TWMS J. Pure Appl. Math. V.11, N.2, 2020, pp.173-188

INVERTIBLE SQUARE MATRICES OVER RESIDUATED LATTICES

S. BALAEI[1], E. ESLAMI[1] AND A. BORUMAND SAEID[1]

Abstract. In this paper, we discuss the invertibility of square matrices over residuated lattices. Some necessary and sufficient conditions under which a matrix is invertible are examined. Invertible matrices are applied to solve special systems of equations with coefficients from residuated lattices. Solvability of these systems, based on the structural properties of the algebra of square matrices over residuated lattices are investigated.

Keywords: residuated lattices, matrices over residuated lattices, invertible matrices, linear system of equations.

AMS Subject Classification: 15A30, 15B99, 15B22.

1. Introduction

Residuated lattices are algebraic models of substructural logics [5]. These logics include many non-classical logics such as fuzzy logic, linear logic, relevant logic, many-valued logics, Intuitionistic logic and Łukasiewicz logic, where the last two are modelled by the classes of Heyting algebras and MV-algebras, respectively. Therefore, the residuated lattices can be thought of as ordered sets of degrees [5].

Matrix theory plays an important role in various areas of science and engineering to represent different types of binary relations. For instance, it appears in formal concept analysis which is initially developed by Wille in 1980's [18]. In applications of formal concept analysis, the relationship between the objects and the attributes is often not binary; zero or one, but rather that there is only a degree to which the object has the attribute [6].
Lattice matrices, i.e. matrices whose entries are elements of a lattice are first developed by Y. Give'on in 1964 [7], while Boolean matrices are given in 1950's by Hohn, R. E and Schissler R. L [8] and Luce, R.D [10].

Matrices over different algebraic structures are also studied ([11], [12], [14], [16], [15], [17]).
The above mentioned articles do not construct an algebra of such matrices. We introduced an algebra very close to a residuated lattice of matrices over residuated lattices in [1]. When matrices are used for solving a system of equations, the principal questions arise how to decide whether a matrix is invertible, and if so, how to compute the inverse matrix. For matrices over fields the answers are well-known: A matrix over a field is invertible if and only if its determinant is non zero and its inverse can be found for example with the help of Gauss-Jordan technique. A similar useful criterion for invertible matrices over arbitrary semirings is not known in general. There are results for invertible matrices over Boolean algebras ([8], [14]). Furthermore, there exist generalizations to matrices over certain ordered algebraic structures [2], and there are

[1]Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran
e-mail: balaeisaber@yahoo.com, eslami@uk.ac.ir, arsham@uk.ac.ir
Manuscript received May 2018.

results for matrices over Brouwerian lattices [19] and distributive lattices [9]. Also for matrices over certain commutative semirings some results are known ([4], [15]).

In this paper we present criteria for invertible matrices over residuated lattices. We consider invertibility of matrices with respect to special residuated lattices. The relations between invertible matrices, filters and ideals of the involved residuated lattices are investigated. The paper is organized as follows:

In section 2, some definitions and theorems that we need in the remaining, are given. In section 3, the concept of an invertible element in $\mathbb{M}_{n\times n}(L)$ is introduced. We characterize invertible $n \times n$ matrices over L. These are needed for solving systems of special equations over residuated lattice L. Finally in section 4, we examine some results on invertibility of square matrices and state some applications of invertible matrices.

2. Preliminaries

In this section, we recall the definition of $\mathbb{M}_{n\times n}(L)$ and some basic properties that we need in the sequel.

Definition 2.1. [1] *An algebra* $L = (L, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 2, 0, 0)$*; is a residuated lattice if it satisfies the following conditions:*

(R1): $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
(R2): $(L, \odot, 1)$ *is a monoid,*
(R3): *for all* $x, y, z \in L$ *we have* $x \odot y \leqslant z \Leftrightarrow x \leqslant y \rightarrow z \Leftrightarrow y \leq x \rightsquigarrow z$*.*

L is called commutative if the operation $\odot$ is commutative. In this case $x \rightarrow y = x \rightsquigarrow y$, for all $x, y \in L$.

Examples 2.1. [13] *For commutative residuated lattices, we may consider three basic t-norm algebras. (1) Lukasiewicz algebra in which* $L = [0, 1]$, $x \odot y = max\{0, x + y - 1\}$, $x \rightarrow y = x \rightsquigarrow y = min\{1, 1 - x + y\}$ *for all* $x, y \in L$*, (2) Gödel algebra where* $L = [0, 1]$, $x \odot y = min\{x, y\}$, $x \rightarrow y = x \rightsquigarrow y = 1$ *if* $x \leq y$ *and* y *otherwise, (3) Product algebra in which* $L = [0, 1]$, $x \odot y = xy$, $x \rightarrow y = x \rightsquigarrow y = 1$ *if* $x \leq y$ *and* $\dfrac{y}{x}$ *elsewhere.*
For non-commutative case consider the lattice $(L, \vee, \wedge, 0, 1)$ *with* $L = \{0, a, b, c, 1\}$*, where* $0 < a < b < c < 1$ *equipped with the operations* [3].

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Definition 2.2. [1] *Let* $M_{n\times n}(L) = \{[a_{ij}]_{n\times n} \mid 1 \leq i, j \leq n, a_{ij} \in L\}$*. Define:*
(1) $[a_{ij}]_{n\times n} \sqcup [b_{ij}]_{n\times n} = [a_{ij} \vee b_{ij}]_{n\times n}$,
(2) $[a_{ij}]_{n\times n} \sqcap [b_{ij}]_{n\times n} = [a_{ij} \wedge b_{ij}]_{n\times n}$,
(3) $[a_{ij}]_{n\times n} \boxdot [b_{ij}]_{n\times n} = [c_{ij}]_{n\times n}$*, such that* $c_{ij} = \bigvee_{t=1}^{n}(a_{it} \odot b_{tj})$,
(4) $[a_{ij}]_{n\times n} \rhd [b_{ij}]_{n\times n} = [c_{ij}]_{n\times n}$*, such that* $c_{ij} = \bigwedge_{t=1}^{n}(a_{jt} \rightarrow b_{it})$,
(5) $[a_{ij}]_{n\times n} \lhd [b_{ij}]_{n\times n} = [c_{ij}]_{n\times n}$*, such that* $c_{ij} = \bigwedge_{t=1}^{n}(a_{ti} \rightarrow b_{tj})$.
Let $A, B \in M_{n\times n}(L)$*. Define:*
(6) $A \preceq B$ *iff* $A \sqcap B = A$ *iff* $A \sqcup B = B$ *iff* $a_{ij} \leq b_{ij}$ *for all* $1 \leq i, j \leq n$,
(7) $\perp = [a_{ij}]_{n\times n}$ *where* $a_{ij} = 0$ *for all* $1 \leq i, j \leq n$,

(8) $\top = [a_{ij}]_{n\times n}$ *where* $a_{ij} = 1$ *for all* $1 \le i, j \le n$,
(9) $I_n = [a_{ij}]_{n\times n}$ *where* $a_{ij} = 1$ *if* $i = j$ *and* $a_{ij} = 0$ *otherwise.*

Based on the above, we introduced $(M_{n\times n}(L), \sqcup, \sqcap, \boxdot, \rhd, \lhd, \perp, \top, I_n)$ as an algebra of type $(2, 2, 2, 2, 2, 0, 0, 0)$ denoted by $\mathbb{M}_{n\times n}(L)$ in [1]. We showed that this algebra behaves like a residuated lattice, but there are some differences. We explored a set of like and unlike properties of $M_{n\times n}(L)$. Below we give some results from [1] that are needed for our purpose in this paper.

