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A SURVEY FOR PARANORMED SEQUENCE SPACES GENERATED BY INFINITE MATRICES

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Abstract. In the present paper, we summarize the recent literature concerning the domains of triangles in Maddox's sequence spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$ and $\ell(p)$, and related topics.

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1. Introduction and notations

We denote the set of all sequences of complex entries by ω. Any vector subspace of ω is called a *sequence space*. We write ℓ_∞, c, c_0 and f, for the spaces of all bounded, convergent, null and almost convergent sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p we denote the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively.

A sequence space λ with linear topology is called a $K-$*space* if each of the maps $r_n : \lambda \to \mathbb{C}$ defined by $r_n(x) = x_n$ is continuous for all $x = (x_n) \in \lambda$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A *Fréchet space* is a complete linear metric space. A K-space λ is called an FK*-space* if λ is a complete linear metric space. A normed FK-space is called a BK*-space*. Given a BK-space λ we denote the n^{th} section of a sequence $x = (x_k) \in \lambda$ by $x^{[n]} = \sum_{k=0}^{n} x_k e^k$ and we say that x is; AK (abschnittskonvergent) when $\lim_{n\to\infty} \left\| x - x^{[n]} \right\|_\lambda = 0$, AB (abschnittsbeschränkt) when $\sup_{n\in\mathbb{N}} \left\| x^{[n]} \right\|_\lambda < \infty$ and AD (abschnittsdicht) when ϕ is dense in λ, where e^n is a sequence whose only non-zero term is 1 in n^{th} place for each $n \in \mathbb{N}$ and ϕ is the set of all finitely non-zero sequences. If one of these properties holds for every $x \in \lambda$, then we said that the space λ has that property. It is trivial that AK implies AB and AD.

Definition 1.1. *Let X be a real or complex linear space, g be a function from X to the set $\mathbb{R}$ of real numbers. Then, the pair (X, g) is called a paranormed space and g is a paranorm for X, if the following axioms are satisfied for all elements $x, y \in X$ and for all scalars α:*

(i) $g(\theta) = 0$ if $x = \theta$, where θ is the zero element of X,
(ii) $g(x) \geq 0$,
(iii) $g(x) = g(-x)$,
(iv) $g(x + y) \leq g(x) + g(y)$,
(v) If (α_n) is a sequence of scalars with $\lim_{n\to\infty} \alpha_n = \alpha$ and (x_n) is a sequence in X with $\lim_{n\to\infty} g(x_n - x) = 0$, then $\lim_{n\to\infty} g(\alpha_n x_n - \alpha x) = 0$.

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A paranorm g is said to be total, if $g(x) = 0$ implies $x = \theta$. Let g be a paranorm on a sequence space λ. If $g(x) \neq g(|x|)$ for at least one sequence in λ, then λ is called a sequence space of non-absolute type; where $|x| = (|x_k|)$.

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We use the notation $O(1)$ as in [28], that is, "$f = O(\phi)$" means "$|f| < m\phi$", where m is a constant.

If a sequence space λ paranormed by g contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} g\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

Following Hamilton and Hill [27], Maddox [35, 36] gave the following definition:

Definition 1.2. *Let $A = (a_{nk})_{n,k\in\mathbb{N}}$ be an infinite matrix over the complex field $\mathbb{C}$ and $p = (p_k)$ be a sequence of positive numbers. Then, a sequence $x \in \omega$ is said to be strongly summable by A to ℓ if*

$$\sum_k a_{nk}|x_k - \ell|^{p_k}$$

exists for each $n \in \mathbb{N}$ and tends to zero as $n \to \infty$, this is denoted by $x_k \to \ell[A, p]$. If $\sum_k a_{nk}|x_k|^{p_k} = O(1)$, then we say that x is *strongly bounded* by A and denoted by $x_k = O(1)[A, p]$.

Let $\mathcal{A}$ denote the class of all infinite matrices $A = (a_{nk})_{n,k\in\mathbb{N}}$ for which there exists a positive integer K such that

(i*) $a_{nk} \geq 0$ for each $n \geq 1$ and for each $k > K$,
(ii) $\lim_{n\to\infty} (|a_{nk}| - a_{nk}) = 0$ for $1 \leq k \leq K$.

Two important subclasses of $\mathcal{A}$ are the nonnegative matrices, and the matrices satisfying (i*) and the condition $a_{nk} \to \alpha_k$ as $n \to \infty$ for $1 \leq k \leq K$, [35]. Uniqueness of strong limit is characterized for matrices in $\mathcal{A}$ by Maddox [35] as:

Lemma 1.1. [35, Theorem 2] *Suppose A is in $\mathcal{A}$ and (p_k) is bounded for all $k \in \mathbb{N}$. Then, the limit of a strongly summable sequence is unique if and only if one (at least) of the following fails to hold:*

(i) $\sum_k a_{nk}$ converges for each $n \in \mathbb{N}$,
(ii) $\lim_{n\to\infty} \sum_k a_{nk} = 0$.

Definition 1.3. [35] *The pair (A, p) consisting of a matrix A and a positive sequence $p = (p_k)$ is said to be a strongly regular method if $x_k \to \ell$ as $k \to \infty$ implies $x_k \to \ell[A, p]$.*

In the case $p_k = p > 0$ for all $k \in \mathbb{N}$ it was shown in [27] that necessary and sufficient conditions for strong regularity are

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N}, \tag{1}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty, \tag{2}$$

that is, (A, p) is strongly regular if and only if A maps null sequences into null sequences.

Using Definition 3.1. and following Hamilton and Hill [27], Maddox [35] gave the following results:

Theorem 1.1. *The following statements hold:*

(i) [35, Theorem 3] *Let m and M be constants such that $0 < m \leq p_k \leq M$ for all $k \in \mathbb{N}$, then (A, p) is strongly regular if and only if the conditions (1) and (2) hold.*

(ii) [35, Theorem 4] *Suppose that (1) and (2) hold and the sequence (p_k) converges to a positive limit. Then, $\lim_{k\to\infty} x_k = \ell$ implies that $x_k \to \ell[A, p]$ uniquely if and only if*

$$\limsup_{n\to\infty} \left| \sum_k a_{nk} \right| > 0.$$

(iii) [35, Result of Theorem 5] *Suppose that $A \in \mathcal{A}$ and $\|A\| < \infty$. Let $0 < p_k \leq q_k$ and q_k/p_k be bounded for all $k \in \mathbb{N}$. Then, $x_k \to \ell[A, q]$ implies $x_k \to \ell[A, p]$.*

2. Maddox's spaces

In this section, we give definitions and some topological properties of Maddox's spaces.

