)\prod_{i=1}^{n}\zeta^{(i)}(y,\tau), & \text{in the case 2.}\\ W_0(y,\tau)\dfrac{a_0(y,\tau)}{\pi_0}, & \text{in the case 3.}\end{cases} \tag{16}$$

Case 1. If $(x,t)\to(y,\tau)=\{(y_1,\tau_1)or(y_2,\tau_2)\}$;

Case 2. If $(x,t)\to(y,\tau)=(y_3,\tau_3)$ and particle is generated, $n\neq 0$;

Case 3. If $(x,t)\to(y,\tau)=(y_3,\tau_3)$ and particle is absorbed.

Here $\zeta^{(i)}(y,\tau)$ are independent realizations of the estimators $\zeta(y,\tau)$.

So called "weights", i.e. multipliers, are multiplied by the estimator at each step, and are determine in the following way:

$$W_0(y,\tau)=\left(1-\nu^{2/m}\exp\left(-\frac{2\xi}{m+2}\right)\right),$$
$$W_n(y,\tau)=W_0(y,\tau)\frac{M\,a_n(y,\tau)}{\overline{a}_n(\alpha-q_{1m})}, \tag{17}$$

where ξ is a Gamma distributed random variable with parameter $(m/2)$, ν is Beta distributed random variable with parameter $(2,2/m)$, which is used to determine the point $(y_3,\tau_3)=\left(y_1(\xi,\nu,\omega),\tau_1(\xi,\nu)\right)$ by formulas (7)-(8).

Let $\Re_k$ be a σ-algebra which generated by the sequence $T_0=(x,t)$, T_1, …, T_k, …. The following statement is valid.

Theorem 2. The sequence $\{\zeta_k(x,t)\}_{k=0}^{\infty}$ forms a martingale on $\{\Re_k\}_{k=0}^{\infty}$. If $M<cq_{1m}$, then $\zeta_k(x,t)$ is a uniformly integrable martingale.

Proof. By definition, $\zeta_k(x,t)$ is $\Re_k$ measurable. From the properties of conditional expectation and formulas (16)-(17) we get

$$E\left(\zeta_{n+1}(x,t)|\,\Re_n\right)=E\left(\Psi\left(\zeta_n(x,t)\right)|\,\Re_n\right)=\alpha_1E\zeta_n(y_1,\tau_1)+$$
$$+\alpha_2E\zeta_n(y_2,\tau_2)+\alpha_3\sum_{i=1}^{\infty}\pi_iEW_i(y_3,\tau_3)\prod_{j=1}^{i}\zeta_n^{(j)}(y_3,\tau_3)=$$
$$=\alpha_1E\zeta_n(y_1,\tau_1)+\alpha_2E\zeta_n(y_2,\tau_2)+$$

$$+\alpha_3\sum_{i=1}^{\infty}E\left(1-\nu^{2/m}\exp\left(-\frac{2\xi}{m+2}\right)\right)\frac{a_i(y_3,\tau_3)}{cq_{1m}}\zeta_n^i(y_3,\tau_3)$$

Using the probabilistic representation (11) it follows that

$$E\left(\zeta_{n+1}(x,t)/\Re_n\right)=\alpha_1 E\zeta_n(y_1,\tau_1)+\alpha_2 E\zeta_n(y_2,\tau_2)+$$

$$+\alpha_3 E\left(1-\nu^{2/m}\exp\left(-\frac{2\xi}{m+2}\right)\right)\frac{f\left(\zeta_n(y_3,\tau_3)\right)}{cq_{1m}}=\zeta_n(x,t).$$

Thus, the sequence $\{\zeta_k(x,t)\}_{k=0}^{\infty}$ forms a martingale on $\{\Re_k\}_{k=0}^{\infty}$. For proving that $\zeta_k(x,t)$ is uniformly integrable it's enough to show that $|\zeta_k(x,t)|\le C$, $(C=const)$. Let the parameter α be chosen from the condition $\frac{M}{cq_{1m}}\le\alpha<1$. Since

$$u(x,t)\in C\left(\overline{D}\times[0,T]\right)\cap C^{2,1}\left(\overline{D}\times[0,T]\right)$$

and $\overline{\Omega}$ is a bounded domain, then $|u(x,t)|\le const$ for $(x,t)\in\overline{\Omega}$.