Lemma 2.1. [1] *Let* L *be a residuated lattice. Then*
(*a*) $(M_{n\times n}(L), \sqcup, \sqcap, \perp, \top)$ *is a bounded lattice,*
(*b*) $(M_{n\times n}(L), \boxdot, I_n)$ *is a monoid,*
(*c*) $(\boxdot, \rhd)$ *and* $(\boxdot, \lhd)$ *are two adjont pairs, i.e.*
$A \boxdot B \preceq C$ *iff* $A \preceq B \rhd C$ *iff* $B \preceq A \lhd C$, *for all* A,B *and* $C \in M_{n\times n}(L)$.

Lemma 2.2. [1] *Let* A, B, C *and* $Z \in M_{n\times n}(L)$. *Then the following properties hold in algebra* $\mathbb{M}_{n\times n}(L)$.
1) $A \preceq B$ *then* $A \boxdot Z \preceq B \boxdot Z$ *and* $Z \boxdot A \preceq Z \boxdot B$,
2) $(A \boxdot B) \rhd C = A \rhd (B \rhd C)$ *and* $(B \boxdot A) \lhd C = B \lhd (A \lhd C)$,
3) $A \boxdot (B \sqcup C) = (A \boxdot B) \sqcup (A \boxdot C)$,
4) $(B \sqcup C) \boxdot A = (B \boxdot A) \sqcup (C \boxdot A)$,
5) $A \boxdot (B \sqcap C) \preceq (A \boxdot B) \sqcap (A \boxdot C)$,
6) $(B \sqcap C) \boxdot A \preceq (B \boxdot A) \sqcap (C \boxdot A)$.

Lemma 2.3. [1] *For every* $A, B \in M_{n\times n}(L)$, $A \preceq B$ *iff* $I_n \preceq A \rhd B$ *iff* $I_n \preceq A \lhd B$.

Definition 2.3. [1] *The element* $A \in M_{n\times n}(L)$ *is called complemented if there is an element* $B \in M_{n\times n}(L)$ *such that* $A \sqcup B = \top$ *and* $A \sqcap B = \perp$. *Such an element* B *is called a complement element of* A.

If L is complemented, then every $A \in M_{n\times n}(L)$ has a unique complement B denoted by $B = A^c$. The set of all complemented elements of $M_{n\times n}(L)$ is denoted by $B(M_{n\times n}(L))$.

Proposition 2.1. [1] L *is complemented iff* $\mathbb{M}_{n\times n}(L)$ *is complemented.*

Let $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$. Define $A^* = [a^*_{ij}]_{n\times n}$, where $a^*_{ij} = a_{ij} \to 0$ for all $1 \le i, j \le n$.

Theorem 2.1. [1] *Let* $C \in B(M_{n\times n}(L))$. *Then* $C^c = C^*$ *and* $C^{**} = C$.

For $A \in M_{n\times n}(L)$, the powers of A, A^n for integers $n \ge 0$ are defined inductively as : $A^0 = I_n$ and $A^n = A^{n-1} \boxdot A$.

Definition 2.4. [1] *An element* $A \in M_{n\times n}(L)$ *is called idempotent iff* $A^2 = A$, *and it is called nilpotent if there exists a nutural number* n *such that* $A^n = \perp$. *The least such* n *is called the nilpotence order of* A *and is denoted by* $ord(A)$; *if there is no such* n *then* $ord(A) = \infty$.

Theorem 2.2. [1] *If* L *is a locally finite, then* $M_{n\times n}(L^*)$ *is locally finite (every* $A \in M_{n\times n}(L^*)$ *is nilpotent), where* $L^* = L \setminus \{1\}$.

Definition 2.5. [1] *A nonempty subset* S *of* $M_{n\times n}(L)$ *is called a filter of* $M_{n\times n}(L)$ *if the following conditions hold:*
1) if $A, B \in S$, *then* $A \boxdot B \in S$,
2) if $A \in S$, $B \in M_{n\times n}(L)$ *and* $A \preceq B$ *then* $B \in S$.
A filter S *of* $M_{n\times n}(L)$ *is proper if* $S \neq M_{n\times n}(L)$. *Clearly,* S *is a proper filter iff* $\perp \notin S$.

Theorem 2.3. [1] *$M_{n\times n}(F)$ for $F \subseteq L$ is a filter of $M_{n\times n}(L)$ iff F is a filter of L.*

Corollary 2.1. [1] *$M_{n\times n}(F)$ for $F \subseteq L$ is a proper filter of $M_{n\times n}(L)$ iff F is a proper filter of L.*

For every $A \in M_{n\times n}(L)$, we put $A^- = A \rhd \perp$ and $A^\sim = A \lhd \perp$. Then,

Proposition 2.2. [1] *If L is a residuated lattice, then the sets $M_{n\times n}(L)^- = \{A \in M_{n\times n}(L) : A^- = \perp\}$ and $M_{n\times n}(L)^\sim = \{A \in M_{n\times n}(L) : A^\sim = \perp\}$ are proper filters of $M_{n\times n}(L)$.*

Definition 2.6. [1] *A subset $S \subseteq M_{n\times n}(L)$ is called an ideal of $M_{n\times n}(L)$ if the following conditions hold:*
i) $\perp \in S$,
ii) if $A, B \in S$, then $A \sqcup B \in S$,
iii) if $A \in M_{n\times n}(L)$, $B \in S$ and $A \preceq B$, then $A \in S$.
An ideal S of $M_{n\times n}(L)$ is called proper if $S \neq M_{n\times n}(L)$. Clearly, S is proper iff $\top \notin S$.

Theorem 2.4. [1] *$M_{n\times n}(I)$ for $I \subseteq L$ is an ideal of $M_{n\times n}(L)$ iff I is an ideal of L.*

Corollary 2.2. [1] *For an ideal I of L, $M_{n\times n}(I)$ is proper iff I is proper.*

Definition 2.7. [1] *For $X \subseteq M_{n\times n}(L)$, $B \in M_{n\times n}(L)$, we define*
$[X, B] = \{A \in M_{n\times n}(L) : A \boxdot B \in X\}$, in particular if $X = \{\perp\}$, then $[\{\perp\}, B]$ is called an annihilator of the matrix B. Generally for $X, Y \subseteq M_{n\times n}(L)$ we define $[X, Y] = \{A \in M_{n\times n}(L) : A \boxdot B \in X,$ for all $B \in Y\}$.

Theorem 2.5. [1] *For subsets $X, Y \subseteq M_{n\times n}(L)$, if X is an ideal of $M_{n\times n}(L)$, then $[X, Y]$ is an ideal of $M_{n\times n}(L)$.*

Corollary 2.3. [1] *For every $B \in M_{n\times n}(L)$, $[\{\perp\}, B]$ is an ideal of $M_{n\times n}(L)$.*

3. Invertible matrices

Throughout this paper, we assume that L is a commutative residuated lattice and $M_{n\times n}(L)$ is the set of all $n \times n$ matrices over L.