Maddox [35, 36] used the notations $[A, p]$, $[A, p]_\infty$ and $[A, p]_0$ for the sets of $x \in \omega$ which are strongly summable, strongly bounded and strongly summable to zero by A, respectively.

Taking A to be the unit matrix I, Maddox [35] introduced the spaces $[I, p]_\infty = \ell_\infty(p)$ given in [58] for the case $0 < p_k \leq 1$ and $[I, p] = c(p)$, $[I, p]_0 = c_0(p)$ as

$$\begin{aligned}
\ell_\infty(p) &:= \left\{ x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty \right\}, \\
c(p) &:= \left\{ x = (x_k) \in \omega : \exists \ell \in \mathbb{C} \text{ such that } \lim_{k\to\infty} |x_k - \ell|^{p_k} = 0 \right\}, \\
c_0(p) &:= \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0 \right\},
\end{aligned}$$

and taking the summation matrix $S = (s_{nk})$ and Cesàro matrix $C = (c_{nk})$ of order one instead of the matrix A, he gave the spaces $[S, p] = \ell(p)$ established in [58] for the case $0 < p_k \leq 1$ and $[C, 1, p] = \omega(p)$, $[C, 1, p]_0 = \omega_0(p)$ and $[C, 1, p]_\infty = \omega_\infty(p)$, respectively, as

$$\begin{aligned}
\ell(p) &:= \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\}, \\
\omega(p) &:= \left\{ x = (x_k) \in \omega : \exists \ell \in \mathbb{C} \text{ such that } \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell|^{p_k} = 0 \right\}, \\
\omega_0(p) &:= \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k} = 0 \right\}, \\
\omega_\infty(p) &:= \left\{ x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k} < \infty \right\},
\end{aligned}$$

where $S = (s_{nk})$ and $C = (c_{nk})$ are

$$s_{nk} = \begin{cases} 1 & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases} \quad \text{and } c_{nk} = \begin{cases} 1/n & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases} \tag{3}$$

for all $k, n \in \mathbb{N}$. In the case (p_k) are constant and equal to $p > 0$ for $k \in \mathbb{N}$ we write $\ell(p) = \ell_p$, $\omega(p) = \omega_p$, etc.

Taking (p_k) is a sequence of real numbers such that $0 < p_k < \sup_{k\in\mathbb{N}} p_k < \infty$, Nanda [53, 55] introduced the spaces $f_0(p)$, $f(p)$ and $\widehat{f}(p)$ by

$$\begin{aligned} f_0(p) &:= \left\{x = (x_k) \in \omega : \lim_{m\to\infty} |t_{mn}(x)|^{p_m} = 0 \text{ uniformly in } n\right\}, \\ f(p) &:= \left\{x = (x_k) \in \omega : \exists \ell \in \mathbb{C} \ni \lim_{m\to\infty} |t_{mn}(x) - \ell|^{p_m} = 0 \text{ uniformly in } n\right\}, \\ \widehat{f}(p) &:= \left\{x = (x_k) \in \omega : \sup_{m,n\in\mathbb{N}} |t_{mn}(x)|^{p_m} < \infty\right\}, \end{aligned}$$

where

$$t_{mn}(x) = \frac{1}{m+1}\sum_{k=0}^{m} x_{n+k}$$

for all $m, n \in \mathbb{N}$. If we take $p_k = p > 0$ for $k \in \mathbb{N}$, then we write

$$\widehat{f}(p) = \widehat{f} = \left\{x \in \omega : \sup_{m,n\in\mathbb{N}} |t_{mn}(x)|^{p} < \infty\right\},$$

(see [55]).

Following him, Başar [14] introduced the spaces $bs(p)$ and $\widehat{bs}(p)$ by

$$\begin{aligned} bs(p) &:= \{x = (x_k) \in \omega : Px \in \ell_\infty(p)\}, \\ \widehat{bs}(p) &:= \left\{x = (x_k) \in \omega : Px \in \widehat{f}(p)\right\}, \end{aligned}$$

where Px denotes the sequence of partial sums of an infinite series $\sum_k x_k$, i.e. $(Px)_n = \sum_{k=0}^{n} x_k$ for all $n \in \mathbb{N}$.

We shall assume throughout that N denotes the finite subsets of $\mathbb{N}$ and $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.

3. Some topological properties of Maddox's spaces

Before Maddox, Bourgin [20], Nakano [50, 51, 52], Landsberg [32] and Simons [58] used the spaces $\ell(p)$ and $\ell_\infty(p)$, as follows:

Let L be a linear topological space, A be a bounded open set in L and $A' = \{\lambda x : |\lambda| \leq 1, x \in A\}$. Define the quasi norm $\|x\|$ by $\|x\| = \inf\{h : x \in hA'\}$.

Lemma 3.1. [20, Theorem 13] *If L is locally bounded, the quasi norm on L satisfies*

$$\|x_1 + x_2\| \leq b_A(\|x_1\| + \|x_2\|)$$

for some $b_A \geq 1$ depending on A and L.

b_A in Lemma 3.1. is called the multiplier of the quasi norm. The quantity

$$\beta_L = \inf\{b_A : \text{A bounded and open in L}\}$$

is a characteristic of L, [20].

Taking $p_k = (1 + \log(k+1)^{-1/2})^{-1}$ for all $k \in \{1, 2, ...\}$, Bourgin [20] considered the linear sequence space $\ell(p)$ with the metric $d(x, y) = \sum_{k=1}^{\infty} |x_k - y_k|^{p_k}$ and he showed that $\beta_{\ell(p)}$ is not a possible multiplier.

For a sequence of positive numbers (p_k) with $p_k \geq 1$, Nakano [51] defined the sequence space $\ell(p_1, p_2, ...)$ consists of the sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} \frac{1}{p_k}|\alpha x_k|^{p_k} < +\infty$ for some $\alpha > 0$. Putting $m(x) = \sum_{k=1}^{\infty} \frac{1}{p_k}|x_k|^{p_k}$ for $x \in \ell(p_1, p_2, ...)$, he obtained a modular (the definiton of modular given in [50]) m on $\ell(p_1, p_2, ...)$, and putting

$$\|x\| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|}, \tag{4}$$

he introduced a norm on $\ell(p_1, p_2, ...)$ which is a complete sequence space with the norm (4).