Further $|W_0(y,\tau)|=\left|1-\nu^{2/m}\exp\left(-\frac{2\xi}{m+2}\right)\right|\le 1$, and taking into account the conditions of the theorem, $|W_n(y,\tau)|=\left|W_0(y,\tau)\frac{M\,a_n(y,\tau)}{\overline{a}_n\alpha cq_{1m}}\right|\le 1$.

Finally, from here we obtain $|\zeta_n(x,t)|\le C<\infty$, $(C=const)$. It follows that the sequence $\{\zeta_n(x,t)\}$ is uniformly integrable, and theorem has been proven.

Further, using the estimator $\zeta_n(x,t)$, we will build a biased estimator which is easy implementable on a computer programs.

We will take ε sufficiently small, and will consider the ε - neighbourhood of the boundary:

$$(\partial\Omega)_\varepsilon=\{D\times[0,\varepsilon]\}\cup\{(\partial D)_\varepsilon\times[0,T]\}.$$

Let N_1 be the time index of the absorption of the process inside the domain, and N_ε the time index of first passage in $(\partial\Omega)_\varepsilon$. Then $N=\min\{N_1,N_\varepsilon\}$ is the stopping time of the process. Then the probability of absorption of a trajectory at a point will be equal to:

$$g(x_n,t_n)=\begin{cases}1, & (if)(x_n,t_n)\in(\partial\Omega)_\varepsilon\\ \pi_0, & (if)(x_n,t_n)\in\overline{\Omega}\setminus(\partial\Omega)_\varepsilon\end{cases}$$

From lemma 1 it follows that $N<\infty$.

Theorem 3. Let us the conditions of theorem 2 be satisfied. Then $\zeta_N(x,t)$ is an unbiased estimator for $u(x,t)$ with finite variance.

Proof. Since $\zeta_n(x,t)$ is a uniformly integrable martingale, and N is a Markov time, then by Doobs optional sampling theorem [8] for the martingale $\{\zeta_k(x,t)\}_{k=0}^{\infty}$, we get $E\zeta_N(x,t) = E\zeta_1(x,t)$. By definition $\zeta_1(x,t)$, the formulas (16)-(17), and the probabilistic representation it follows that

$$E\left(\zeta_1(x,t)\right) = \alpha_1 Eu(y_1,\tau_1) + \alpha_2 Eu(y_2,\tau_2) +$$
$$+ \alpha_3 E\left[\left(1-\nu^{2/m}\exp\left(-\frac{2\xi}{m+2}\right)\right)\frac{f(u(y_3,\tau_3))}{cq_{1m}}\right] = u(x,t).$$

In accordance with the conditions of theorem 2 $E\left(\zeta_N(x,t)\right)^2 < \infty$ and thus it's variance is finite. The theorem has been proven.

Further, by using $\zeta_N(x,t)$ we will build a biased, but practically implementable estimator $\zeta_N^*(x,t)$ in the following way.

Let $\Phi(x,t) = y(x,t)$ when $x \in \partial D$, $t \in [0,T]$ and $\Phi(x,0) = y_0(x)$ when $x \in \overline{D}$, (x^*,t^*) is the closest point to the bound $\partial\Omega$. $\zeta_N^*(x,t)$ is obtained by substituting $u(x_N,t_N)$ into $\zeta_N(x,t)$ along $\Phi\left(x_N^*,t_N^*\right)$. Let's estimate the bias in $\zeta_N^*(x,t)$. It's clear that

$$\left|E\zeta_N(x,t) - u(x,t)\right| \le E\left|\zeta_N^*(x,t) - \zeta_N(x,t)\right|.$$

If $N = N_1$, then $T_N = \{\theta\}$ and the process is terminates without hitting $(\partial\Omega)_\varepsilon$. In this case $\zeta_N^*(x,t) = \zeta_N(x,t)$ and the bias is equal to the zero. If $N = N_\varepsilon$, then

$$T_N = (x_N^1, t_N^1, n_N^1; x_N^2, t_N^2, n_N^2; \ldots; x_N^k, t_N^k, n_N^k),$$

where $(x_N^i, t_N^i) \in \partial\Omega$, $i = 1,2,\ldots,k.$ and the number k does not depend on N. Taking into account that $\left|W_n(y,\tau)\right| \le 1$ for arbitrary n, and $(y,\tau) \in \overline{\Omega}$, we have

$$\zeta_N^*(x,t) - \zeta_N(x,t) \le \left(\prod_{i=1}^{k}\left[\Phi\left((x_N^i)^*,(t_N^i)^*\right)\right]^{n_N^i} - \prod_{i=1}^{k}\left[u\left((x_N^i)^*,(t_N^i)^*\right)\right]^{n_N^i}\right).$$