Definition 3.1. *Let $A \in M_{n\times n}(L)$. Then $B \in M_{n\times n}(L)$ is called a right inverse (left inverse) of matrix A if $A \boxdot B = I_n (B \boxdot A = I_n)$. Also, if $A \boxdot B = B \boxdot A = I_n$, then B is called an inverse of A.*

It is easy to see that:

Lemma 3.1. *If $A \in M_{n\times n}(L)$ has a left inverse B and a right inverse C, then $B = C$.*

Corollary 3.1. *If $A \in M_{n\times n}(L)$ has an inverse, it is unique.*

If the inverse of A exists we denote it by A^{-1} and we say that A is invertible. The set of all invertible matrices in $M_{n\times n}(L)$ is denoted by $S_{n\times n}(L)$. It can be shown that $S_{n\times n}(L)$ is closed under $\boxdot$ as follows:

Lemma 3.2. *If $A, B \in M_{n\times n}(L)$ are invertible, then $A \boxdot B$ is also invertible and the inverse is $(B^{-1} \boxdot A^{-1})$.*

As it is said in the Introduction section, the invertible matrices over some algebraic structures such as fields, Boolean algebras and certain commutative semirings are known. To investigate the invertibility of a matrix in our case, i.e. $A \in M_{n\times n}(L)$, we employ some conventions and notations defined as below:
For matrix $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ we define $R_i(A)$ and $C_i(A)$, i-th row and i-th column of matrix A, respectively. Also, we define $R_i(B) \otimes C_j(A) = (b_{i1} \odot a_{1j}) \vee (b_{i2} \odot a_{2j}) \vee ... \vee (b_{in} \odot a_{nj})$.

Remark 3.1. *Let $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ have a left inverse $B = [b_{ij}]_{n\times n} \in M_{n\times n}(L)$. i.e.*

$$\begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ . & & & \\ . & & & \\ . & & & \\ b_{n1} & b_{n2} & ... & b_{nn} \end{pmatrix} \boxdot \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} = I_n.$$

Therefore, the following conditions are necessary conditions for matrix B being a left inverse of A.

I) $R_i(B) \otimes C_j(A) = \bigvee_{t=1}^{n} (b_{it} \odot a_{tj}) = 0$ for all $i \neq j$.

II) $R_i(B) \otimes C_j(A) = \bigvee_{t=1}^{n} (b_{it} \odot a_{tj}) = 1$ for all $i = j$.

Note that by (I) $a_{tj} \leq b_{it}^*$ for all $1 \leq i, j, t \leq n$ where $i \neq j$, since in residuated lattice L, $a \odot b = 0$ iff $a \leq b^*$, for all $a, b \in L$.

Definition 3.2. *If $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$, for all $1 \leq i, j \leq n$, define*
$E_i^l(A) = \bigvee_{j=1}^{n} ((a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)}) \odot a_{ji})$. Such that

$$\triangle(u) = \begin{cases} k+1 & \text{if } u = n+k \\ u+1 & \text{if } 1 \leq u < n \\ 1 & \text{if } u = n. \end{cases}$$

Examples 3.1. [1] *Let $L = \{0, a, b, 1\}$ with $0 < a, b < 1$ such that a, b are incomparable. Then L is a commutative residuated lattice relative to the following operations.*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

For $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_{2\times 2}(L)$, $E_1^l(A) = [(b^* \odot a) \vee (a^* \odot b)] = 1$ and $E_2^l(A) = [(a^* \odot b) \vee (b^* \odot a)] = 1$.

Theorem 3.1. *The matrix $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ is left invertible iff*
$E^l(A) = \bigwedge_1^n E_i^l(A) = 1$.

Proof. Let $E^l(A) = \bigwedge_1^n E_i^l(A) = 1$. Consider $B = [b_{ij}]_{n\times n}$, such that $b_{ij} = (a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)})$ for all $1 \leq i, j \leq n$. It is enough to show $B \boxdot A = I_n$, i.e. the conditions (I) and (II) of Remark 3.1 hold.

(II) holds since
$\bigvee_{t=1}^{n}(b_{it} \odot a_{ti}) = \bigvee_{t=1}^{n}((a^*_{t\triangle(i)} \wedge a^*_{t\triangle(i+1)} \wedge ... \wedge a^*_{t\triangle(i+n-2)}) \odot a_{ti}) = E^l_i(A) = 1$.
To prove (I), for all $1 \leq i, j, t \leq n$ such that $i \neq j$, we have
$b_{it} \odot a_{tj} = (a^*_{t\triangle(i)} \wedge a^*_{t\triangle(i+1)} \wedge ... \wedge a^*_{t\triangle(i+n-2)}) \odot a_{tj} = 0$, since for all $1 \leq j \leq n$ there is $i \leq k \leq i+n-2$ such that $\triangle(k) = j$ (if $1 < j \leq n$ take $k = j-1$ and if $j = 1$ take $k = n$) and $a^*_{tj} \odot a_{tj} = 0$. Therefore $\bigvee_{t=1}^{n} b_{it} \odot a_{tj} = 0$.
Conversely, let $A = [a_{ij}]_{n\times n}$ have a left inverse $B = [b_{ij}]_{n\times n}$, we show that $E^l(A) = 1$, i.e. $E^l_i(A) = 1$ for all $1 \leq i \leq n$. It is enough to show:
($\star$) $((a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)}) \odot a_{ji}) \geq b_{ij} \odot a_{ji}$,
because if ($\star$) holds for all $1 \leq i, j \leq n$ then by (II) of Remark 3.1 we get $E^l_i(A) \geq \bigvee_{j=1}^{n}(b_{ij} \odot a_{ji}) = 1$, for all $1 \leq i, j \leq n$.
On the other hand by (I) of Remark 3.1 we have $a_{jk} \leq b^*_{ij}$ for all $i \neq k$. Then $a^*_{jk} \geq b^{**}_{ij} \geq b_{ij}$. By definition of $\triangle(i)$, we have $i \notin \{\triangle(i), \triangle(i+1), ..., \triangle(i+n-2)\}$, so $(a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)}) \geq b_{ij}$. Therefore ($\star$) is satisfied and we get
$\bigvee_{j=1}^{n}((a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-1)}) \odot a_{ji}) \geq \bigvee_{j=1}^{n} b_{ij} \odot a_{ji} = 1$. □

Corollary 3.2. *If $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ is left invertible, then its left inverse is $A^{-1} = [a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)}]_{n\times n}$.*

Similar discussions can be given for the right invertible matrices as follows:

Definition 3.3. *If $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$, for all $1 \leq i, j \leq n$, define $E^r_i(A) = \bigvee_{j=1}^{n}(a_{ij} \odot (a^*_{\triangle(i)j} \wedge a^*_{\triangle(i+1)j} \wedge ... \wedge a^*_{\triangle(i+n-2)j}))$. Such that*

$$\triangle(u) = \begin{cases} k+1 & \text{if } u = n+k \\ u+1 & \text{if } 1 \leq u < n \\ 1 & \text{if } u = n. \end{cases}$$

Theorem 3.2. *The matrix $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ is right invertible iff $E^r(A) = \bigwedge_1^n E^r_i(A) = 1$.*

Corollary 3.3. *If $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ is right invertible, then its right inverse is $A^{-1} = [a^*_{\triangle(i)j} \wedge a^*_{\triangle(i+1)j} \wedge ... \wedge a^*_{\triangle(i+n-2)j}]^t_{n\times n}$.*

Corollary 3.4. *$A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ is invertible iff $E^r(A) = E^l(A) = E(A) = 1$.*

Proof. The proof follows from Theorems 3.1, 3.2 and Lemma 3.1. □

Theorem 3.3. *If $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$ invertible, then A^* is also invertible, and $(A^{-1})^* = (A^*)^{-1}$, where $A^* = [a^*_{ij}]_{n\times n}$.*

