TWMS J. Pure Appl. Math., V.11, N.1, 2020, pp.72-88

AFFINE FACTORABLE SURFACES IN ISOTROPIC SPACES

MUHITTIN EVREN AYDIN[1], AYLA ERDUR[2], MAHMUT ERGUT[2]

Abstract. In this paper, we study the problem of finding affine factorable surfaces in a $3-$dimensional isotropic space $\mathbb{I}^3$ with prescribed Gaussian (K) or mean (H) curvatures. Because the absolute figure of $\mathbb{I}^3$, by permutation of coordinates two different types of these surfaces appear. We firstly classify the affine factorable surfaces of type 1 with K, H constants. Afterwards, we provide the affine factorable surfaces of type 2 with $K = const.$ or $H = 0$. Besides in some particular cases, the affine factorable surfaces of type 2 with $H = const$ were obtained.

Keywords: Isotropic space, affine factorable surface, mean curvature, Gaussian curvature.

AMS Subject Classification: 53A35, 53A40, 53B25.

1. Introduction

Let $\mathbb{R}^3$ be a 3-dimensional Euclidean space with usual coordinates (x, y, z) and

$$w : \mathbb{R}^2 \to \mathbb{R}, \ \ (x, y) \mapsto, w(x, y)$$

a smooth real-valued function of 2 variables. Then, the graph $z = w(x, y)$ is a smooth surface with an atlas that only consists of the following patch

$$\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3, \ \ (x, y) \mapsto (x, y, w(x, y)).$$

Notice that every surface in $\mathbb{R}^3$ is locally a part of the graph $z = w(x, y)$ if its normal is not parallel to the $xy-$plane. Otherwise, the regularity assures that it is a part of the graph $x = w(y, z)$ or $y = w(x, z)$. See [37, p. 119]. These graphs are also called *surfaces of Monge type* [17, p. 302].

In the differential geometry of surfaces, one of the challenging problems has been to obtain explicit equations of surfaces with prescribed Gaussian (K) or mean (H) curvatures. In this manner, it is naturally reasonable to concern the graphs. For a graph, a problem of this kind is indeed to solve an equation of *Monge-Ampère type* given by ([39, 42])

$$w_{xx}w_{yy} - w_{xy}^2 = K(x, y) W^2, \tag{1}$$

and an equation of *mean curvature type* ([27, 39])

$$\left(1 + w_x^2\right) w_{yy} - 2w_x w_y w_{xy} + \left(1 + w_y^2\right) w_{xx} = 2H(x, y) W^{\frac{3}{2}}, \tag{2}$$

where $w_x = \frac{\partial w}{\partial x}$, $w_{xx} = \frac{\partial^2 w}{\partial x^2}$, etc. and $W = 1 + w_x^2 + w_y^2$.

The equations (1) and (2) also arise in economics, meteorology, oceanography etc. [7, 8, 9, 11].

In a 3-dimensional isotropic space $\mathbb{I}^3$, by separation of variables we study the graphs

$$z = w(x, y) = f_1(x) f_2(y),$$

[1]Firat University, Department of Mathematics, Faculty of Science, Elazig, Turkey
[2]Namik Kemal University, Department of Mathematics, Tekirdag, Turkey
e-mail: meaydin@firat.edu.tr, aerdur@nku.edu.tr, mergut@nku.edu.tr
Manuscript received January 2018.

so-called *factorable* or *homothetical surfaces.* Here f_1 and f_2 are smooth functions of a single variable. Many results on the factorable surfaces in other 3-dimensional spaces were obtained so far, see [1-4, 18, 20, 25, 28, 43, 48].

This kind of surfaces also appears as invariant surfaces in the 3-dimensional space $\mathbb{H}^2 \times \mathbb{R}$ which is one of the eight homogeneous geometries of Thurston. More clearly, a certain type of translation surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is the graph of $z = f_1(x) f_2(y)$, see [45, p. 1547]. For further details, we refer to [5, 6, 19, 22-27, 32, 41, 46, 47].

Recently, Zong et al. [49] defined *affine factorable surfaces* in $\mathbb{R}^3$ as the graphs

$$z = f_1(x) f_2(y + ax), \ a \in \mathbb{R}, \ a \neq 0.$$

They obtained these surfaces with $K = 0$ and $H = const.$ It is clear that this class of surfaces is more general than the factorable surfaces.

In this paper, the problem of determining affine factorable surfaces in $\mathbb{I}^3$ with K or H constant is considered. Because the absolute figure of $\mathbb{I}^3$ (for details see Preliminaries section), by permutation of the coordinates two different types of these surfaces exist, i.e. the graphs of

$$z = f_1(x) f_2(y + ax) \text{ and } x = f_1(y + az) f_2(z).$$

We call the surfaces *affine factorable surfaces* of *type 1* and *2*, respectively. Point out also that such surfaces reduce to the factorable surfaces in $\mathbb{I}^3$ when $a = 0$.

In this sense, our first concern is to obtain affine factorable surfaces of type 1 with K or H constant. And then, we present some results relating to the affine factorable surfaces of type 2 with $K = const.$ or $H = 0$. Furthermore, in some particular cases, the affine factorable surfaces of type 2 with $H = const.$ were found.

2. Preliminaries

In this section, we provide some fundamental properties of isotropic geometry from [10, 12-16, 29-33, 35, 36, 38, 49]. For basics of Cayley-Klein geometries see also [21, 34, 44].

Let $(x_0 : x_1 : x_2 : x_3)$ denote the homogenous coordinates in a 3-dimensional real projective space $P(\mathbb{R}^3)$. A *3-dimensional isotropic space* $\mathbb{I}^3$ is a Cayley-Klein space defined in $P(\mathbb{R}^3)$ with the absolute figure $\{\omega, l_1, l_2\}$, where ω is an *absolute plane* and l_1, l_2 two *absolute lines* in ω. These are respectively parameterized by $x_0 = 0$ and $x_0 = x_1 \pm ix_2 = 0$. The intersection point of these complex-conjugate lines is called *absolute point*, $(0 : 0 : 0 : 1)$.

The group of motions of $\mathbb{I}^3$, which leave the absolute figure invariant, is given by the $6-$ parameter group

$$(x, y, z) \longmapsto (\tilde{x}, \tilde{y}, \tilde{z}) : \begin{cases} \tilde{x} = \theta_1 + x\cos\theta - y\sin\theta, \\ \tilde{y} = \theta_2 + x\sin\theta + y\cos\theta, \\ \tilde{z} = \theta_3 + \theta_4 x + \theta_5 y + z, \end{cases} \tag{3}$$

where (x, y, z) denote the affine coordinates and $\theta, \theta_1, ..., \theta_5 \in \mathbb{R}$. The *isotropic metric* induced by the absolute figure is given by $ds^2 = dx^2 + dy^2$.

Due to the absolute figure there are two types of lines and planes: the *isotropic lines* and *planes* which are parallel to $z-$ axis and others called *non-isotropic lines and planes.* As an example the equation $ax + by + cz = d$ determines a non-isotropic (isotropic) plane if $c \neq 0$ $(c = 0)$, $a, b, c, d \in \mathbb{R}$.

Note that the plane $z = 0$, so-called *basic* (or *top-view*) *plane* , is non-isotropic (or Euclidean) and therefore the $2d$ Euclidean metric is used in it.

Two non-isotropic lines are orthogonal if their projections onto the top-view plane are perpendicular up to the Euclidean metric. Nevertheless, an isotropic line is orthogonal to some non-isotropic line. As a consequence, each non-isotropic plane is orthogonal to the isotropic one. Besides, two isotropic planes are orthogonal if their projections onto the top-view plane are perpendicular.