Let $L(\varepsilon)$ be the absolute value of continuity of the function $u(x,t)$, then the following is valid

$$\left|E\zeta_N^*(x,t) - u(x,t)\right| \le L(\varepsilon)E\left(n_N^1 + n_N^2 + \cdots + n_N^k\right).$$

Since the average number of particles in the N-th generation is

$$E\left(n_N^1 + n_N^2 + \cdots + n_N^k\right) = K_N(x,t;\Omega) \le 1$$

(see section 4), we will find, that, the bias doesn't exceed $L(\varepsilon)$. Finiteness of the variance follows from $\left|W_n(y,\tau)\right| \le 1$.

Following cases $f(u) = g\exp(u)$, $f(u) = g\cos(u)$, $f(u) = g\sinh(u)$, $f(u) = g\cosh(u)$ (g is constant), have been in detail considered in [10] and the corresponding branching probabilities have been obtained.

References

1. Ermakov S.M., Nekrutkin V.V., Sipin A.S., Random Processes for Classical Equations of Mathematical Physics, Kluwer Acad. Publ., 1989, 282p.
2. Ermakov S.M., Rasulov A.S., Bakoev M.T., Veselovskaya A.Z., Effective Algorithms of Monte-Carlo Method, Tashkent University, 1992, 132p. (in Russian).
3. Friedman A., Partial Differential Equations of Parabolic Type, Prentice-Hall Publ., 1964, 347p.
4. Harris T.E., The Theory of Branching Processes, Springer-Verlag, Berlin, 1963, 230p.
5. Kuptzov L.P., Feature of average and the principle of maximum for parabolic equations of the second order, DAN SSSR, vol.242, N.3, 1978, p.529-532 (in Russian).
6. Kurbanmuradov O., Methods of random walks on balloids for solving the equation of thermal conductivity, Theory and algorithms of statistical modelling, Proc. VC SOAN SSSR, Novosibirsk, 1984, pp.67-78 (in Russian).
7. Ladyzenskaja O.A., Solonnikov V.A., Uralceva N.N., Linear and Quasi-linear Equations of Parabolic Type, Translations of Mathematical Monographs, vol. 23, 1968, Providence, RI: American Mathematical Society, 648p.
8. Meyer P.A., Probability and Potentials, Blaisdell Pub. Co., 1966, 266 p.
9. Rasulov A.S., Monte-Carlo Method for Solving Nonlinear Problems, Tashkent, Fan, 1992, 104p. (in Russian).
10. Rasulov A.S., Mascagni M., Raimova G.M., Monte-Carlo Methods for Solution of Linear and Nonlinear Boundary Value Problems, Tashkent, UWED, 2006, 348p.

Qeyri-xətti parabolik tənlik üçün qoyulmuş başlanğic-sərhəd məsələsinin statistik modelləşmə üsulu ilə həlli

Ş. Formanov, A. Rasulov, Q. Raimova

XÜLASƏ

İşdə məchul funksiyaya nəzərən polinomial qeyri-xətti olan parabolik tənlik üçün başlanğıc-sərhəd məsələsinə baxılır. Bu məsələnin həllinin ifadəsi riyazi gözləmə formasında verilir. Sonra məlum nəticələrdən istifadə edərək baxılan məsələnin ehtimal dilində ifadəsi verilir.

Açar sözlər: başlanğıc sərhəd məsələsi, parabolik tənlik,statistik modelləşmə üsulu.

Решение начально-краевой задачи для нелинейного параболического уравнения методом статического моделирования

Ш. Форманов, А. Расулов, Г. Раимова

РЕЗЮМЕ

В работе рассматривается начально-краевая задача для параболического уравнения с полиноминальной относительно неизвестной функции нелинейностью. Представление решения данной задачи дается в форме математического ожидания. Далее используя известные результаты, выведено вероятностное представление для данной задачи.

Ключевые слова: начально-краевая задача, параболическое уравнение, метод статического моделирования.