Proof. It is enough to show that $(A^{-1})^*$ is the inverse of A^*. By Theorem 3.2, $A^{-1} = [b_{ij}]_{n\times n}$ where $b_{ij} = a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)}$ such that $\bigvee_{j=1}^{n} b_{ij} \odot a_{ji} = 1$. Set $A^* = [a^*_{ij}]_{n\times n}$. We prove that for all $1 \leq i \leq n$, $E^l_i(A^*) = 1$, i.e. $\bigvee_{j=1}^{n} b^*_{ij} \odot a^*_{ji} = 1$.

$$\begin{aligned} \bigvee_{j=1}^{n} b^*_{ij} \odot a^*_{ji} &= \bigvee_{j=1}^{n} (a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)})^* \odot a^*_{ji} \\ &= \bigvee_{j=1}^{n} (a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)})^{**} \odot a_{ji} \end{aligned}$$

$$\geq \bigvee_{j=1}^{n} (a^*_{j\triangle(i)} \wedge a^*_{j\triangle(i+1)} \wedge ... \wedge a^*_{j\triangle(i+n-2)}) \odot a_{ji} = 1.$$

Then $E^l_i(A^*) = 1$ for all $1 \leq i \leq n$ and A^* is left invertible. The proof for right invertibility of A^* is similar. □

We end up this section by some results about invertibility of 2×2 matrices over residuated lattices.

Definition 3.4. *The residuated lattice L is called a domain if $a \odot b = 0$ implies $a = 0$ or $b = 0$, for all $a, b \in L$.*

Examples 3.2. *The Gödel and product residuated lattices are domain. But Łukasiewiczs residuated lattices as well as the residuated lattice in Example 3.1, are not domain.*

Proposition 3.1. *Let L be a domain. If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times 2}(L)$ is invertible, then $a_{11} \odot a_{22} \neq a_{12} \odot a_{21}$.*

Proof. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times 2}(L)$ have an inverse $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_{2\times 2}(L)$ and $a_{11} \odot a_{22} = a_{12} \odot a_{21}$. Then $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \boxdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = I_2$. So we have
(i) $(a_{11} \odot b_{11}) \vee (a_{12} \odot b_{21}) = 1$,
(ii) $(a_{21} \odot b_{12}) \vee (a_{22} \odot b_{22}) = 1$,
(iii) $(a_{11} \odot b_{12}) \vee (a_{12} \odot b_{22}) = 0$ which implies $(a_{11} \odot b_{12}) = (a_{12} \odot b_{22}) = 0$,
(iv) $(a_{21} \odot b_{11}) \vee (a_{22} \odot b_{21}) = 0$ which implies $(a_{21} \odot b_{11}) = (a_{22} \odot b_{21}) = 0$.
Since L is a domain, from (iii) we realize four cases. Each case contradicts (i) or (ii). Similarly we have four cases for (iv), again each contradicts (i) or (ii). □

In the following theorem we show that there are only two invertible 2×2 matrices over Łukasiewicz, product or Gödel residuated lattices.

Theorem 3.4. *If L is a Łukasiewicz (product or Gödel) residuated lattice, then only $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_{2\times 2}(L)$ are right invertible.*

Proof. Let L be a Łukasiewicz residuated lattice. By Theorem 3.2, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is right invertible iff $E^r(A) = 1$. On the other hand $a^* = a \to 0 = min\{1, 1 - a\} = 1 - a$, for all $a \in L$. Then

$$\begin{aligned} E^r(A) = 1 \Leftrightarrow & [(a \odot (1 - c)) \vee (b \odot (1 - d))] \wedge [(c \odot (1 - a)) \vee (d \odot (1 - b))] = 1 \\ \Leftrightarrow & [max\{0, a - c\} \vee max\{0, b - d\}] \wedge [max\{0, c - a\} \vee max\{0, d - b\}] = 1 \\ \Leftrightarrow & max\{0, a - c, b - d\} \wedge max\{0, c - a, d - b\} = 1 \\ \Leftrightarrow & min\{max\{a - c, b - d\}, max\{c - a, d - b\}\} = 1. \end{aligned}$$

Then $a - c = d - b = 1$ or $b - d = c - a = 1$, i.e. $a = d = 1$ and $c = b = 0$ or $b = c = 1$ and $a = d = 0$.
So only $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_{2\times 2}(L)$ are right invertible. We can similary prove that only these matrices are left invertible. The proof for product and Gödel residuated lattices is simillar. □

Remark 3.2. *The convers of the Theorem 3.3 is not generally true. For instance, if L is a Gödel residuated lattice and* $A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}$, *then by Theorem 3.4,* $A^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *which is invertible and* $(A^*)^{-1} = A^*$, *but A is not invertible.*

4. Some results on invertibility of square matrices

In this section, first we state some results on invertibility of diagonal matrices. Then we investigate necessary and sufficient conditions for invertibility of general square matrices. In the sequel, using structural properties of $M_{n\times n}(L)$ given in the previous sections, we consider invertibility of matrices with respect to special residuated lattices, the relations between invertible matrices themselves, and the relationships between invertible matrices and some special filters and ideals of $M_{n\times n}(L)$ are also investigated.
In the following, some results for invertible diagonal matrices over residuated lattices are given.

Theorem 4.1. $A = diag(a_{11}, a_{22}, ..., a_{nn}) \in M_{n\times n}(L)$ *is invertible iff* $a_{ii} = 1$, *for every* $1 \le i \le n$.

Proof. If $A \boxdot B = [c_{ij}]_{n\times n}$, then $c_{ij} = \bigvee_{t=1}^{n}(a_{it} \odot b_{tj}) = 0$ for $i \ne j$. This implies that for $i \ne j$, $(a_{it} \odot b_{tj}) = 0$ for all $1 \le t \le n$. Furthermore, $c_{ii} = \bigvee_{t=1}^{n}(a_{it} \odot b_{ti}) = 1$ for every $1 \le i \le n$. Since $a_{ij} = 0$ for all $i \ne j$, $1 \le i, j \le n$, we have $c_{ii} = (a_{ii} \odot b_{ii}) = 1$ for every $1 \le i \le n$. This means that each a_{ii} is invertible and therefore $a_{ii} = 1$ for $1 \le i \le n$. □

Proposition 4.1. *If* $A \in M_{n\times n}(L)$ *has a right(left) inverse B and* $a_{ii} = 1$, *for all* $1 \le i \le n$. *Then B is a diagonal matrix.*

Proof. By assumption we have $A \boxdot B = I_n$. Then for $1 \le i, j \le n$

$$(a_{ii} \odot b_{ij}) \vee (\bigvee_{1\le t\ne i\le n}^{n} a_{it} \odot b_{tj}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

So for $i \ne j$ we have $(a_{ii} \odot b_{ij}) = 0$, but since $a_{ii} = 1$, $b_{ij} = 0$, i.e. B is diagonal matrix.
Similarly if $A \in M_{n\times n}(L)$ has a left inverse B and $a_{jj} = 1$, for all $1 \le j \le n$. Then B is diagonal matrix and $B \boxdot A = I_n$. □

Corollary 4.1. *If* $A \in M_{n\times n}(L)$ *is invertible with* $a_{ii} = 1$, *for* $1 \le i \le n$, *then* $A = I_n$.

Corollary 4.2. *If A is invertible and for some D,* $A \sqcup D = I_n$, *then* $A = I_n$ *and D is a diagonal matrix.*

Examples 4.1. *Let* $A = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \in M_{2\times 2}(L)$, *where L is a residuated lattice defined in Example 3.1. If A is invertible, then by Theorem 3.1 or Corollary 4.1, we see that,* $a = b = 0$.

In the following Theorem, we give some necessary and sufficient conditions for a square matrix to be invertible.