A surface is said to be *admissible* if nowhere it has isotropic tangent planes. If some admissible surface is locally parameterized by

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

then the Jacobian determinant satisfies

$$\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u \neq 0,$$

where $x_u = \frac{\partial x}{\partial u}$, etc.

We may introduce an isotropic scalar product between two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ as

$$\langle \mathbf{u}, \mathbf{v} \rangle_i = \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} = u_1 v_1 + u_2 v_2,$$

where $\tilde{\mathbf{u}}$ denotes the top view of $\mathbf{u}$ and $\cdot$ the Euclidean scalar product in $\mathbb{R}^2$.

Denote g and h the first and second fundamental forms, respectively. Then the components of g are calculated by the induced metric from $\mathbb{I}^3$, namely

$$g_{11} = \langle \mathbf{r}_u, \mathbf{r}_u \rangle_i, \ g_{12} = \langle \mathbf{r}_u, \mathbf{r}_v \rangle_i, \ g_{22} = \langle \mathbf{r}_v, \mathbf{r}_v \rangle_i, \ \mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}.$$

The unit normal vector is $(0,0,1)$ because it is orthogonal to all non-isotropic vectors. The components of h are given by

$$h_{11} = \frac{\det(\mathbf{r}_{uu}, \mathbf{r}_u, \mathbf{r}_v)}{\sqrt{\det g}}, \ h_{12} = \frac{\det(\mathbf{r}_{uv}, \mathbf{r}_u, \mathbf{r}_v)}{\sqrt{\det g}}, \ h_{22} = \frac{\det(\mathbf{r}_{vv}, \mathbf{r}_u, \mathbf{r}_v)}{\sqrt{\det g}},$$

where $\mathbf{r}_{uu} = \frac{\partial^2 \mathbf{r}}{\partial u \partial u}$, etc. Therefore, the *isotropic Gaussian* (or *relative*) and *mean curvatures* are respectively defined by

$$K = \frac{\det h}{\det g}, \ H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2 \det g}.$$

For convenience, we call these *Gaussian* and *mean curvatures.*

By a *flat* (*minimal*) *surface* we mean a surface with vanishing Gaussian (mean) curvature.

Notice that h_{ij}'s are proportional to the corresponding Euclidean coefficient of the surface; namely, it is possible to define *elliptic, hyperbolic* and *parabolic points.* So, we can interpret the sign of K in the same way we do in Euclidean geometry.

In the particular case that the surface is the graph $z = w(x,y)$, the Gaussian and mean curvatures turn to

$$K = w_{xx}w_{yy} - w_{xy}^2, \ H = \frac{w_{xx} + w_{yy}}{2}. \tag{4}$$

Accordingly; if it is the graph $x = w(y,z)$, then these curvatures are formulated by

$$K = \frac{w_{yy}w_{zz} - w_{yz}^2}{w_z^4}, \ H = \frac{w_z^2 w_{yy} - 2w_y w_z w_{yz} + \left(1 + w_y^2\right) w_{zz}}{2w_z^3}, \tag{5}$$

where $w_z \neq 0$ because of the admissibility.

3. Affine factorable surfaces of type 1

An *affine factorable surface of type 1* in $\mathbb{I}^3$ is a graph

$$z = w(x, y) = f_1(x) f_2(y + ax), \ a \neq 0,$$

for smooth functions f_1 and f_2. Let us put $u_1 = x$ and $u_2 = y + ax$. By (4), we get the Gaussian curvature as

$$K = f_1 f_2 f_1'' f_2'' - (f_1' f_2')^2, \tag{6}$$

where $f_1' = \frac{df_1}{du_1}$ and $f_2' = \frac{df_2}{du_2}$ and so on.

Theorem 3.1. *Let an affine factorable surface of type 1 in $\mathbb{I}^3$ have constant Gaussian curvature K_0. Then, for $b, c_0, c_1, c_2 \in \mathbb{R}$, we have*

(1) *if $K_0 = 0$, then*
  (a) $w(x, y) = c_0 f_2(y + ax)$ *or* $w(x, y) = c_0 f_1(x)$;
  (b) $w(x, y) = c_0 e^{c_1 x + c_2(y+ax)}$;
  (c) $w(x, y) = c_0 \left[\frac{(y+ax+c_1)^b}{x+c_2}\right]^{\frac{1}{b-1}}, \ b \neq 1$;
  (d) $w(x, y) = c_0 \left[\frac{(x+c_1)^b}{y+ax+c_2}\right]^{\frac{1}{b-1}}, \ b \neq 1$.
(2) *Otherwise, i.e. $K_0 \neq 0$ then K_0 is negative and*
  (a) $w(x, y) = c_0 \left(\sqrt{-K_0} x + c_1\right)(y + ax + c_2)$.
  (b) $w(x, y) = c_0 (x + c_1) \left[\sqrt{-K_0}(y + ax) + c_2\right]$.

Proof. We have two cases:

(1) $K_0 = 0$. (6) reduces to

$$f_1 f_2 f_1'' f_2'' - (f_1' f_2')^2 = 0. \tag{7}$$

(7) holds when f_1 or f_2 is a constant. This proves the item (1.a) of the theorem. If $f_1' f_2' \neq 0$, then (7) implies $f_1'' f_2'' \neq 0$. Thereby, (7) can be rewritten by dividing $f_2 f_2'' (f_1')^2$ as

$$\frac{f_1 f_1''}{(f_1')^2} = \frac{(f_2')^2}{f_2 f_2''},$$

where the left-hand side is a function of u_1 whereas the right-hand side is a function of u_2. This is possible in case both sides are only constant, i.e.

$$\frac{f_1 f_1''}{(f_1')^2} = b = \frac{(f_2')^2}{f_2 f_2''}, \tag{8}$$

where $b \in \mathbb{R}$, $b \neq 0$. If $b = 1$, after solving (8), we obtain

$$f_1(u_1) = c_0 \exp(c_1 u_1), \ f_2(u_2) = c_2 \exp(c_3 u_2), \ c_0, ..., c_3 \in \mathbb{R},$$

which proves the item (1.b) of the theorem. Otherwise, i.e. $b \neq 1$, by solving (8), we derive

$$f_1(u_1) = [(1 - b)(c_4 u_1 + c_5)]^{\frac{1}{1-b}}, \ f_2(u_2) = \left[\left(\frac{b}{b-1}\right)(c_6 u_2 + c_7)\right]^{\frac{b}{b-1}},$$

for $c_4, ..., c_7 \in \mathbb{R}$. This is the proof of the item (1.c) of the theorem. The item (1.d) of the theorem can be proved in a similar way by taking $\frac{1}{b}$ instead of b.

(2) $K_0 \neq 0$. (6) can be rewritten as

$$K_0 = f_1 f_2 f_1'' f_2'' - \left(f_1' f_2'\right)^2. \tag{9}$$

If f_1 or f_2 is constant in (9) then $K_0 = 0$. Thus we may assume $f_1' f_2' \neq 0$. To solve (9) we have two sub-cases:

(i) f_1 or f_2 is linear function. Without loss of generality we may assume $f_1 = c_0 u_1 + c_1$, $c_0, c_1 \in \mathbb{R}$, $c_0 \neq 0$. By (9), we get $K_0 = -c_0^2 \left(f_2'\right)^2$or

$$f_2(u_2) = \sqrt{\frac{-K_0}{c_0^2}} u_2 + c_2, \ c_2 \in \mathbb{R},$$

which proves the item (2.b) of the theorem. In a similar way the proof of the item (2.a) of the theorem can be done.