Taking $p_k < 1$ and $x \in \ell(p)$ and putting $m(x) = \sum_{k=1}^{\infty} |x_k|^{p_k}$, Nakano [52] obtained a concave modular $m(x)$ on $\ell(p)$. Also, he gave the following result: "Every bounded linear functional φ on $\ell(p)$ is represented in the form

$$\varphi(x) = \sum_{k=1}^{\infty} a_k x_k,$$

where $a = (a_k) \in \ell_\infty$ and $x = (x_k) \in \ell(p)$.

Definition 3.1. [32] *The following statements hold:*

(i) If $0 < r \leq 1$, a non-void subset U of a linear space is said to be *absolutely $r-$convex* provided that

$$|\lambda|^r + |\mu|^r \leq 1 \text{ imply that } \lambda x + \mu y \in U, \quad (x, y \in U),$$

or equivalently,

$$\sum_{i=1}^{n} |\lambda_i|^r \leq 1 \text{ imply that } \sum_{i=1}^{n} \lambda_i x_i \in U, \quad (x_1, ..., x_n \in U).$$

(ii) A linear topological space is said to be *$r-$convex* if there is a neighbourhood base of 0 that consists of absolutely $r-$convex sets.

Let L be a linear sequence space containing all finite sequences, and (p_k) be a sequence of real numbers with $0 < p_k \leq 1$ and $0 < \liminf_{k\to\infty} p_k < 1$ for all $k \in \mathbb{N}$. All $x = (x_n) \in L$ with $d(x) = \sum_k |x_k|^{p_k} < +\infty$ form a linear sequence space $\ell(L; (p_k))$, which is defined by the metric $d(x-y)$ for $x, y \in \ell(L; (p_k))$, becomes a linear topological space. The space $\ell(L; (p_k))$ is $r-$convex for every r with $0 < r < \liminf_{k\to\infty} p_k$, but can not be $s-$convex for any s with $\liminf_{k\to\infty} p_k < s \leq 1$, Landsberg [32]. If we take $L = \omega$, we have the space $\ell(p)$.

Writing τ_p and τ_p^∞ for the topology introduced on $\ell(p)$ and $\ell_\infty(p)$ by the metrics $d(x, y) = g(x - y)$ and $d_1(x, y) = g_1(x - y)$, respectively, defined by

$$g(x) = \sum_k |x_k|^{p_k} \quad \text{and} \quad g_1(x) = \sup_k |x_k|^{p_k},$$

Simons [58] gave the following results:

Theorem 3.1. *The following statements hold:*

(i) [58, Lemma 1] $(\ell(p), \tau_p)$ *is a complete linear topological space.*

(ii) [58, Lemma 2] *If* $0 < p_k \leq q_k \leq 1$ *for all* $k \in \mathbb{N}$, *then*

(1) $\ell(p) \subset \ell(q)$,

(2) *The identity map* $(\ell(p), \tau_p) \to (\ell(q), \tau_q)$ *is continuous,*

(3) $\ell(p)$ *is dense in* $(\ell(q), \tau_q)$.

(iii) [58, Theorem 1] *If* $0 < p_k \le q_k \le 1$ *for all* $k \in \mathbb{N}$*, then the following four conditions are equivalent:*
   (1) τ_p *is the topology induced on* $\ell(p)$ *by* τ_q.
   (2) *If* $(x^n)_{n\in\mathbb{N}} \in \ell(p)$ *and* $x^n \to 0$ *in* τ_q *as* $n \to \infty$*, then* $x^n \to 0$ *in* τ_p *as* $n \to \infty$.
   (3) $\ell(p)$ *is closed in* $(\ell(q), \tau_q)$.
   (4) $\ell(p) = \ell(q)$.

(iv) [58, Theorem 3] *Let* $0 < p_k \le 1$ *for all* $k \in \mathbb{N}$ *and* $1/p_k + 1/q_k = 1$*. Then, the following two conditions are equivalent:*
   (1) $\ell(p) = \ell_1$.
   (2) $\sum_k B^{q_k} < \infty$ *for some integer* $B > 1$.

(v) [58, Theorem 5] *The following four conditions on* (p_k) *are equivalent:*
   (1) $(\ell(p), \tau_p)$ *is locally convex.*
   (2) $\ell(p) = \ell_1$.
   (3) τ_p *is identical with the topology induced on* $\ell(p)$ *by* τ_1.
   (4) $\ell(p)$ *is closed in* (ℓ_1, τ_1).

(vi) [58, Theorem 7] *The following three conditions on* (ζ_k) *are equivalent:*
   (1) *The map* $(x_n) \to \sum_k x_k\zeta_k$ *is a continuous linear functional on* $(\ell(p), \tau_p)$.
   (2) $\sum_k x_k\zeta_k$ *is convergent for all* $(x_k) \in \ell(p)$.
   (3) $(\zeta_k) \in \ell_\infty(p)$.

(vii) [58, Theorem 8] *If* $0 < p_k \le q_k \le 1$ *for all* $k \in \mathbb{N}$*, then the following conditions are equivalent:*
   (1) τ_q^∞ *is the topology induced on* $\ell_\infty(q)$ *by* τ_p^∞.
   (2) *The identity map* $(\ell_\infty(q), \tau_q^\infty) \to (\ell_\infty(q), \tau_p^\infty)$ *is continuous.*
   (3) *There exists* $B > 1$ *such that* $Bp_k \ge q_k$ *for all* $k \in \mathbb{N}$.
   (4) $\ell_\infty(p) = \ell_\infty(q)$.
   (5) $\ell_\infty(q)$ *is dense in* $(\ell_\infty(p), \tau_p^\infty)$.

(viii) [58, Theorem 9] *The following five conditions on* (p_k) *are equivalent:*
   (1) τ^∞ *is the topology induced on* ℓ_∞ *by* τ_p^∞*, where* τ^∞ *is the topology on* ℓ_∞ *defined by the supremum metric.*
   (2) *The identity map* $(\ell_\infty, \tau^\infty) \to (\ell_\infty, \tau_p^\infty)$ *is continuous.*
   (3) $\inf_{k\in\mathbb{N}} p_k > 0$.
   (4) ℓ_∞ *is dense in* $(\ell_\infty(p), \tau_p^\infty)$.
   (5) $(\ell_\infty(p), \tau_p^\infty)$ *is a linear topological space.*

If we take $0 < p_k \le q_k$ for all $k \in \mathbb{N}$, then it is true that $\ell(p) \subset \ell(q)$. We note that no restriction such as boundedness has to be placed on the sequences (p_k), (q_k) for the validity of the inclusion. But the inclusion $\omega(p) \subset \omega(q)$ does not hold when $0 < p_k \le q_k$. This brings out an immediate distinction between the spaces $\ell(p)$ and $\omega(p)$, [35].