Theorem 4.2. *Let* $A = [a_{ij}]_{n\times n} \in M_{n\times n}(L)$. *Then A is invertible iff the following conditions holds.*
i) for every $1 \le i, j \le n$ *there are* $x_{ij}, b_{ij}, y_{ij} \in L$ *such that*
$(a_{ij} \odot b_{ji}) \vee x_{ij} = (b_{ji} \odot a_{ij}) \vee y_{ij} = 1$,
ii) for every $1 \le i \le n$,
$(a_{i1} \odot b_{1i}) \vee ... \vee (a_{in} \odot b_{ni}) = 1$ *and* $a_{ij} \odot b_{ji} \odot a_{ik} \odot b_{ki} = 0$, *for* $j \ne k$, *and*
iii) for every $1 \le j \le n$
$(b_{j1} \odot a_{1j}) \vee ... \vee (b_{jn} \odot a_{nj}) = 1$ *and* $b_{ji} \odot a_{ij} \odot b_{jk} \odot a_{kj} = 0$, *for* $i \ne k$.

Proof. Let $A \in M_{n\times n}(L)$ be invertible. Then by Remark 3.1, there exists a matrix B such that $A \boxdot B = B \boxdot A = I_n$. So we have

$$(1)\begin{cases} \bigvee_{t=1}^{n}(a_{it}\odot b_{tj}) = 0 & \text{if } i\neq j\\ \bigvee_{t=1}^{n}(a_{it}\odot b_{tj}) = 1 & \text{if } i = j\end{cases}$$

$$(2)\begin{cases} \bigvee_{t=1}^{n}(b_{it}\odot a_{tj}) = 0 & \text{if } i\neq j\\ \bigvee_{t=1}^{n}(b_{it}\odot a_{tj}) = 1 & \text{if } i = j\end{cases}$$

Then (ii) and (iii) holds. To prove (i) we put $x_{ij} = \bigvee_{k\neq j}(a_{ik}\odot b_{ki})$ and $y_{ij} = \bigvee_{k\neq i}(b_{ik}\odot a_{ki})$. It is obvious that

$$\begin{cases} (a_{ij}\odot b_{ji}) \vee (\bigvee_{k\neq j}(a_{ik}\odot b_{ki})) = 1,\\ (b_{ji}\odot a_{ij}) \vee (\bigvee_{k\neq i}(b_{ik}\odot a_{kj})) = 1.\end{cases}$$

Conversely, let the conditions (i), (ii) and (iii) hold. We prove that A is invertible or equivalently (1) and (2) are satisfied.
It is clear that $\bigvee_{t=1}^{n}(a_{it}\odot b_{tj}) = 1$, if $i = j$ by (ii). To show that $\bigvee_{t=1}^{n}(a_{it}\odot b_{tj}) = 0$ if $i\neq j$, we have:

$$\begin{aligned} a_{ik}\odot b_{kj} =&(a_{ik}\odot \bigvee_{k=1}^{n}(a_{ik}\odot b_{ki}))\odot(b_{kj}\odot(\bigvee_{k=1}^{n}(a_{jk}\odot b_{kj}))\\ =&\bigvee_{k=1}^{n}(a_{ik}\odot b_{ki}\odot a_{ik})\odot\bigvee_{k=1}^{n}(b_{kj}\odot a_{jk}\odot b_{kj})\\ =&\bigvee_{k=1}^{n}(a_{ik}\odot b_{ki}\odot a_{ik}\odot b_{kj}\odot a_{jk}\odot b_{kj}) = 0.\end{aligned}$$

Similarly we can prove that A is left invertible or equivalently (2) holds. □

In the following we show some results about invertible matrices over residuated lattice L, these are obtained from propositions and theorems that were proved in previous sections.

Proposition 4.2. *In the algebraic structure $\mathbb{M}_{n\times n}(L)$, the following properties hold:*
1) If $A \preceq B$, then $A \rhd B (A \lhd B)$ is invertible iff $A \rhd B = I_n (A \lhd B = I_n)$.
2) If A and A^{-1} are invertible, then for every B such that $A \preceq B$, we have B is invertible iff $B = A$.
3) If $A \in M_{n\times n}(L)$ has an inverse B, then A^t is invertible and therefore $A \boxdot A^t$ and $A^t \boxdot A$ are invertible ($A^t = [a_{ji}]_{n\times n}$ is the transpose of $A = [a_{ij}]_{n\times n}$).

Proof. (1) It is readily proved by Lemma 2.3 and Corollary 4.1.
(2) Let $A \preceq B$ and A, A^{-1} and B are invertible. Then $I_n = A \boxdot A^{-1} = A^{-1} \boxdot A \preceq A^{-1} \boxdot B$, but since A^{-1} and B are invertible, $A^{-1} \boxdot B$ is invertible and $(A^{-1} \boxdot B)_{ii} = 1$ for all $1 \leq i \leq n$. So by Corollary 4.1, we get $A^{-1} \boxdot B = I_n$. Similarly $I_n = A^{-1} \boxdot A = A \boxdot A^{-1} \preceq B \boxdot A^{-1}$ and $(B \boxdot A^{-1})_{ii} = 1$, for all $1 \leq i \leq n$. Again by using Corollary 4.1, we get $A^{-1} \boxdot B = B \boxdot A^{-1} = A \boxdot A^{-1} = A^{-1} \boxdot A = I_n$. On the other hand by Corollary 3.1 the inverse is unique, therefore we get $A = B$. The converse is obvious.
(3) Let $A \boxdot B = B \boxdot A = I_n$. Then $\bigvee_{t=1}^{n}(a_{it}\odot b_{tj}) = \bigvee_{t=1}^{n}(b_{it}\odot a_{tj}) = 0$, and $\bigvee_{t=1}^{n}(a_{it}\odot b_{ti}) =$

$\bigvee_{t=1}^{n}(b_{it} \odot a_{ti}) = 1$ for all $1 \leq i \neq j \leq n$. So

$$(A^t \boxdot B^t)_{ij} = \bigvee_{k=1}^{n}(a_{ik} \odot b_{kj}) = \bigvee_{k=1}^{n}(a_{ki} \odot b_{jk})$$
$$= \bigvee_{k=1}^{n}(b_{jk} \odot a_{ki}) = 0$$

and $\bigvee_{k=1}^{n}(b_{ik} \odot a_{ki}) = 1$. Therefore $A^t \boxdot B^t = I_n$, similarly $B^t \boxdot A^t = I_n$ and we see that A^t is invertible and $(A^t)^{-1} = B^t$. By Lemma 3.2, $A \boxdot A^t$ and $A^t \boxdot A$ are invertible. □

In the following proposition we show that if $A \in M_{n\times n}(L)$ is invertible, then $[\{\bot\}, A] = \{\bot\}$, i.e. the annihilator of A consists only of the trivial element $\bot$. We show also that if there is some filter containing A, then it includes A^{-1} too.