(ii) Neither f_1 nor f_2 are a linear function. After dividing (9) with $f_1 f_1'' \left(f_2'\right)^2$, we can write

$$\frac{K_0}{f_1 f_1''} \left(\frac{1}{f_2'}\right)^2 = \frac{-\left(f_1'\right)^2}{f_1 f_1''} + \frac{f_2 f_2''}{\left(f_2'\right)^2}. \tag{10}$$

By taking partial derivative of (10) with respect to u_2 we derive

$$\frac{K_0}{f_1 f_1''} = -\left[\frac{f_2 f_2''}{\left(f_2'\right)^2}\right]' \left[\frac{\left(f_2'\right)^3}{2 f_2''}\right],$$

which means $f_1 f_1'' = c_3$, $c_3 \in \mathbb{R}$, $c_3 \neq 0$. Considering it into (10) gives

$$\frac{1}{c_3} \left(f_1'\right)^2 = \frac{f_2 f_2''}{\left(f_2'\right)^2} - \frac{K_0}{c_3} \left(\frac{1}{f_2'}\right)^2.$$

This yields $f_1' = const.$ and contradicts with $f_1'' \neq 0$.

□

From (4), the mean curvature follows

$$2H = \left(1 + a^2\right) f_1 f_2'' + 2a f_1' f_2' + f_1'' f_2. \tag{11}$$

Theorem 3.2. *Let an affine factorable surface of type 1 in $\mathbb{I}^3$ be minimal. Then, for $b, c_0, c_1, c_2 \in \mathbb{R}$, either*

(1) *it is a non-isotropic plane; or*

(2) $w(x, y) = c_0 e^{bx + \frac{-ba}{1+a^2}(y+ax)} \left[c_1 \sin\left(\frac{b}{1+a^2}(y + ax)\right) + c_2 \cos\left(\frac{b}{1+a^2}(y + ax)\right)\right]$; *or*

(3) $w(x, y) = c_0 e^{by} \left[c_1 \sin(bx) + c_2 \cos(bx)\right]$.

Proof. (11) reduces to

$$\left(1 + a^2\right) f_1 f_2'' + 2a f_1' f_2' + f_1'' f_2 = 0. \tag{12}$$

If f_1 or f_2 is a constant function in (12), we have immediately the first item of the theorem. Suppose then that $f_1' f_2' \neq 0$ in (12). If $f_1 = c_0 u_1 + c_1$, $c_0, c_1 \in \mathbb{R}$, $c_0 \neq 0$, (12) gives the following polynomial equation in u_1

$$\left[\left(1 + a^2\right) f_2''\right] c_1 + 2a c_0 f_2' + c_0 \left[\left(1 + a^2\right) f_2''\right] u_1 = 0,$$

which yields $f_2' = f_2'' = 0$. This is not our case and we deduce $f_1'' \neq 0$. In a similar way $f_2'' \neq 0$ can be shown. Next we divide (12) by $f_1' f_2'$

$$-2a = \left(1 + a^2\right) \left(\frac{f_1}{f_1'}\right) \left(\frac{f_2''}{f_2'}\right) + \left(\frac{f_1''}{f_1'}\right) \left(\frac{f_2}{f_2'}\right). \tag{13}$$

The partial derivative of (13) with respect to u_1 gives

$$\left(1+a^2\right)\left(\frac{f_1}{f_1'}\right)' f_2'' + \left(\frac{f_1''}{f_1'}\right)' f_2 = 0. \tag{14}$$

We have three cases to solve (14):

(1) $f_1' = bf_1$, $b \in \mathbb{R}$, $b \neq 0$. That is a solution for (14) and thus (13) reduces to

$$\frac{1+a^2}{b} f_2'' + 2af_2' + bf_2 = 0, \tag{15}$$

which is a homogenous linear second-order ODE with constant coefficients. The characteristic equation of (15) has complex roots $\frac{-b}{1+a^2}(a \pm i)$, so its solution turns to

$$f_2(u_2) = e^{\left(\frac{-ba}{1+a^2}\right)u_2}\left[c_1 \cos\left(\frac{b}{1+a^2}u_2\right) + c_2 \sin\left(\frac{b}{1+a^2}u_2\right)\right]. \tag{16}$$

Considering (16) with the assumption of Case 1 gives the proof of the second item of the theorem.

(2) $f_2' = bf_2$, $b \in \mathbb{R}$, $b \neq 0$. After taking partial derivative of (13) with respect to u_2 the proof of the last item of the theorem is same with previous case.

(3) $(f_1/f_1')'(f_2/f_2')' \neq 0$. Hence, (14) yields that f_2 and f_2'' can not be linearly independent, i.e. $f_2'' = c_0 f_2$, $c_0 \in \mathbb{R}$, $c_0 \neq 0$. Substituting this into (13) gives

$$-2a\frac{f_2'}{f_2} = c_0\left(1+a^2\right)\frac{f_1}{f_1'} + \frac{f_1''}{f_1'}. \tag{17}$$

Because $(f_2/f_2')' \neq 0$ the left-hand side of (17) is a function of u_2 whereas the right-hand side is a function of u_1 or a constant. This is a contradiction.

□

Theorem 3.3. *Let an affine factorable surface of type 1 in $\mathbb{I}^3$ have nonzero constant mean curvature H_0. Then, for $c_0, c_1, c_2 \in \mathbb{R}$, we have either*

(i) $w(x,y) = \frac{H_0}{1+a^2}(y+ax)^2 + y + ax + c_0$ *or*

(ii) $w(x,y) = (c_0 x + c_1)\left[\frac{H_0}{ac_0}(y+ax) + c_2\right]$.

Proof. (11) can be rewritten as

$$2H_0 = \left(1+a^2\right) f_1 f_2'' + 2af_1'f_2' + f_1''f_2. \tag{18}$$

In order to solve (18) we have two cases:

(1) Case $f_1'' = 0$. If $f_1' = 0$, then (18) proves the item (i) of the theorem. If $f_1 = c_0u_1 + c_1$, $c_0, c_1 \in \mathbb{R}$, $c_0 \neq 0$, then by (18) we get a polynomial equation in f_1

$$-2H_0 + 2ac_0 f_2' + \left[\left(1+a^2\right) f_2''\right] f_1 = 0,$$

which implies that $f_2'' = 0$ and

$$f_2' = \frac{H_0}{ac_0}.$$

This proves the item (ii) of the theorem.

(2) Case $f_1'' \neq 0$. By assuming $f_2'' = 0$ and applying above process we achieve the contradiction $f_1'' = 0$. Then $f_1''f_2'' \neq 0$ and (18) can be rearranged as

$$2H_0 = \left(1+a^2\right) f_1 p_2 \dot{p}_2 + 2ap_1p_2 + f_2 p_1 \dot{p}_1, \tag{19}$$

where $p_i = \frac{df_i}{du_i}$ and $\dot{p}_i = \frac{dp_i}{df_i} = \frac{f_i''}{f_i'}$, $i = 1, 2$. Thus (19) can be divided by $f_2 p_1$ as

$$\frac{2H_0}{f_2 p_1} = \left(1 + a^2\right) \left(\frac{f_1}{p_1}\right) \left(\frac{p_2 \dot{p}_2}{f_2}\right) + 2a \frac{p_2}{f_2} + \dot{p}_1. \tag{20}$$