Also, one can find that the boundedness of $p = (p_k)$ is sufficient for the spaces $[A, p]$ and $[A, p]_\infty$ to be linear spaces in Theorem 1 of [35]. So, the argument of [35] shows that $[A, p]_0$ is linear when $p = (p_k)$ is bounded. It was also noted in [35] that $p_k = O(1)$ is necessary for the linearity of the spaces $\ell(p)$ and $\omega(p)$. In [36], Maddox showed that $c(p)$ is a linear space only if $p_k = O(1)$. In general, $p_k = O(1)$ is not necessary for $[A, p]$, $[A, p]_0$ and $[A, p]_\infty$ to be linear spaces.

In the case, $0 < p_k \le 1$ for all $k \in \mathbb{N}$, the inequality $|x_k + y_k|^{p_k} \le |x_k|^{p_k} + |y_k|^{p_k}$ suggests the natural paranorm

$$g(x) = \sup_{n\in\mathbb{N}} \sum_k a_{nk}|x_k|^{p_k} \tag{5}$$

for the spaces $[A,p]_\infty$ and $[A,p]_0$. In general $[A,p]$ is not a subset of $[A,p]_\infty$ so that (5) is not suitable for $[A,p]$. In the more general case $p_k = O(1)$, a suitable paranorm for $[A,p]_\infty$ and $[A,p]_0$ is

$$g_A(x) = \sup_{n\in\mathbb{N}} \left(\sum_k a_{nk}|x_k|^{p_k}\right)^{1/M}, \tag{6}$$

where $M = \max\{1, p_k\}$, which gives (5) when $0 < p_k \leq 1$ for all $k \in \mathbb{N}$, [36].

For arbitrary A and (p_k), we have the inclusions $[A,p]_0 \subset [A,p]$ and $[A,p]_0 \subset [A,p]_\infty$. For the inclusion $[A,p] \subset [A,p]_\infty$ holds the necessary condition is that

$$\|A\| = \sup_{n\in\mathbb{N}} \sum_k a_{nk} < \infty, \tag{7}$$

whether (p_k) is bounded or not. If (p_k) is bounded then (7) is sufficient for $[A,p] \subset [A,p]_\infty$. Thus, in this case we have that $[A,p]$ is a subset of $[A,p]_\infty$ if and only if (7) holds, and then we may do the space $[A,p]$ a paranormed space with the paranorm (6). Also, the spaces $[A,p]_0$ and $[A,p]_\infty$ are complete, [39].

Theorem 3.2. *The following statements hold:*

(i) [36, Theorem 1] *For any nonnegative matrix A and any bounded sequence $p = (p_k)$, the space $[A,p]_0$ is paranormed space by the paranorm (6).*

(ii) [36, Corollary 2 of Theorem 1] *If A is a nonnegative matrix and $0 < \inf p_k \leq \sup p_k < \infty$ for all $k \in \mathbb{N}$, the space $[A,p]_\infty$ is paranormed space by the paranorm (6).*

(iii) [36, Theorem 2] *$\omega_\infty(p)$ is paranormed space by the paranorm (6) if and only if $0 < \inf p_k \leq \sup p_k < \infty$.*

In 1969, Maddox [39, 40] studied some topological properties of the spaces $[A,p]$, $[A,p]_0$ and $[A,p]_\infty$ as:

Theorem 3.3. *Define the set S by $S = \{k : 0 < \sup_{n\in\mathbb{N}} a_{nk} < \infty\}$ and let $A = (a_{nk})$ be a lower semi-matrix such that $a_{nk} \to 0$ as $n \to \infty$ for all fixed $k \in \mathbb{N}$. Then, the following statements hold:*

(i) [40, Theorem] *$[A,p]_0$ and $[A,p]$ are linear if and only if $\sup_{k\in S} p_k < \infty$.*

(ii) [39, Theorem 3] *Let $a_{nk} \leq M$ for all $n, k \in \mathbb{N}$ and $\liminf_{n\to\infty} \sum_k a_{nk} > 0$. Then, $[A,p]$ is linear if and only if $\sup_{k\in\mathbb{N}} p_k < \infty$.*

(iii) [39, Theorem 4] *Let $M_k = \sup_{n\in\mathbb{N}} a_{nk} > 0$ for each $k \in \mathbb{N}$. Then, $[A,p]_\infty$ is paranormed space by the paranorm (6).*

(iv) [39, Theorem 1] *For an arbitrary A, $[A,p]_\infty$ is linear if and only if $\sup_{k\in S} p_k < \infty$.*

(v) [39, Theorem 5] *Let $p_k = O(1)$ and $\|A\| < \infty$ for an arbitrary A. Then, either of the following conditions is sufficient for $[A,p]$ to be complete:*

  (1) $\limsup_{n\to\infty} \sum_k a_{nk} = 0$.

  (2) $\limsup_{n\to\infty} \sum_k a_{nk} > 0$ *and* $\inf p_k > 0$.

(vi) [39, Theorem 6] *Let $p_k = O(1)$. Then $c(p)$ and $\omega(p)$, equipped with their natural paranorms are complete.*