Proposition 4.3. *In the algebraic structure $\mathbb{M}_{n\times n}(L)$, we have*
1) *If A is invertible and $A \in M_{n\times n}(L)^{-}$, then $A^{-1} \in M_{n\times n}(L)^{-}$.*
2) *If A is invertible, then $A \notin [\{\bot\}, B]$, where $[\{\bot\}, B]$ is annihilator of the arbitrary matrix $B \in M_{n\times n}(L)$ and $B \neq \bot$.*

Proof. (1) Let $A \in M_{n\times n}(L)^{-}$ be invertible. Then $A \rhd \bot = \bot$ and $A \boxdot A^{-1} = I_n$. We know that $A \boxdot A^{-1} = A^{-1} \boxdot A = I_n$. So $(A^{-1} \boxdot A) \rhd \bot = I_n \rhd \bot = \bot$, since $I_n \in M_{n\times n}(L)^{-}$. By Lemma 2.2, we have $(A^{-1} \boxdot A) \rhd \bot = A^{-1} \rhd (A \rhd \bot) = \bot$ but we assume $A \in M_{n\times n}(L)^{-}$, then $A^{-1} \rhd \bot = \bot$, i.e. $A^{-1} \in M_{n\times n}(L)^{-}$.
Similarly if $A \in M_{n\times n}(L)^{\sim}$ we can show that $A^{-1} \in M_{n\times n}(L)^{\sim}$.
(2) Let A be invertible and $A \in [\{\bot\}, B]$ for arbitrary matrix $B \in M_{n\times n}(L)$ such that $B \neq \bot$. Then $A \boxdot B = \bot$, so $A^{-1} \boxdot A \boxdot B = A^{-1} \boxdot \bot = \bot$, therefore $B = \bot$. Which contradicts $B \neq \bot$. □

In what follows, we examine some results about invertibility in $M_{n\times n}(L)$, where L is a special residuated lattice.

Proposition 4.4. *If the residuated lattice L is complemented, then we have:*
1) *For a proper ideal S of $M_{n\times n}(L)$, if $A = [a_{ij}]_{n\times n} \in S$, then $A^c \notin S$.*
2) *If $A \in M_{n\times n}(L)$ has an inverse B, then $A \boxdot B^c \succeq (I_n)^c$.*

Proof. (1) if $A^c \in S$, then by definition of an ideal, we have $A \sqcup A^c = \top \in S$, that is a contradiction.
(2) Let $A \in M_{n\times n}(L)$ have an inverse B. Then by Lemma 2.2,

$$\begin{aligned}(A \boxdot \top)_{ij} =& (A \boxdot (B \sqcup B^c))_{ij}\\ =& ((A \boxdot B) \sqcup (A \boxdot B^c))_{ij}\\ =& (I_n \sqcup (A \boxdot B^c))_{ij}\end{aligned}$$

for all $1 \leq i, j \leq n$. Therefore $(A \boxdot \top)_{ii} = \bigvee_{t=1}^{n} a_{it} = 1$ and if $i \neq j$ we get $(A \boxdot B^c))_{ij} = (I_n \sqcup (A \boxdot B^c))_{ij} = (A \boxdot \top)_{ij} = \bigvee_{t=1}^{n} a_{it} = 1$. Then $A \boxdot B^c \succeq (I_n)^c$. □

We note that in Boolean algebras, $A^c = A^*$ (see Theorem 2.1). So for proper ideal S of $M_{n\times n}(L)$, if $A \in S$, then $A^c \notin S$. On the other hand if $A \in M_{n\times n}(L)$ has an inverse B, then $A \boxdot B^* \succeq (I_n)^c$.

Proposition 4.5. *If L is locally finite, then*
1) *No element $A \in M_{n\times n}(L^*)$ is invertible, where $L^* = L \setminus \{1\}$.*
2) *There is no proper filter S of $M_{n\times n}(L)$, such that $S \subseteq M_{n\times n}(L^*)$.*

Proof. For (1), if k is the nilpotence order of a matrix A and A is invertible then $A^{k-1} \boxdot A \boxdot A^{-1} = A^{k-1} = \perp$. Which contradicts that k is the nilpotence order of A.
(2) Let S be a proper filter of $M_{n\times n}(L^*)$. We know that every $A \in M_{n\times n}(L^*)$ is nilpotent, then if k is the nilpotence order of matrix $A \in S$, $A^k = \perp \in S$, which contradicts $\perp \notin S$. □

We note that by Theorem 2.3, $M_{n\times n}(L\backslash\{0\})$ is a filter of $M_{n\times n}(L)$(since $L\backslash\{0\}$ is a filter of L). In the following proposition we show that if L is a domain, then no element of this filter is invertible.

Proposition 4.6. *If L is a domain, and $A \in M_{n\times n}(L\backslash\{0\})$, then A is not invertible.*

Proof. Let $A \in M_{n\times n}(L\backslash\{0\})$ be invertible. Then $A \boxdot B = I_n$, for some B and $(A \boxdot B)_{ij} = \bigvee_{t=1}^{n} a_{it} \odot b_{tj} = 0$, for all $1 \leq i, j, t \leq n$, where $i \neq j$. Then $a_{it} \odot b_{tj} = 0$, for all $1 \leq i, j, t \leq n$, where $i \neq j$. But $a_{it} \in L\backslash\{0\}$ and L is a domain, then $b_{tj} = 0$ for all $1 \leq t, j \leq n$ which contradicts invertibility of B. □

Definition 4.1. *An element a in a bounded lattice L is said be an atom if a is a nonzero minimal element of L. The dual of an atom is called a molecule, i.e. an element b of a bounded lattice L is a molecule if $b \neq 1$ and is a maximal element of L.*

Using the above definition we prove:

Proposition 4.7. *If L has only one molecule, then no element $A \in M_{n\times n}(L^*)$ is invertible (where $L^* = L\backslash\{1\}$).*

Proof. Let $A \in M_{n\times n}(L^*)$ be invertible and c be a molecule of L. Then $A \boxdot B = I_n$, for some B and $(A \boxdot B)_{ii} = \bigvee_{t=1}^{n} a_{it} \odot b_{ti} = 1$, for all $1 \leq i, t \leq n$. Since $a_{it} \in L^*$, $a_{it} \odot b_{ti} \neq 1$, for all $1 \leq i, t \leq n$. But c is a molecule of L, therefore $\bigvee_{t=1}^{n} a_{it} \odot b_{ti} \leq c \neq 1$, for all $1 \leq i, t \leq n$, which contradicts $(A \boxdot B)_{ii} = 1$. □

Similarly, we have:

Proposition 4.8. *If L is G-algebra and has only one atom, then no element $A \in M_{n\times n}(L\backslash\{0\})$ is invertible.*

Proof. Let $A \in M_{n\times n}(L\backslash\{0\})$ be invertible and c be an atom of L. Then $A \boxdot B = I_n$, for some B and $(A \boxdot B)_{ij} = \bigvee_{t=1}^{n} a_{it} \odot b_{tj} = 0$, for all $1 \leq i, j, t \leq n$, where $i \neq j$. Then $a_{it} \odot b_{tj} = 0$, for all $1 \leq i, j, t \leq n$, where $i \neq j$. But L is a G-algebra, then $a_{it} \wedge b_{tj} = 0$, for all $1 \leq i, j, t \leq n$ and $a_{it} \in L\backslash\{0\}$. Since c is an atom of L, $b_{tj} = 0$ for all $1 \leq t, j \leq n$ (we note that if $b_{tj} \neq 0$, then $a_{it} \wedge b_{tj} \geq c \neq 0$) which contradicts invertibility of B. □

Examples 4.2. *Let $L = \{0, a, b, c, e, d, 1\}$. Such that $0 < a < b < e < 1$ and $0 < c < d < e < 1$. Then L is a residuated lattice with the following operations. But L has only one molecule e, then by Proposition 4.7, no element $A \in M_{n\times n}(\{o, a, b, c, d, e\})$ is invertible.*

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	d
d	0	0	0	c	c	c	d
e	0	a	a	c	c	e	e
1	0	a	b	c	d	e	1

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	e	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	e	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

Definition 4.2. *We define* $C(M_{n\times n}(L)) = \{A \in M_{n\times n}(L) : A \boxdot B = B \boxdot A,$ *for all* $B \in M_{n\times n}(L)\}$, *Center of* $M_{n\times n}(L)$, *and for every* $X \subseteq M_{n\times n}(L)$ *define* $C(X) = \{A \in M_{n\times n}(L) : A \boxdot B = B \boxdot A,$ *for all* $B \in X\}$.