The partial derivative of (20) with respect to f_1 leads to

$$\frac{d}{df_1} \left(\frac{1}{p_1}\right) \left(\frac{2H_0}{f_2}\right) = \left(1 + a^2\right) \frac{d}{df_1} \left(\frac{f_1}{p_1}\right) \left(\frac{p_2 \dot{p}_2}{f_2}\right) + \ddot{p}_1, \tag{21}$$

where $\ddot{p}_1 = \frac{d^2 p_1}{df_1^2}$. If $p_1 = c_3 f_1$, $c_3 \in \mathbb{R}$, $c_3 \neq 0$, then the right-hand side of (21) becomes zero, which is no possible. Thereby, (21) can be rewritten by dividing $\frac{d}{df_1}\left(\frac{f_1}{p_1}\right)$ as

$$\underbrace{\frac{\frac{d}{df_1}\left(\frac{1}{p_1}\right)}{\frac{d}{df_1}\left(\frac{f_1}{p_1}\right)}}_{A(f_1)} \left(\frac{2H_0}{f_2}\right) = \underbrace{\frac{\ddot{p}_1}{\frac{d}{df_1}\left(\frac{f_1}{p_1}\right)}}_{B(f_1)} + \frac{\left(1 + a^2\right) p_2 \dot{p}_2}{f_2}, \tag{22}$$

where $A(f_1)$ and $B(f_1)$ are a function of f_1. After taking partial derivatives of (22) with respect to f_1 and f_2 we can deduce A is a constant $A_0 \neq 0$ because $\frac{d}{df_2}\left(\frac{2H_0}{f_2}\right) \neq 0$. This follows from (22) that B is also a constant B_0. Therefore we write

$$\frac{d}{df_1}\left(\frac{1}{p_1}\right) = A_0 \frac{d}{df_1}\left(\frac{f_1}{p_1}\right) \text{ and } \ddot{p}_1 = B_0 \frac{d}{df_1}\left(\frac{f_1}{p_1}\right). \tag{23}$$

An integration of first equation in (23) gives

$$\frac{1}{p_1} = A_0 \frac{f_1}{p_1} + c_4,\ c_4 \in \mathbb{R},\ c_4 \neq 0, \text{ or } p_1 = \frac{1}{c_4} - \frac{A_0}{c_4} f_1. \tag{24}$$

It follows from (24) that $\ddot{p}_1 = B_0 = 0$ and thus (22) implies

$$\left(1 + a^2\right) p_2 \dot{p}_2 = 2A_0 H_0. \tag{25}$$

On the other hand, if we take partial derivative of (19) with respect to f_1 and consider (25) into it then we have

$$p_2 = \frac{A_0}{2ac_4} f_2 + \frac{c_4 H_0}{a}. \tag{26}$$

Comparing (25) and (26) gives a contradiction.

□

4. Affine factorable surfaces of type 2

An affine factorable surface of type 2 in $\mathbb{I}^3$ is a graph

$$z = w(x, y) = f_1(y + az) f_2(z),\ a \neq 0,$$

for smooth functions f_1, f_2. Put $u_1 = y + az$ and $u_2 = z$. From (5), the Gaussian curvature follows

$$K = \frac{f_1 f_2 f_1'' f_2'' - \left(f_1' f_2'\right)^2}{\left(a f_1' f_2 + f_1 f_2'\right)^4}, \tag{27}$$

where $f_1' = \frac{df_1}{du_1}$, $f_2' = \frac{df_2}{du_2}$. Notice that the admissibility refers to $a f_1' f_2 + f_1 f_2' \neq 0$.

Theorem 4.1. *Let an affine factorable surface of type 2 in $\mathbb{I}^3$ have constant Gaussian curvature K_0. Then it is flat (i.e. $K_0 = 0$) and one of the following occurs:*

a. $w(y, z) = c_0 f_1(y + az)$, $\frac{\partial f_1}{\partial z} \neq 0$ *or* $w(y, z) = c_0 f_2(z)$, $\frac{df_2}{dz} \neq 0$;

b. $w(y,z) = c_0 e^{c_1(y+az)+c_2 z}$;

c. $w(x,y) = c_0 \left[\frac{(y+az+c_1)^b}{z+c_2} \right]^{\frac{1}{b-1}}$, $b \neq 1$;

d. $w(x,y) = c_0 \left[\frac{(z+c_1)^b}{y+az+c_2} \right]^{\frac{1}{b-1}}$, $b \neq 1$, *where* $b, c_0, c_1, c_2 \in \mathbb{R}$.

Proof. If $K_0 = 0$ in (27), then the proofs of the items (a),...,(d) of the theorem are similar with the first four items of Theorem 3.1. The continuation of the proof is by contradiction. Suppose that $K_0 \neq 0$ and then (27) turns to

$$K_0 = \frac{f_1 f_2 f_1'' f_2'' - (f_1' f_2')^2}{(a f_1' f_2 + f_1 f_2')^4}, \tag{28}$$

where f_1, f_2 must be non-constants. Afterwards, we use the property that the roles of f_1, f_2 are symmetric in (28). If $f_1 = c_0 u_1 + c_1$, $c_0, c_1 \in \mathbb{R}$, $c_0 \neq 0$, then (28) turns to a polynomial equation in f_1

$$\xi_1(u_2) + \xi_2(u_2) f_1 + \xi_3(u_2) f_1^2 + \xi_4(u_2) f_1^3 + \xi_5(u_2) f_1^4 = 0,$$

where

$$\begin{aligned} \xi_1(u_2) &= K_0 a^4 c_0^4 f_2^4 + c_0^2 (f_2')^2, \\ \xi_2(u_2) &= 4K_0 a^3 c_0^3 f_2^3 f_2', \\ \xi_3(u_2) &= 6K_0 a^2 c_0^2 f_2^2 (f_2')^2, \\ \xi_4(u_2) &= 4K_0 a c_0 f_2 (f_2')^3, \\ \xi_5(u_2) &= K_0 (f_2')^4. \end{aligned}$$

The fact that each coefficient ξ_i, $i = 1, ..., 5$, must vanish contradicts with $f_2 \neq const$. Thereby, we conclude $f_1'' \neq 0$ (and so $f_2'' \neq 0$ by symmetry). Next, put $\omega_1 = f_1 f_1''$, $\omega_2 = (f_1')^2$, $\omega_3 = f_1'$, $\omega_4 = f_1$ in (28). After taking partial derivative of (28) with respect to u_1, it can be rewritten as

$$\mu_1 f_2^2 f_2'' + \mu_2 f_2 f_2' f_2'' - \mu_3 f_2 (f_2')^2 - \mu_4 (f_2')^3 = 0, \tag{29}$$

where

$$\begin{aligned} \mu_1 &= a(\omega_1' \omega_3 - 4\omega_1 \omega_3'), \\ \mu_2 &= \omega_1' \omega_4 - 4\omega_1 \omega_4', \\ \mu_3 &= a(\omega_2' \omega_3 + 4\omega_2 \omega_3'), \\ \mu_4 &= -\omega_2' \omega_4 + 4\omega_2 \omega_4', \end{aligned} \tag{30}$$

for $\omega_i' = \frac{d\omega_i}{du_1}$, $i = 1, ..., 4$. Notice that ω_i and μ_i are function of the variable u_1. By dividing (29) with $f_2^2 f_2'$, we deduce

$$\frac{f_2''}{f_2'} \left(\mu_1 + \mu_2 \frac{f_2'}{f_2} \right) = \left(\mu_3 \frac{f_2'}{f_2} + \mu_4 \left(\frac{f_2'}{f_2} \right)^2 \right). \tag{31}$$

For (31) we have to distinguish several cases:

(1) $\frac{f_2'}{f_2} = c_2 \neq 0$, $c_2 \in \mathbb{R}$. Substituting it into (28) leads to the polynomial equation in f_2

$$c_2^2 \left(f_1 f_1'' - (f_1')^2 \right) - K_0 (a f_1' + c_2 f_1)^4 f_2^2 = 0,$$

where the coefficients must vanish, namely

$$a f_1' + c_2 f_1 = 0 \text{ and } f_1 f_1'' - (f_1')^2 = 0.$$

The first equality however contradicts with the admissibility, i.e. the assumption $a f_1' f_2 + f_1 f_2' \neq 0$.