Thus, in the light of above information we can write: Let (p_k) be a bounded sequence of strictly positive real numbers with $\sup_{k\in\mathbb{N}} p_k = H$ and $M = \max\{1, H\}$. $\ell(p)$ is a linear space if and only if $H < \infty$ and it is a complete paranormed space (cf. [35, 39]) with

$$g(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M}.$$

The sets $c_0(p)$, $c(p)$ and $\ell_\infty(p)$ are linear spaces if and only if $p = (p_k) \in \ell_\infty$. If $p = (p_k) \in \ell_\infty$ and $\inf_{k\in\mathbb{N}} p_k > 0$ then the sets $c_0(p)$, $c(p)$ and $\ell_\infty(p)$ reduce to the classical sets c_0, c and ℓ_∞, respectively. The identities $c_0(p) = c_0$, $c(p) = c$ and $\ell_\infty(p) = \ell_\infty$ are satisfied if and only if $0 < \inf_{k\in\mathbb{N}} p_k$ and $\sup_{k\in\mathbb{N}} p_k < \infty$. The function

$$g_1(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M}$$

on the spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ introduced a topology τ_{g_1} via the corresponding metric $d(x, y) = g_1(x - y)$. Then, $c(p)$ and $c_0(p)$ are complete paranormed spaces paranormed by g_1 if $p = (p_k) \in \ell_\infty$. Also, $\ell_\infty(p)$ is a complete paranormed space by g_1 if and only if $\inf_{k\in\mathbb{N}} p_k > 0$. In $\ell_\infty(p)$, g_1 is a paranorm and τ_{g_1} is a linear topology only in the trivial case $\inf_{k\in\mathbb{N}} p_k > 0$, when $\ell_\infty(p) = \ell_\infty$. Indeed the natural topology of $\ell_\infty(p)$ is not metrizable, hence not paranormable unless $\ell_\infty(p) = \ell_\infty$. In $c_0(p)$, g_1 is a paranorm (without the restriction $\inf_{k\in\mathbb{N}} p_k > 0$) and τ_{g_1} is an FK topology, so that by the uniqueness of FK topologies [62, Corollary 4.4.2] τ_{g_1} coincides with the projective limit topology. In $c(p)$, again g_1 is a paranorm and τ_{g_1} is a linear topology only if $\inf_{k\in\mathbb{N}} p_k > 0$, when $c(p) = c$. But, in contrast to $\ell_\infty(p)$, the natural topology of $c(p)$ can be induced by a paranorm. A convenient one is $g_2(x) = g_1(x - \xi e)$, where ξ is the unique number with $x - \xi e \in c_0(p)$ and $e = (1, 1, 1, \ldots)$, (cf. [58, 35, 36, 38, 41]).

Theorem 3.4. *Nanda* [53, 55] *gave the following results:*

(i) [53, Proposition 1] *The inclusions* $c_0(p) \subset f_0(p)$, $c(p) \subset f(p)$ *and* $f_0(p) \subset f(p)$ *hold.*

(ii) [53, Proposition 2] *If* $0 < p_k \le q_k < \infty$ *for all* $k \in \mathbb{N}$, *then the inclusions* $f_0(p) \subset f_0(q)$ *and* $f(p) \subset f(q)$ *hold.*

(iii) [53, Theorem 1] *The space* $f_0(p)$ *is a complete linear topological space paranormed by* g *defined by*

$$g(x) = \sup_{m,n\in\mathbb{N}} |t_{mn}(x)|^{p_m/M}. \tag{8}$$

If $\inf_{m\in\mathbb{N}} p_m > 0$, *then* $f(p)$ *is a complete linear topological space with respect to the paranormed* g.

(iv) [53, Proposition 3] *The spaces* $f_0(p)$ *and* $f(p)$ *are 1-convex.*

(v) [55, Theorem 1] *Let* $\inf_{k\in\mathbb{N}} p_k > 0$ *for all* $k \in \mathbb{N}$. *Then, the space* $\widehat{f}(p)$ *is a complete linear topological space paranormed by* g *defined as in (8).*

(ii) [55, Proposition 1] $\widehat{f}(p)$ *is* $1-$*convex.*

(iii) [55, Theorem 2] *Let* $0 < p_k \le q_k < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $\widehat{f}(q)$ *is a closed subspace of* $\widehat{f}(p)$.

Başar [14] obtained that: The space $\widehat{bs}(p)$ is linearly isomorphic to the space $\widehat{f}(p)$. Following him, Başar and Altay [16] gave the following results:

Theorem 3.5. *The following statements hold:*

(i) [16, Theorem 2.1] *The space* $bs(p)$ *is a complete linear metric space paranormed by* g *defined by*

$$g(x) = \sup_{k\in\mathbb{N}} \left| \frac{1}{k+1} \sum_{i=0}^{k} x_i \right|^{p_k/M} \quad \textit{iff} \ \inf_{k\in\mathbb{N}} p_k > 0.$$

(ii) [16, Theorem 2.2]

(1) $bs(p) \subset bs$ *if and only if* $h = \inf_{k\in\mathbb{N}} p_k > 0$.

(2) $bs(p) \supset bs$ *if and only if* $H = \sup_{k\in\mathbb{N}} p_k > 0$.

(3) $bs(p) = bs$ *if and only if* $0 < h \leq H < \infty$.

4. Some new Maddox's spaces

In this section, we assume that $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $\sup_{k\in\mathbb{N}} p_k = H$ and $M = \max\{1, H\}$ unless stated otherwise.

Let $\mathcal{U}$ denotes the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. Define the matrices difference $\Delta = (d_{nk})$, Riesz $R^t = (r^t_{nk})$, Nörlund $N^t = (u^t_{nk})$, generalized weighted mean or factorable $G(u, \nu) = (g_{nk})$, generalized difference $B(r, s) = (b_{nk}(r, s))$, double sequential band $B(\widetilde{r}, \widetilde{s}) = (b_{nk}(r_k, s_k))$, triple band $B(r, s, t) = (b_{nk}(r, s, t))$, double band $F = (f_{nk})$, $A^r = (a^r_{nk})$ and $A^u = (a^u_{nk})$ by

$$d_{nk} = \begin{cases} (-1)^{n-k} & , \ n-1 \leq k \leq n, \\ 0 & , \ \text{otherwise} \end{cases}, \qquad r^t_{nk} = \begin{cases} t_k/T_n & , \ 0 \leq k \leq n, \\ 0 & , \ k > n \end{cases}$$

$$u^t_{nk} = \begin{cases} t_{n-k}/T_n & , \ 0 \leq k \leq n, \\ 0 & , \ k > n \end{cases}, \qquad g_{nk} = \begin{cases} u_n \nu_k & , \ 0 \leq k \leq n, \\ 0 & , \ \text{otherwise} \end{cases}$$

$$b_{nk}(r, s) = \begin{cases} r & , \ k = n, \\ s & , \ k = n-1, \\ 0 & , \ \text{otherwise} \end{cases}, \qquad b_{nk}(r_k, s_k) = \begin{cases} r_k & , \ k = n, \\ s_k & , \ k = n-1, \\ 0 & , \ \text{otherwise} \end{cases}$$