It is obvious that $\{\perp, I_n\} \subseteq C(M_{n\times n}(L))$ and $C(M_{n\times n}(L))$ is closed under $\boxdot, \sqcup$, and $\sqcap$.

Theorem 4.3. *If for* $A \in M_{n\times n}(L)$, $\langle I_n, A\rangle = \{B \in M_{n\times n}(L) : A \boxdot B = B \boxdot A = I_n\}$. *Then* $\bigcup_{A\in M_{n\times n}(L)} \langle I_n, A\rangle = \bigcup_{A\in M_{n\times n}(L)} [I_n, C(A)] = S_{n\times n}(L)$.

Proof. By Lemma 3.1, it is clear that $\langle I_n, A\rangle$ has only one element or it is empty. Then the equality is obtained by definitions. □

Corollary 4.3. *For* $A, B \in M_{n\times n}(L)$, *we have* $A \boxdot B = I_n$ *iff* $B \boxdot A = I_n$, *then* $\langle I_n, A\rangle = [I_n, A]$.

5. Applications

In this section, we give some applications of invertible matrices for solving the systems of especial equations over residuated lattices.

Definition 5.1. *Let* $A = [a_{ij}]$ *and* $X = [x_j]$ *be* $n \times n$ *and* $n \times 1$ *matrices over* L, *respectively. Define the matrix*
$A \boxdot X = [\bigvee_{j=1}^{n} (a_{ij} \odot x_j)]_{n\times 1}$ *for all,* $1 \le i \le n$.

By a linear system of equations $A \boxdot X = B$ over L, we mean the following system of equations

$$\begin{cases} (a_{11} \odot x_1) \vee (a_{12} \odot x_2) \vee \ldots \vee (a_{1n} \odot x_n) = b_1 \\ (a_{21} \odot x_1) \vee (a_{22} \odot x_2) \vee \ldots \vee (a_{2n} \odot x_n) = b_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad\qquad\qquad . \\ (a_{n1} \odot x_1) \vee (a_{n2} \odot x_2) \vee \ldots \vee (a_{nn} \odot x_n) = b_n \end{cases}$$

where $a_{ij}, x_j, b_i \in L$ for all $1 \le i, j \le n$.
A sufficient condition for the system of equations $A \boxdot X = B$ over L having exactly one solution is given in the following theorem.

Theorem 5.1. *Let* A, X *and* B *be* $n \times n$, $n \times 1$ *and* $n \times 1$ *matrices over* L, *respectively. Then the above linear system has exactly one solution if* $E(A) = 1$.

Proof. Let $E(A) = 1$. Then by Corollary 3.4, A is an invertible matrix of $M_{n\times n}(L)$. Therefore $A^{-1} \boxdot (A \boxdot X) = A^{-1} \boxdot B$, i.e. $X = A^{-1} \boxdot B$ is a solution for our system. On the other hand let X_1 and X_2 are two solution for our system. Then we have $A \boxdot X_1 = A \boxdot X_2 = Y$. Thus $A^{-1} \boxdot (A \boxdot X_1) = A^{-1}(A \boxdot X_2)$ and we conclude that $X_1 = X_2$. □

Examples 5.1. *Let* $L = \{0, a, b, 1\}$ *be the residuated lattice in Example 3.1. In this case the following system of equations has a solution.*

$$\begin{cases} (a \odot x) \vee (b \odot y) = 1 \\ (b \odot x) \vee (a \odot y) = 0 \end{cases}$$

We know that $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_{2\times 2}(L)$ *such that* $a^2 = a \neq b = b^2$.
and $E(A) = [(a \odot b^*) \vee (b \odot a^*)] \wedge [(b \odot a^*) \vee (a \odot b^*)] = 1$. *Then by Corollary 3.2, we get* $A^{-1} = \begin{pmatrix} b^* & a^* \\ a^* & b^* \end{pmatrix}$ *and* $A^{-1} \boxdot A = A \boxdot A^{-1} = I_2$.

Then $A^{-1}\boxdot\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}b^* & a^*\\ a^* & b^*\end{pmatrix}\boxdot\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}(b^*\odot 1)\vee(a^*\odot 0)\\(a^*\odot 1)\vee(b^*\odot 0)\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$*. So by Theorem 5.1, the system of equations has exactly one solution* $x=a$ *and* $y=b$*. We note that the residuated lattice* L *is not a domain, because* $a\odot b=0$ *but* $a\neq 0$ *and* $b\neq 0$*.*

Remark 5.1. *We note that a system of equations* $A\boxdot X=B$ *may have a solution, but the matrix* A *is not invertible. For example let* $L=\{0,a,b,c,d,1\}$*, such that* $0<a<b<1$*,* $0<c<d<1$ *where* a *and* c *as well as* b *and* d *are incomparable. It is easy to see that* $0^*=1$*,* $a^*=d$*,* $b^*=c$*,* $c^*=b$ *and* $d^*=a$*. Then* L *is a residuated lattice with the following operations.*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

The following system has a solution $x_1=a, x_2=1$ *and* $x_3=a$*.*

$$\begin{cases}(a\odot x_1)\vee(b\odot x_2)\vee(1\odot x_3)=b\\(b\odot x_1)\vee(0\odot x_2)\vee(a\odot x_3)=a\\(0\odot x_1)\vee(b\odot x_2)\vee(b\odot x_3)=b\end{cases}$$

But the matrix $A=\begin{pmatrix}a & b & 1\\ b & 0 & a\\ 0 & b & b\end{pmatrix}$ *is not invertible. In fact, if it is invertible, by Theorem 3.2,*
$A^{-1}=\begin{pmatrix}c & d & c\\ c & c & c\\ c & 0 & 0\end{pmatrix}$*, but it is obvious that* $A\boxdot A^{-1}\neq I_3$*.*

We showed in Theorem 5.1, $E(A)=1$ is a sufficient condition for the system of equations $A\boxdot X=B$ over L for having exactly one solution $X=A^{-1}\boxdot B$. We note that the system of equations may have a solution, but the matrix A is not invertible, i.e. $E(A)\neq 1$ (see Remark 5.1). i.e. the above system has more than one solution($x_1=0, x_2=0$ and $x_3=b$ is another solution for above system). Following we try to find a set of columns that include all of the solutions for this system. To find these set we use the property of algebra $\mathbb{M}_{n\times n}(L)$ that we defined in [1].

Proposition 5.1. *Every solution* $X^{'}$ *of the system* $A\boxdot X=B$*, if it exists, satisfies* $[0]_{n\times 1}\preceq X^{'}\preceq[\bigwedge_{k=1}^{n}a_{kj}\rightarrow b_k]_{n\times 1}$*, for all* $1\leq j\leq n$*.*

Proof. For all $1\leq i,j\leq n$, we put $A=[a_{ij}]_{n\times n}$,

$$B^{'}=[b_{ij}]_{n\times n}=\begin{cases}b_{ij}=b_i & \text{if } j=1\\ b_{ij}=0 & \text{if } j\neq 1.\end{cases}$$

$$X^{'}=[x_{jt}]_{n\times n}=\begin{cases}x_{jt}=x_j & \text{if } t=1\\ x_{jt}=0 & \text{if } t\neq 1.\end{cases}$$

Then we get $A\boxdot X^{'}=B^{'}$, i.e.