(2) $\mu_i = 0$, $i = 1, ..., 4$. Because $\mu_3 = 0$, we conclude $6 (f_1')^2 f_1'' = 0$, which is not our case.

(3) $\mu_1 + \mu_2 \dfrac{f_2'}{f_2} \neq 0$. (31) follows

$$\frac{f_2''}{f_2'} = \frac{\mu_3 \frac{f_2'}{f_2} + \mu_4 \left(\frac{f_2'}{f_2}\right)^2}{\mu_1 + \mu_2 \frac{f_2'}{f_2}}. \tag{32}$$

The partial derivative of (32) with respect to u_1 gives a polynomial equation in $\frac{f_2'}{f_2}$ and the fact that each coefficient must vanish yields the following system:

$$\begin{cases} \mu_2'\mu_4 - \mu_2\mu_4' = 0, \\ \mu_2'\mu_3 - \mu_2\mu_3' + \mu_1'\mu_4 - \mu_1\mu_4' = 0, \\ \mu_1'\mu_3 - \mu_1\mu_3' = 0. \end{cases} \tag{33}$$

By (33), we deduce that $\mu_3 = c_3\mu_1$, $\mu_4 = c_4\mu_2$, $c_3, c_4 \in \mathbb{R}$, and

$$(c_4 - c_3)\left(\mu_1'\mu_2 - \mu_1\mu_2'\right) = 0. \tag{34}$$

We have to consider two sub-cases:

(i) $c_3 = c_4$. Put $c_3 = c_4 = c$ and thus c must be nonzero due to the assumption of Case 3. Then (31) leads to

$$\frac{f_2''}{f_2'} = c\frac{f_2'}{f_2},$$

which implies $f_2' = c_5 f_2^c$, $c_5 \in \mathbb{R}$, $c5 \neq 0$. Note that $c \neq 1$ due to Case 1. Hence, (28) turns to

$$\frac{K_0}{c_5^2\left(cf_1f_1'' - (f_1')^2\right)} = \left(\frac{f_2^c}{(af_1'f_2 + c_5f_1f_2^c)^2}\right)^2. \tag{35}$$

The partial derivative of (35) with respect to f_2 concludes

$$a\,(c-2)\,f_1' - cc_5f_1f_2^{c-1} = 0. \tag{36}$$

Again partial derivative of (36) with respect to f_2 leads to either $c = 0$ or $c = 1$ or $c_5 = 0$. However, none of these is possible.

(ii) $c_3 \neq c_4$. It follows from (34) that $\mu_1 = c_6\mu_2$, $c_6 \in \mathbb{R}$. On the other hand, plugging $\omega_1 = \omega_3'\omega_4$ and $\omega_3 = \omega_4'$ into the equation $\mu_3 - c_3\mu_1 = 0$ yields

$$(6 - c_3)\,\omega_3^2\omega_3' - c_3\omega_3\omega_3''\omega_4 + 4c_3\left(\omega_3'\right)^2\omega_4 = 0. \tag{37}$$

Dividing (37) by $\omega_3\omega_4\omega_3'$ gives

$$(6 - c_3)\,\frac{\omega_4'}{\omega_4} - c_3\frac{\omega_3''}{\omega_3'} + 4c_3\frac{\omega_3'}{\omega_3} = 0. \tag{38}$$

Integrating of (38) leads to

$$\omega_3' = c_7\omega_3^4\omega_4^{\frac{6-c_3}{c_3}}, \; c_7 \in \mathbb{R}, \; c_7 \neq 0. \tag{39}$$

By producting (39) with ω_4, we get

$$\omega_1 = c_7\omega_3^4\omega_4^{\frac{6}{c_3}}. \tag{40}$$

On the other hand, $\mu_1 - c_4\mu_2 = 0$ implies

$$\frac{\omega_1'}{\omega_1} - 4\frac{a\omega_3' - c_4\omega_4'}{a\omega_3 - c_4\omega_4} = 0. \tag{41}$$

From integrating of (41), we derive

$$\omega_1 = c_8 \left(a\omega_3 - c_6\omega_4\right)^4, \ c_8 \in \mathbb{R}, \ c_8 \neq 0. \tag{42}$$

Comparing (40) and (42) leads to

$$c_7\omega_3^4\omega_4^{\frac{6}{c_3}} = c_8 \left(a\omega_3 - c_6\omega_4\right)^4. \tag{43}$$

Without of loss generality, we may assume that the terms are positive in (43). Then we can obtain ω_3 from (43) as follows:

$$\omega_3 = \frac{-c_6\omega_4}{\left(\frac{c_7}{c_8}\right)^{\frac{1}{4}}\omega_4^{\frac{3}{2c_3}} - a}. \tag{44}$$

Revisiting (39) and integrating it gives

$$\omega_3^2 = \frac{1}{c_9\omega_4^{\frac{6}{c_3}} + c_{10}}, \ c_9, c_{10} \in \mathbb{R}, \ c_9 \neq 0. \tag{45}$$

After equalizing (44) and (45), we obtain an equation of the form

$$c_6^2\omega_4^{\frac{6+2c_3}{c_3}} - \left(\frac{c_7}{c_8}\right)^{\frac{1}{2}}\omega_4^{\frac{3}{c_3}} + 2a\left(\frac{c_7}{c_8}\right)^{\frac{1}{4}}\omega_4^{\frac{3}{2c_3}} + c_{10}c_4^2\omega_4^2 - a^2 = 0.$$

This equation leads to a contradiction because $\omega_4 = f_1$ is an arbitrary non-constant function.

□

By (5) the mean curvature is

$$2H = \frac{\left(f_1'f_2\right)^2 f_1f_2'' - 2\left(f_1'f_2'\right)^2 f_1f_2 + \left(f_1f_2'\right)^2 f_2f_1'' + f_1f_2'' + 2af_1'f_2' + a^2f_1''f_2}{\left(af_1'f_2 + f_1f_2'\right)^3}. \tag{46}$$

Theorem 4.2. *There does not exist a minimal affine factorable surface of type 2 in $\mathbb{I}^3$, except non-isotropic planes.*

Proof. The proof is by contradiction. (46) follows

$$\left(f_1'f_2\right)^2 f_1f_2'' - 2\left(f_1'f_2'\right)^2 f_1f_2 + \left(f_1f_2'\right)^2 f_2f_1'' + f_1f_2'' + 2af_1'f_2' + a^2f_1''f_2 = 0. \tag{47}$$

If f_1 or f_2 is a constant, then (47) deduces that the surface is a non-isotropic plane. Assume that f_1, f_2 are non-constant. If $f_1'' = 0$, then (47) gives a polynomial equation in f_1 :

$$2af_1'f_2' + \left[\left(f_1'f_2\right)^2 f_2'' - 2\left(f_1'f_2'\right)^2 f_2 + f_2''\right] f_1 = 0,$$

which is no possible because $af_1'f_2' \neq 0$. Then we have $f_1'' \neq 0$ and so $f_2'' \neq 0$ by symmetry. Henceforth we deal with the case $f_1''f_2'' \neq 0$. Dividing (47) with $\left(f_1'f_2'\right)^2 f_1f_2$ leads to

$$\begin{aligned} &\frac{f_2f_2''}{\left(f_2'\right)^2} + \frac{f_1f_1''}{\left(f_1'\right)^2} + \left(\frac{1}{\left(f_1'\right)^2}\right)\left(\frac{f_2''}{f_2\left(f_2'\right)^2}\right) + \\ &+2a\left(\frac{1}{f_1f_1'}\right)\left(\frac{1}{f_2f_2'}\right) + a^2\left(\frac{f_1''}{f_1\left(f_1'\right)^2}\right)\left(\frac{1}{\left(f_2'\right)^2}\right) = 2. \end{aligned} \tag{48}$$