$$f_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & , \ k = n-1, \\ -\frac{f_n}{f_{n+1}} & , \ k = n, \\ 0 & , \ \text{otherwise} \end{cases} \qquad b_{nk}(r, s, t) = \begin{cases} r & , \ n = k, \\ s & , \ n = k+1, \\ t & , \ n = k+2, \\ 0 & , \ \text{otherwise} \end{cases}$$

$$a^r_{nk} = \begin{cases} \frac{1+r^k}{n+1} \upsilon_k & , \ 0 \leq k \leq n, \\ 0 & , \ k > n \end{cases} \qquad a^u_{nk} = \begin{cases} (-1)^{n-k} u_k & , \ n-1 \leq k \leq n, \\ 0 & , \ \text{otherwise} \end{cases} \tag{9}$$

for all $k, n \in \mathbb{N}$, respectively; where (t_k) is a sequence of positive numbers, $T_n = \sum_{k=0}^{n} t_k = \sum_{k=0}^{n} t_{n-k}$ for all $n \in \mathbb{N}$, $r, s, t \in \mathbb{R}\backslash\{0\}$, $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ are the convergent sequences whose entries either constants or distinct non-zero numbers for all $k \in \mathbb{N}$, $\upsilon, u, \nu \in \mathcal{U}$ and (f_n) is a sequence of Fibonacci numbers defined by the linear recurrence relations

$$f_n = \begin{cases} 1 & , \ n = 0, 1, \\ f_{n-1} + f_{n+1} & , \ n \geq 2 \end{cases}$$

and denote the Euler matrix of order r with $E^r = (e^r_{nk})$ defined by

$$e^r_{nk} = \begin{cases} \binom{n}{k}(1-r)^{n-k} r^k & , \ 0 \leq k \leq n \\ 0 & , \ \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$, where $0 < r < 1$.

The *summability domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}. \tag{10}$$

Taking (p_k) not necessarily bounded, Ahmad and Mursaleen [1] and Malkowsky [44] introduced the spaces $\Delta\ell_\infty(p)$, $\Delta c(p)$ and $\Delta c_0(p)$ as

$$\begin{aligned}
\Delta\ell_\infty(p) &:= \{x = (x_k) \in \omega : \Delta x \in \ell_\infty(p)\}, \\
\Delta c(p) &:= \{x = (x_k) \in \omega : \Delta x \in c(p)\}, \\
\Delta c_0(p) &:= \{x = (x_k) \in \omega : \Delta x \in c_0(p)\}.
\end{aligned}$$

Following them, Choudhary and Mishra [22] defined the same spaces with bounded (p_k) and gave the following results:

(i) [22, Properties] $\Delta\ell_\infty(p)$ and $\Delta c(p)$ are paranormed spaces with the paranorm

$$g(x) = \sup_{k\in\mathbb{N}} |\Delta x|^{p_k/M} \tag{11}$$

if and only if $0 < \inf_{k\in\mathbb{N}} p_k \le H < \infty$ for all $k \in \mathbb{N}$.

(ii) [22, Properties] If $p = (p_k)$ is a bounded sequence, then $\Delta c_0(p)$ is a paranormed space with the paranorm (11).

Altay and Başar [2, 4] defined the *Riesz sequence spaces* $r^t(p)$, $r^t_\infty(p)$, $r^t_c(p)$ and $r^t_0(p)$ as the domain of the Riesz matrix in the spaces $\ell(p)$, $\ell_\infty(p)$, $c(p)$ and $c_0(p)$, respectively, as

$$\begin{aligned}
r^t(p) &:= \{x = (x_k) \in \omega : Rx \in \ell_\infty(p)\}, \\
r^t_\infty(p) &:= \{x = (x_k) \in \omega : Rx \in \ell_\infty(p)\}, \\
r^t_c(p) &:= \{x = (x_k) \in \omega : Rx \in c(p)\}, \\
r^t_0(p) &:= \{x = (x_k) \in \omega : Rx \in c_0(p)\}.
\end{aligned}$$

If we take $(p_k) = e$ for all $k \in \mathbb{N}$ the spaces $r^t_\infty(p)$, $r^t_c(p)$ and $r^t_0(p)$ are reduced the spaces r^t_∞, r^t_c and r^t_0 introduced by Malkowsky [46]. One can find the following results in their papers:

Theorem 4.1. *The following statements hold:*

(i) [2, Theorem 2.1] $r^t(p)$ *is a complete linear metric space paranormed by* g*, defined by*

$$g(x) = \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^{k} t_j x_j\right|^{p_k}\right)^{1/M} \quad \text{with } 0 < p_k \le H < \infty.$$

(ii) [2, Theorem 2.3] *The Riesz sequence space* $r^t(p)$ *of non-absolute type is linearly isomorphic to the space* $\ell(p)$*, where* $0 < p_k \le H < \infty$.

(iii) [4, Theorem 2.1] $r^t_\infty(p)$*,* $r^t_c(p)$ *and* $r^t_0(p)$ *are the complete linear metric spaces paranormed by* g*, defined by*

$$g(x) = \sup_{n\in\mathbb{N}} \left|\frac{1}{T_k}\sum_{j=0}^{k} t_j x_j\right|^{p_k/M}.$$

g *is a paranorm for the spaces* $r^t_\infty(p)$ *and* $r^t_c(p)$ *only in the trivial case* $\inf_{k\in\mathbb{N}} p_k > 0$ *when* $r^t_\infty(p) = r^t_\infty$ *and* $r^t_c(p) = r^t_c$.

(iv) [4, Theorem 2.3] *The Riesz sequence spaces* $r^t_\infty(p)$*,* $r^t_c(p)$ *and* $r^t_0(p)$ *of non-absolute type are linearly isomorphic to the spaces* $\ell_\infty(p)$*,* $c(p)$ *and* $c_0(p)$*, respectively, where* $0 < p_k \le H < \infty$.

Using the notation $\lambda(u,\nu;p)$ for $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$, Altay and Başar [3, 5] defined the spaces $\lambda(u,\nu;p)$ by

$$\lambda(u,\nu;p) \quad := \quad \left\{ x = (x_k) \in \omega : y = \left(\sum_{j=0}^{k} u_k \nu_j x_j \right) \in \lambda(p) \right\},$$

called *generalized weighted mean sequence spaces*.