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ . & & \\ . & & \\ . & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \boxdot \begin{pmatrix} x_{11} & \dots & 0 \\ x_{21} & \dots & 0 \\ . & & \\ . & & \\ . & & \\ x_{n1} & \dots & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & 0 \\ b_{21} & \dots & 0 \\ . & & \\ . & & \\ . & & \\ b_{n1} & \dots & 0 \end{pmatrix}$$

then by Lemma 2.1 we have

$$A \boxdot X^{'} \preceq B^{'} \Leftrightarrow X^{'} \preceq A \lhd B^{'}$$
$$\Leftrightarrow (X^{'})_{jt} \leq \bigwedge_{k=1}^{n} (a_{kj} \rightarrow b_{kt}).$$

Then if $t = 1$, $x_j \leq \bigwedge_{k=1}^{n}(a_{kj} \rightarrow b_k)$. We conclude that the solutions of $A \boxdot X = B$, are between two column matrices $[0]_{n\times 1}$ and $[\bigwedge_{k=1}^{n} a_{kj} \rightarrow b_k]_{n\times 1}$, for all $1 \leq j \leq n$. □

Examples 5.2. *Let L be residuated lattice defined in Remark 5.1. Consider the following linear system of 3 equations and 3 unknowns over L.*

$$\begin{cases} (a \odot x_1) \vee (b \odot x_2) \vee (d \odot x_3) = 1 \\ (c \odot x_1) \vee (1 \odot x_2) \vee (d \odot x_3) = 1 \\ (b \odot x_1) \vee (c \odot x_2) \vee (a \odot x_3) = 0 \end{cases}$$

Then by Proposition 5.1 the above system has a solution between two columns $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} c \\ b \\ d \end{pmatrix}$*. Because*

$x_1 \leq (a \rightarrow 1) \wedge (c \rightarrow 1) \wedge (b \rightarrow 0) = c$
$x_2 \leq (b \rightarrow 1) \wedge (1 \rightarrow 1) \wedge (c \rightarrow 0) = b$
$x_3 \leq (d \rightarrow 1) \wedge (d \rightarrow 1) \wedge (a \rightarrow 0) = d.$

We can easily check that $\begin{pmatrix} 0 \\ b \\ d \end{pmatrix}$*, is a solution of this system.*

We note that the matrix $A = \begin{pmatrix} a & b & d \\ c & 1 & d \\ b & c & a \end{pmatrix}$ *is not invertible, since by Theorem 3.3, if it is invertible then* $A^{-1} = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$*. But it is obvious that* $A \boxdot A^{-1} \neq I_3$*.*

6. Conclusion

We have given a necessary and sufficient condition under which an element $A \in M_{n\times n}(L)$ is invertible. Using invertible matrices we solved a system of equations $A \boxdot X = B$ over the residuated lattice L. We showed that this system has exactly one solution if $E(A) = 1$. Given algebraic properties of $\mathbb{M}_{n\times n}(L)$, we established some relations between invertible matrices and the set of filters and ideals of $\mathbb{M}_{n\times n}(L)$, respectively. Particularly in special residuated lattices, we found some filters (ideals) in which no matrix is invertible. Regarding the matrix operations

of $\mathbb{M}_{n\times n}(L)$, we characterized invertible matrices over some special residuated lattices.
Some other issues for future work are:

1) Let $A \in M_{n\times n}(L)$. Is it necessary that $\langle I_n, A\rangle = [I_n, A]$?

2) How to characterize invertible matrices over special residuated lattices, such as Łukasiewicz, product or Gödel algebras?

3) Under what condition(s) the system of equations $A \boxdot X = B$ over L, when $A \in M_{n\times n}(L)$ is not invertible, does have a solution?

4) Under what condition(s) the system of equations $A \boxdot X = B$ over L, does not have any solution?

7. Acknowledgement

The authors would like to express their special thanks of gratitude to the referee for the comments that greatly improved our manuscript.

References

[1] Balaei, S., Eslami, E. and Borumand Saeid, A., (2019), Algebraic structure of square matrices over residuated lattices, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68(2), pp.2216-2237.
[2] Blyth, T. S., (1964), Matrices over ordered Algebraic structures, J. London Math. Soc., 34(1), pp.427-432.
[3] Ciungu, L. C., (2014), Non-commutative Multiple-valued Logic Algebras, Springer International publishing Switzerland, 276p.
[4] Dolzan, D., Oblak, P., (2004), Invertible and Nilpotent Matrices over Antirings, Linear Algebra Appl., 430(1), pp.271-278.
[5] Galatos, N., Jipsen, P., Kowalski, T., and Ono, H., (2007), Residuated Lattices: An Algebraic Gelimpse at Substructural Logics, Volume 151 of studies in Logic and the Foundations of Mathematics, Elsevier, Amesterdam, 532p.
[6] Ganter, B., Wille, R., (1990), Formal Concepts Analysis, Mathematical Foundations. Springer, Berlin, 284p.
[7] Give'on, Y., (1964), Lattice mattrices, Information and control, 7(4), pp.477-484.
[8] Hohn, R. E., and Schissler, R. L., (1955), Boolean matrices and the design of combinatorial relay switching circuits, Bell System Tech. J., 34, pp.177-202.
[9] Kim, K. H., (1982), Boolean Matrix Theory and Applications, Marcel Dekker, Inc., New York, 228p.
[10] Luce, R. D., (1952), A Note on Boolean Matrix Theory, Proc. Amer. Math. Soc., 3(3), pp.382-388.
[11] Pal, M., (2001), Intuitionistic fuzzy determinat, V.U.J. physical sciences, 7(1), pp.87-93.
[12] Pal, M., Khan, S. K., and Shyama, A. K., (2002), Intuitionistic fuzzy matrices, Notes IFS, 8(2), pp.51- 62.
[13] Piciu, D., (2007), Algebras of Fuzzy Logic, Ed. Universitaria Craiova.
[14] Rutherford, D. E., (1963), Inverses of Boolean matrices, proc. Glasgow Math. Assoc., 6(1), pp.49-53.
[15] Tan, Y., (2007), On invertible matrices over antirings, Linear Algebra Appl., 423(2), pp.428-444.
[16] Thomason, M. G., (1977), Convergence of powers of a fuzzy matrix, J. Math. Anal. Appl., 57(2), pp.476-480.
[17] Wilding, D., (2014), Linear Algebra over Semiring, University of Manchester, 135p.
[18] Wille, R., (1982), Restructuring lattice theory: An approach based on concepts. In I. Rival, editor, Ordered sets, 83(3), pp.445-470.
[19] Zhao, C. K., (1990), Inverses of L-fuzzy Matrices, Fuzzy Sets Syst., 34(1), pp.103-116.

Saber Balaei received his Ph.D. degree in Pure Mathematics in 2019 from Shahid Bahonar University of Kerman, Kerman, Iran. His major research interests include logical algebras and algebraic structures.

Esfandiar Eslami He is a professor of Mathematics at the Department of Pure Mathematics in Shahid Bahonar University of Kerman, Kerman, Iran. His interests are mainly in non-classical logic and algebraic logics together with their applications.

Arsham Borumand Saeid received his Ph.D. in Mathematics, Iran in 2003. He is currently a professor at the Department of Pure Mathematics in Shahid Bahonar University of Kerman, Iran. His current research interests include fuzzy sets and applications, fuzzy logic, algebraic logic and algebraic structures.