The partial derivative of (48) with respect to u_1 and u_2 yields

$$\underbrace{\left(\frac{1}{(f_1')^2}\right)'}_{\omega_1}\underbrace{\left(\frac{f_2''}{f_2(f_2')^2}\right)'}_{\omega_2}+2a\underbrace{\left(\frac{1}{f_1f_1'}\right)'}_{\omega_3}\underbrace{\left(\frac{1}{f_2f_2'}\right)'}_{\omega_4} +a^2\underbrace{\left(\frac{f_1''}{f_1(f_1')^2}\right)'}_{\omega_5}\underbrace{\left(\frac{1}{(f_2')^2}\right)'}_{\omega_6}=0, \tag{49}$$

where $\omega_1\omega_6 \neq 0$ because $f_1''f_2'' \neq 0$. For (49), we consider two cases:

(1) Case $\omega_3 = 0$. It follows $f_1f_1' = c_0$, $c_0 \in \mathbb{R}$, $c_0 \neq 0$. Then, we have $\omega_1 = \frac{2}{c_0}$, $\omega_5 = \frac{-1}{f_1^2}$ and hence (49) reduces to the following polynomial equation in f_1

$$-\frac{c_0}{2}a^2\omega_6 + \omega_2 f_1^2 = 0$$

which is not possible because $\omega_6 \neq 0$.

(2) Case $\omega_3 \neq 0$. After dividing (49) by $\omega_1\omega_6$, we write

$$A(u_2) + 2aB(u_1)C(u_2) + a^2 D(u_1) = 0, \tag{50}$$

where

$$A(u_2) = \frac{\omega_2}{\omega_6},\ B(u_1) = \frac{\omega_3}{\omega_1},\ C(u_2) = \frac{\omega_4}{\omega_6},\ D(u_1) = \frac{\omega_5}{\omega_1}.$$

Notice also that all A, B, C and D in (50) must be constant for every u_1, u_2. Therefore, being B and D constants lead to respectively

$$f_1 = \frac{f_1'}{c_1 + c_2(f_1')^2} \tag{51}$$

and

$$\frac{f_1''}{f_1'} = f_1\left(\frac{c_3}{f_1'} + c_4 f_1'\right), \tag{52}$$

where $c_1, ..., c_4 \in \mathbb{R}$, $c_1 \neq 0$ because $\omega_3 \neq 0$. Put $p_1 = \frac{df_1}{du_1}$ and $\dot{p}_1 = \frac{dp_1}{df_1} = \frac{f_1''}{f_1'}$ in (51) and (52). The derivative of (51) with respect to f_1 gives

$$\dot{p}_1 = \frac{\left(c_1 + c_2p_1^2\right)^2}{c_1 - c_2(p_1)^2}. \tag{53}$$

Nevertheless, plugging (51) into (52) leads to

$$\dot{p}_1 = \frac{c_3 + c_4p_1^2}{c_1 + c_2p_1^2}. \tag{54}$$

Equalizing (53) and (54) refers to the polynomial equation in p_1

$$\xi_1 + \xi_2p_1^2 + \xi_3p_1^4 + \xi_4p_1^6 = 0,$$

in which the following coefficients

$$\begin{aligned}
\xi_1 &= c_1^3 - c_1c_3,\\
\xi_2 &= 3c_1^2c_2 - c_1c_4 + c_2c_3,\\
\xi_3 &= 3c_1c_2^2 + c_2c_4,\\
\xi_4 &= c_2^3
\end{aligned}$$

must vanish. Being $\xi_2 = \xi_4 = 0$ implies $c_2 = c_4 = 0$ and thus from (51) and (53) we get $f_1'' = c_1 f_1' = c_1^2 f_1$. Considering these into (47) leads to the polynomial equation in f_1

$$c_1^2 \left[f_2^2 f_2'' - f_2 \left(f_2'\right)^2 \right] f_1^3 + \left[f_2'' + 2ac_1 f_2' + a^2 c_1^2 f_2 \right] f_1 = 0,$$

where the coefficients must vanish, namely

$$f_2^2 f_2'' - f_2 \left(f_2'\right)^2 = 0, \tag{55}$$

and

$$f_2'' + 2ac_1 f_2' + a^2 c_1^2 f_2 = 0. \tag{56}$$

Integrating of (55) leads to $f_2' = c_5 f_2$ and thus $f_2'' = c_5^2 f_2$, $c_5 \in \mathbb{R}$, $c_5 \neq 0$. Substituting it into (56) gives $ac_1 + c_5 = 0$. This however contradicts with the admissibility, i.e. the assumption $af_1' f_2 + f_1 f_2' = f_1 f_2 (ac_1 + c_5) \neq 0$.

□

Theorem 4.3. *Let an affine factorable surface of type 2 in* $\mathbb{I}^3$ *have nonzero constant mean curvature* H_0. *If* f_2 *is a linear function then, for* $c_0, c_1, c_2 \in \mathbb{R}$, *we have*

$$w(y,z) = \frac{\pm 1}{2H_0 c_0} \sqrt{c_1 - 4H_0 c_0^2 (y + az)} + c_2. \tag{57}$$

Proof. If $f_2 = c_0$, $c_0 \in \mathbb{R}$, then (46) follows

$$2H_0 c_0^2 = \frac{f_1''}{\left(f_1'\right)^3}. \tag{58}$$

Solving (58) concludes

$$f_1(u_1) = \frac{\pm 1}{2H_0 c_0^2} \sqrt{-4H_0 c_0^2 u_1 + c_1} + c_2,$$

for $c_1, c_2 \in \mathbb{R}$. This provides (57). If $f_2' = c_4 \neq 0$, $c_4 \in \mathbb{R}$, then (46) reduces to

$$2H_0 \left(af_1' f_2 + c_4 f_1\right)^3 = 2ac_4 f_1' + \left[-2\left(c_4 f_1'\right)^2 f_1 + \left(c_4 f_1\right)^2 f_1'' + a^2 f_1'' \right] f_2,$$

which is a polynomial equation in f_2. It is easy to see that the the coefficient of the term of degree 3 is $2H_0 \left(f_1'\right)^3$ which cannot vanish. This completes the proof. □

Now let us assume that f_1 and f_2 to be polynomials of degree m and n, respectively. So, we get

$$f_1(u_1) = \alpha_m u_1^m + \alpha_{m-1} u_1^{m-1} + ... + \alpha_1 u_1 + \alpha_0, \; f_2(u_2) = \beta_n u_2^n + \beta_{n-1} u_2^{n-1} + ... + \beta_1 u_2 + \beta_0, \tag{59}$$

where $\alpha_m \beta_n \neq 0$. Note that if we take $m \leq 1$ or $n \leq 1$, we already derive a non-existence result because of the previous lemma. Therefore, it is assumed that $m \geq 2$ and $n \geq 2$. Next we have

Theorem 4.4. *Let an affine factorable surface of type 2 in* $\mathbb{I}^3$ *given by*

$$z = w(x,y) = f_1(y + az) f_2(z), \; a \neq 0,$$

where f_1 *and* f_2 *are polynomials of degrees greater than or equal to 2. Then it cannot have nonzero constant mean curvature* H_0.