It is natural that these spaces may also be redefined with the notation of (10) that

$$\lambda(u,\nu;p) \quad = \quad \{\lambda(p)\}_{G(u,\nu)}.$$

If $p_k = 1$ for all $k \in \mathbb{N}$, we write $\lambda(u,\nu)$ instead of $\lambda(u,\nu;p)$ introduced by Malkowsky and Savaş [49]. Following them, Altay and Başar [3, 5] gave the following results:

Theorem 4.2. *The following statements hold:*

(i) [3, Theorem 2.1(a)] $\lambda(u,\nu;p)$ *are the complete linear metric spaces paranormed by* g*, defined by*

$$g(x) = \sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} u_k \nu_j x_j \right|^{p_k/M}.$$

g *is a paranorm for the spaces* $\ell_\infty(u,\nu;p)$ *and* $c(u,\nu;p)$ *only in the trivial case* $\inf_{k\in\mathbb{N}} p_k > 0$ *when* $\ell_\infty(u,\nu;p) = \ell_\infty(u,\nu)$ *and* $c(u,\nu;p) = c(u,\nu)$.

(ii) [3, Theorem 2.1(b)] *The sets* $\lambda(u,\nu)$ *are the Banach spaces with the norm* $\|x\|_{\lambda(u,\nu)} = \|y\|_\lambda$.

(iii) [3, Theorem 2.2] *The generalized weighted mean sequence spaces* $\ell_\infty(u,\nu;p)$, $c(u,\nu;p)$ *and* $c_0(u,\nu;p)$ *of non-absolute type are linearly isomorphic to the spaces* $\ell_\infty(p)$, $c(p)$ *and* $c_0(p)$*, respectively, where* $0 < p_k \le H < \infty$.

(iv) [3, Theorem 2.3] *The sequence space* $c_0(u,\nu)$ *has AD property whenever* $u \in c_0$.

(v) [5, Theorem 2.1(a)] $\ell(u,\nu;p)$ *is a complete linear metric spaces paranormed by* g*, defined by*

$$g(x) = \left(\sum_k \left| \sum_{j=0}^{k} u_k \nu_j x_j \right|^{p_k} \right)^{1/M}.$$

(vi) [5, Theorem 2.1(b)] *Let* $1 \le p < \infty$*. Then,* $\ell_p(u,\nu)$ *is a Banach space with the norm* $\|x\|_{\ell_p(u,\nu)} = \|y\|_{\ell_p}$.

(vii) [5, Theorem 2.2] *The sequence space* $\ell(u,\nu;p)$ *of non-absolute type is linearly isomorphic to the space* $\ell(p)$*, where* $0 < p_k \le H < \infty$.

(viii) [5, Theorem 2.3] *Let* $u \in \ell_1$ *and* $1 \le p < \infty$*. Then, the sequence space* $\ell(u,\nu;p)$ *has AD property.*

Aydın and Başar [9, 10] defined the spaces $a_0^r(\upsilon,p)$, $a_c^r(\upsilon,p)$ and $a^r(\upsilon,p)$ as the domain of the A^r matrix in the spaces $c_0(p)$, $c(p)$ and $\ell(p)$, respectively, as

$$\begin{aligned} a_0^r(\upsilon,p) &:= \{x = (x_k) \in \omega : A^r x \in c_0(p)\}, \\ a_c^r(\upsilon,p) &:= \{x = (x_k) \in \omega : A^r x \in c(p)\}, \\ a^r(\upsilon,p) &:= \{x = (x_k) \in \omega : A^r x \in \ell(p)\}. \end{aligned}$$

In the case $(\upsilon_k) = (p_k) = e$ for all $k \in \mathbb{N}$ the spaces $a_0^r(\upsilon,p)$ and $a_c^r(\upsilon,p)$ are reduced the spaces a_0^r and a_c^r introduced by Aydın and Başar [11] and in the cases $p_k = p$ for all $k \in \mathbb{N}$ and $(\upsilon_k) = e$, the space $a^r(\upsilon,p)$ is reduced the spaces $a_p^r(\upsilon)$ and a_p^r, respectively, where a_p^r is introduced by Aydın and Başar [12].

Theorem 4.3. *The following statements hold:*

(i) [9, Theorem 2.1] *The spaces* $a_0^r(\upsilon,p)$ *and* $a_c^r(\upsilon,p)$ *are the complete linear metric spaces paranormed by* g*, defined by*

$$g(x) = \sup_{k\in\mathbb{N}} \left| \frac{1}{k+1} \sum_{j=0}^{k} (1+r^j) \upsilon_j x_j \right|^{p_k/M}.$$

g is a paranorm for the space $a_c^r(\upsilon,p)$ only in the trivial case $\inf_{k\in\mathbb{N}} p_k > 0$ *when* $a_c^r(\upsilon,p) = a_c^r$.

(ii) [9, Theorem 2.2] *The sequence spaces $a_0^r(\upsilon,p)$ and $a_c^r(\upsilon,p)$ of non-absolute type are linearly isomorphic to the spaces $c_0(p)$ and $c(p)$, respectively, where* $0 < p_k \leq H < \infty$.

(iii) [10, Theorem 2.1] *$a^r(\upsilon,p)$ is a complete linear metric spaces paranormed by g, defined by*

$$g(x) = \left(\sum_k \left|\frac{1}{k+1}\sum_{j=0}^{k}(1+r^j)\upsilon_j x_j\right|^{p_k}\right)^{1/M},$$

where $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$.

(iv) [10, Theorem 2.2] *$a_p^r(\upsilon)$ is the linear space under the coordinatewise addition and scalar multiplication, which is the $BK-$space with the norm*

$$\|x\| = \left(\sum_k \left|\frac{1}{k+1}\sum_{j=0}^{k}(1+r^j)\upsilon_j x_j\right|^{p}\right)^{1/p},$$

where $1 \leq p < \infty$.

(ii) [10, Theorem 2.3] *The sequence space $a^r(\upsilon,p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$.