Proof. (46) can be rewritten as

$$\begin{aligned} &\left(f_1' f_2\right)^2 f_1 f_2'' - 2\left(f_1' f_2'\right)^2 f_1 f_2 + \left(f_1 f_2'\right)^2 f_2 f_1'' + f_1 f_2'' + 2af_1' f_2' + a^2 f_1'' f_2 - \\ &-2H_0 a^3 \left(f_1' f_2\right)^3 - 6H_0 a^2 \left(f_1' f_2\right)^2 f_1 f_2' - 6H_0 a f_1' f_2 \left(f_1 f_2'\right)^2 - 2H_0 \left(f_1 f_2'\right)^3 = 0. \end{aligned} \tag{60}$$

Replacing (59) into (60), we derive a polynomial equation in u_1 and u_2 in which all coefficients vanish identically. The term of highest degree up to u_1 comes from f_1^3 and its coefficient is $2H_0 \alpha_m^3 \left(f_2'\right)^3$. This implies $f_2' = 0$ which contradicts with $n \geq 2$. □

5. Some examples

We provide and illustrate several examples for affine factorable surfaces of both types with K, H constants.

Example 5.1. *Let us consider the following surfaces:*

(1) *a minimal affine factorable surface of type 1, which is the graph*

$$z = e^{x-\frac{1}{2}(y+x)} \left[\sin \frac{1}{2}(y+x) + \cos \frac{1}{2}(y+x) \right], x, y \in [0, 2\pi];$$

(2) *an affine factorable surface of type 1 with $K = -H = -1$, which is the graph*

$$z = x(y+x), x, y \in [-\pi, \pi];$$

(3) *a flat affine factorable surface of type 2, which is the graph*

$$x = \cos(y+2z), y, z \in [0, \pi];$$

(4) *an affine factorable surface of type 2 with $H = 1$, which is the graph*

$$x = \sqrt{y+2z}, y, z \in [0, 10].$$

We can draw these surfaces as in Fig. 1,..., Fig. 4, respectively.

6. Conclusion

In the present paper, affine factorable surfaces with constant Gaussian and mean curvature were classified. Without imposing some extra conditions, finding affine factorable surfaces of type 2 with $H = const \neq 0$ is still an open problem. When f_1 or f_2 is a linear function, and both of them are polynomials, the problem was partially solved in this paper.

7. Acknowledgement

The authors would like to express their thanks to the referees for the constructive comments and recommendations which help to improve the readability and quality of the paper. The figures in this paper are plotted by using Wolfram Mathematica 11.0.

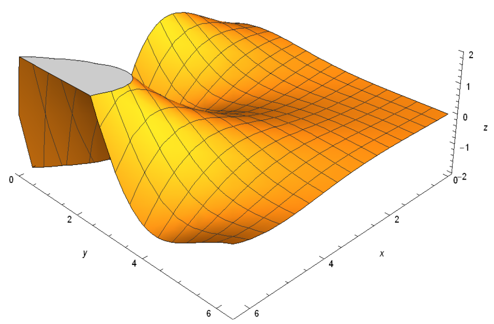


Figure 1. A minimal affine factorable surface of type 1.

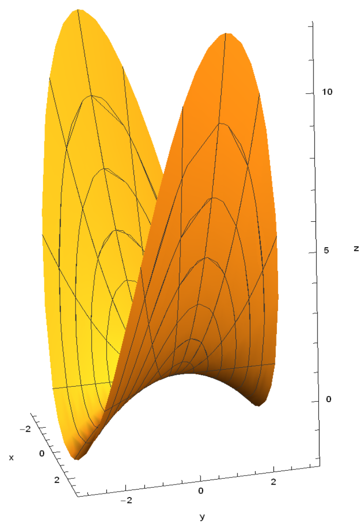


Figure 2. An affine factorable surface of type 1 with $K = -H = -1$.

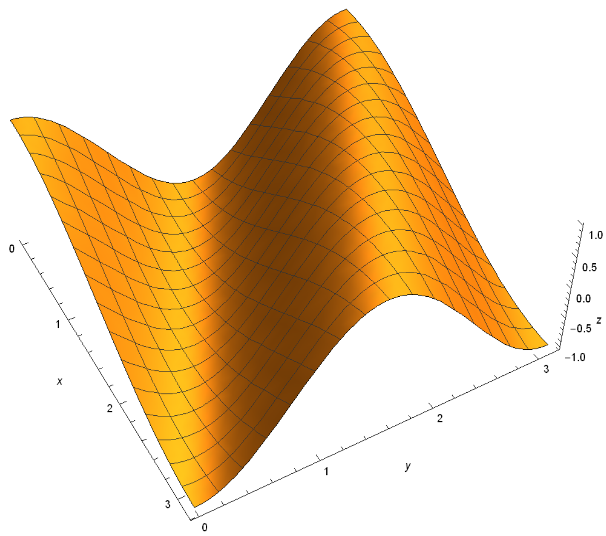


Figure 3. A flat affine factorable surface of type 2.

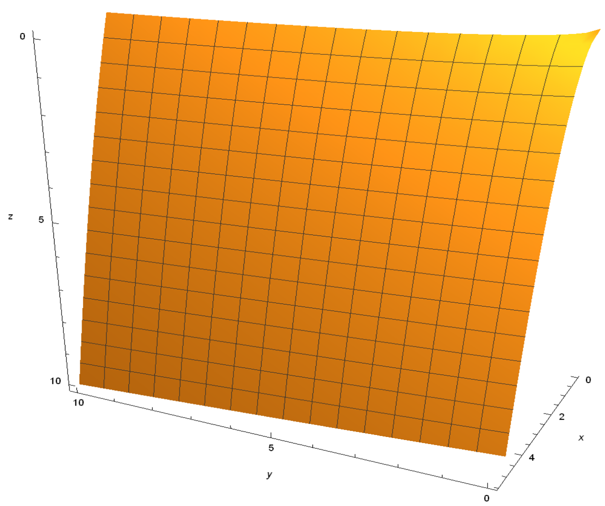


Figure 4. An affine factorable surface of type 2 with $H = 1$.

References

[1] Aydin, M. E., Ogrenmis, A.O., (2015), Homothetical and translation hypersurfaces with constant curvature in the isotropic space, BGS Proceedings, 23, pp.1-10.

[2] Aydin, M.E., Ergut, M., (2016), Isotropic geometry of graph surfaces associated with product production functions in economics, Tamkang J. Math., 47(4), pp.433-443.

[3] Aydin, M.E., (2018), Constant curvature factorable surfaces in 3-dimensional isotropic space, J. Korean Math. Soc., 55(1), pp.59-71.

[4] Bekkar, M., Senoussi, B., (2012), Factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces satisfying $\triangle r_i = \lambda_i r_i$, J. Geom., 103, pp.17-29.

[5] Bozok, H.G., Ergut M., (2019), Polynomial affine translation surfaces in Euclidean 3-space, Bol. Soc. Paran. Mat., 37(3), pp.195-202.

[6] Chekeev, A.A., Rakhmankulov, B.Z., Chanbaeva,A.I., (2018), On C_u^* - C_u and embedded uniform spaces, TWMS J. Pure Appl. Math., 9(2), pp.173-189.

[7] Chen, B.Y., Vîlcu, G. E., (2013), Geometric classifications of homogeneous production functions, Appl. Math. Comput., 225, pp.345-351.

[8] Chen, B.-Y., (2012), A note on homogeneous production models, Kragujevac J. Math., 36(1), pp.41-43.

[9] Chen, B.-Y., (2014), Solutions to homogeneous Monge-Ampère equations of homothetic functions and their applications to production models in economics, J. Math. Anal. Appl., 411, pp.223-229.

[10] Chen, B.-Y., Decu, S., Verstraelen, L., (2014), Notes on isotropic geometry of production models, Kragujevac J. Math., 8(1), pp.23-33.

[11] Cullen, M.J.P., Douglas, R.J., (1997), Applications of the Monge-Ampère equation and Monge transport problem to meterology and oceanography, In: L. A. Caffarelli, M. Milman (eds.), NSF-CBMS Conference on the Monge Ampère Equation, Applications to Geometry and Optimization, July 9-13, Florida Atlantic University, pp.33-54.

[12] Da Silva, L.C.B., (2019), Rotation minimizing frames and spherical curves in simply isotropic and semi-isotropic 3-spaces, To appear in Tamkang J. Math., arXiv:1707.06321.

[13] Da Silva, L.C.B., (2019), The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces, J. Geom. 110(31), https://doi.org/10.1007/s00022-019-0488-9.

[14] Decu, S., Verstraelen, L., (2013), A note on the isotropical geometry of production surfaces, Kragujevac J. Math., 37(2), pp.217-220.

[15] Divjak, B., (1996), The n-dimensional simply isotropic space, Zb.Rad.(Varadin), 21, pp.33-40.

[16] Erjavec, Z., Divjak, B., Horvat, D., (2011), The general solutions of Frenet's system in the equiform geometry of the Galilean, pseudo-Galilean, simple isotropic and double isotropic space, Int. Math. Forum., 6(1), pp.837-856.

[17] Gray, A., (1998), Modern Differential Geometry of Curves and Surfaces with Mathematica, CRC Press LLC.

[18] Goemans, W., Van de Woestyne, I., (2011), Translation and homothetical lightlike hypersurfaces of semi-Euclidean space, Kuwait J. Sci. Eng., 38(2A), pp.35-42.

[19] Inoguchi, J., López, R., Munteanu, M.I., (2012), Minimal translation surfaces in the Heisenberg group Nil_3, Geom. Dedicata, 161, pp.221-231.

[20] Jiu, L., Sun, H., (2007), On minimal homothetical hypersurfaces, Colloq. Math., 109, pp.239-249.

[21] Klawitter, D., (2015), Clifford Algebras: Geometric Modelling and Chain Geometries with Application in Kinematics, Springer Spektrum.

[22] Levent, A., Sahin, B., (2019), Beta Bezier Curves, Appl. Comput. Math., 18(1), pp.79-94.

[23] Liu, H., Yu, Y., (2013), Affine translation surfaces in Euclidean 3-space, Proc. Japan Acad. Ser. A Math. Sci., 89, pp.111-113 .

[24] López, R., Munteanu, M.I., (2012), Minimal translation surfaces in Sol_3, J. Math. Soc. Japan, 64(3), pp.985-1003.

[25] López, R., Moruz, M., (2015), Translation and homothetical surfaces in Euclidean space with constant curvature, J. Korean Math. Soc., 52(3), pp.523-535.

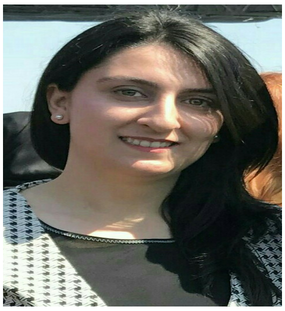
[26] López, R., (2011), Minimal translation surfaces in hyperbolic space, Beitr. Algebra Geom., 52(1), pp.105-112.

[27] López, R., (2016), Separation of variables in equation of mean curvature type, Proc. R. Soc. Edinb. Sect. A Math.,146(5), pp.1017-1035.

[28] Meng, H., Liu, H., (2009), Factorable surfaces in Minkowski space, Bull. Korean Math. Soc., 46(1), pp.155-169.

[29] Milin-Sipus, Z., (2014), Translation surfaces of constant curvatures in a simply isotropic space, Period. Math. Hung., 68, pp.160-175.
[30] Milin-Sipus, Z., Divjak, B., (1998), Curves in $n-$dimensional $k-$ isotropic space, Glas. Mat.Ser.III, 33(53), pp.267-286.
[31] Milin-Sipus, Z., Divjak, B., (1999), Involutes and evolutes in $n-$ dimensional simply isotropic space, Zbornik radova, 23(1), pp.71-79.
[32] Munteanu, M.I., Nistor, A.I., (2009), Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space, In: Diff. Geo., Proceedings of the VIII International Colloquium, Santiago de Compostela, Spain, World Scientific, Hackensack, pp.316-320.
[33] Ogrenmis, A.O., (2016), Rotational surfaces in isotropic spaces satisfying Weingarten conditions, Open Physics, 14(9), pp.221-225.
[34] Onishchick, A., Sulanke, R., (2006), Projective and Cayley-Klein Geometries, Springer.
[35] Pottmann, H., Grohs, P., Mitra, N.J., (2009), Laguerre minimal surfaces, isotropic geometry and linear elasticity, Adv. Comput. Math.,31, pp.391-419.
[36] Pottmann, H., Opitz, K., (1994), Curvature analysis and visualization for functions defined on Euclidean spaces or surfaces, Comput. Aided Geom. Design, 11, pp.655-674.
[37] Pressley, A., (2012), Elementary Differential Geometry, Springer-Verlag, London.
[38] Sachs, H., (1990), Isotrope Geometrie des Raumes, Vieweg Verlag, Braunschweig.
[39] Simon, L., (1977), Equations of mean curvature type in 2 independent variables, Pacific J. Math., 69(1), pp.245-268.
[40] Strubecker, K., (1977), Über die isotropen Gegenstücke der Minimalfläche von Scherk, J. Reine Angew. Math., 293, pp.22-51.
[41] Thurston, W.P., (1997), Three-dimensional geometry and topology, Princenton Math., Ser. 35, Princenton Univ. Press, Princenton, NJ.
[42] Ushakov, V., (2000), The explicit general solution of trivial Monge-Ampère equation, Comment. Math. Helv., 75(1), pp.125-133.
[43] Van de Woestyne, I., (1995), Minimal homothetical hypersurfaces of a semi-Euclidean space, Results. Math., 27, pp.333-342.
[44] Yaglom, I. M., (1979), A simple non-Euclidean Geometry and Its Physical Basis, An elementary account of Galilean geometry and the Galilean principle of relativity, Heidelberg Science Library. Translated from the Russian by Abe Shenitzer. With the editorial assistance of Basil Gordon. Springer-Verlag, New York-Heidelberg.
[45] Yoon, D.W., (2013), Minimal translation surfaces in $\mathbb{H}^2 \times \mathbb{R}$, Taiwanese J. Math., 17(5), pp.1545-1556.
[46] Yoon, D.W., Lee, J.W., (2014), Translation invariant surfaces in the 3-dimensional Heisenberg group, Bull. Iranian Math. Soc., 40(6), pp.1373-1385.
[47] Yoon, D.W., Lee, C. W., Karacan, M.K., (2013), Some translation surfaces in the 3-dimensional Heisenberg group, Bull. Korean Math. Soc., 50(4), pp.1329-1343.
[48] Yu, Y., Liu, H., (2007), The factorable minimal surfaces, Proceedings of the Eleventh International Workshop on Differential Geometry, Kyungpook Nat. Univ. Taegu, 11, pp.33-39.
[49] Zong, P., Xiao, L., Liu, H.L., (2015), Affine factorable surfaces in three-dimensional Euclidean space, (Chinese) Acta Math. Sinica (Chin. Ser.), 58(2), pp.329-336.

Muhittin Evren Aydin was born in 1986 in Elazig, Turkey. He received M.Sc. and Ph.D. degrees from the Firat University, Elazig. Dr. Aydin is currently an associated professor of Department of Mathematics at Firat University. His research interests include economics and differential geometry.



Ayla Erdur was born in 1992 in Elazig, Turkey. She received M.Sc.degree from the Firat University, Elazig. She is currently doctorating in Namik Kemal University, Tekirdag, Turkey. Her research interests include differential geometry.

Mahmut Ergut, for a photograph and biography, see TWMS J. Pure Appl. Math., V.1, N.1, 2010, p.85.