Asma and Çolak [7] and Başar et al. [18] defined the spaces $\lambda(u,\Delta,p)$ and $bv(u,p)$ as the set of all sequences such that A^u-transforms of them are in the spaces $\lambda(p)$ and $\ell(p)$, respectively, where $\lambda \in \{c_0, c, \ell_\infty\}$ that is

$$\begin{aligned}
\ell_\infty(u,\Delta,p) = bv_\infty(u,p) &:= \{x=(x_k)\in\omega : \{u_k\Delta x_k\}\in \ell_\infty(p) < \infty\},\\
c(u,\Delta,p) &:= \{x=(x_k)\in\omega : \{u_k\Delta x_k\}\in c(p)\},\\
c_0(u,\Delta,p) &:= \{x=(x_k)\in\omega : \{u_k\Delta x_k\}\in c_0(p)\},\\
bv(u,p) &:= \{x=(x_k)\in\omega : \{u_k\Delta x_k\}\in \ell(p)\}, (0 < p_k \leq H < \infty).
\end{aligned}$$

Then, they obtained the following results:

(i) [7, Theorem 1.1] Let (p_k) be a bounded sequence of strictly positive real numbers and $u \in \mathcal{U}$. Then, $c_0(u,\Delta,p)$ is a paranormed space with paranorm $g(x) = \sup_{k\in\mathbb{N}} |u_k\Delta x_k|^{p_k/M}$. If $\inf_{k\in\mathbb{N}} p_k > 0$, then $\ell_\infty(u,\Delta,p)$ and $c(u,\Delta,p)$ are paranormed space with the same paranorm.

(ii) [18, Theorem 2.1] The space $bv(u,p)$ is a complete linear metric space paranormed by g defined by

$$g(x) = \left(\sum_k |u_k\Delta x_k|^{p_k}\right)^{1/M},$$

where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

(iii) [18, Theorem 2.3] The sequence spaces $bv(u,p)$ and $bv_\infty(u,p)$ of non-absolute type are linearly isomorphic to the spaces $\ell(p)$ and $\ell_\infty(p)$, respectively, where $0 < p_k \leq H < \infty$.

Kara et al. [30] defined the *Euler sequence space* $e^r(p)$ as the domain of the Euler matrix of order r, E^r in the space $\ell(p)$ as

$$e^r(p) := \{x=(x_k)\in\omega : E^r x \in \ell(p)\}, (0 < p_k \leq H < \infty).$$

Then, they gave the following results:

(i) [30, Theorem 1] $e^r(p)$ is a complete linear topological space paranormed by g defined by

$$g(x) = \left(\sum_k \left|\sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}r^j x_j\right|^{p_k}\right)^{1/M},$$

where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

(ii) [30, Theorem 2] The Euler sequence space $e^r(p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leq H < \infty$.

Başar and Çakmak [19] introduced the spaces $\lambda(B,p)$ as the domain of the triple band matrix $B(r,s,t)$ in the spaces $\lambda(p)$, where $\lambda \in \{c_0, c, \ell_\infty\}$, as

$$\lambda(B,p) \;:=\; \{x=(x_k)\in\omega : y=(tx_{k-2}+sx_{k-1}+rx_k)\in\lambda(p)\}.$$

If λ is any normed or paranormed sequence space then we call the matrix domain $\lambda_{B(r,s,t)}$ as the *generalized difference space of sequences*. If $p_k=1$ for all $k\in\mathbb{N}$, we write $\lambda(B)$ instead of $\lambda(B,p)$.

Theorem 4.4. *Başar and Çakmak* [19] *gave the following results:*

(i) [19, Theorem 2.1(a)] *The spaces $\lambda(B,p)$ are the complete linear metric spaces paranormed by g, defined by*

$$g(x)=\sup_{k\in\mathbb{N}}|tx_{k-2}+sx_{k-1}+rx_k|^{p_k/M}.$$

g is a paranorm for the spaces $\ell_\infty(B,p)$ and $c(B,p)$ only in the trivial case $\inf_{k\in\mathbb{N}} p_k>0$ *when $\ell_\infty(B,p)=\ell_\infty(B)$ and $c(B,p)=c(B)$.*

(ii) [19, Theorem 2.1(b)] *The sets $\lambda(B)$ are Banach spaces with the norm $\|x\|_{B(r,s,t)}=\|y\|_\lambda$.*

(iii) [19, Theorem 2.2] *The generalized difference space of sequences $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ of non-absolute type are paranormed isomorphic to the spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$, respectively, where $0<p_k\le H<\infty$.*

(iv) [19, Theorem 2.3] *Suppose that $|-s+\sqrt{s^2-4tr}|>2r$. Then, the sequence space $c_0(B)$ has AD-property.*

Nergiz and Başar [56] and Özger and Başar [59] defined the spaces $\lambda(\widetilde{B},p)$ as the set of all sequences whose $B(\widetilde{r},\widetilde{s})-$transforms are in the spaces $\ell(p)$ and $\lambda(p)$, respectively, where $\lambda\in\{\ell_\infty,c,c_0\}$, that is

$$\begin{aligned}
\ell(\widetilde{B},p) &:= \left\{x=(x_k)\in\omega : \sum_k |r_kx_k+s_{k-1}x_{k-1}|^{p_k}<\infty\right\}, (0<p_k\le H<\infty),\\
\ell_\infty(\widetilde{B},p) &:= \left\{x=(x_k)\in\omega : \sup_{k\in\mathbb{N}} |r_kx_k+s_{k-1}x_{k-1}|^{p_k}<\infty\right\},\\
c(\widetilde{B},p) &:= \left\{x=(x_k)\in\omega : \lim_{k\to\infty} |r_kx_k+s_{k-1}x_{k-1}-\ell|^{p_k}=0 \text{ for some } \ell\in\mathbb{R}\right\},\\
c_0(\widetilde{B},p) &:= \left\{x=(x_k)\in\omega : \lim_{k\to\infty} |r_kx_k+s_{k-1}x_{k-1}|^{p_k}=0\right\}.
\end{aligned}$$

and they obtained the following results:

(i) [56, Theorem 1] The spaces $\ell(\widetilde{B},p)$ is a complete linear metric spaces paranormed by g, defined by

$$g(x)=\left(\sum_k |r_kx_k+s_{k-1}x_{k-1}|^{p_k}\right)^{1/M}.$$

(ii) [56, Theorem 2] Convergence in $\ell(\widetilde{B},p)$ is stronger than coordinatewise convergence.

(iii) [56, Corollray 4] The sequence space $\ell(\widetilde{B},p)$ of non-absolute type is linearly paranorm isomorphic to the space $\ell(p)$, where $0<p_k\le H<\infty$.

(iv) [56, Theorem 5] The space $\ell(\widetilde{B},p)$ is has AK.

(v) [59, Theorem 3.1] The spaces $\lambda(\widetilde{B},p)$ are the complete linear metric spaces paranormed by g, defined by $g(x)=\sup_{k\in\mathbb{N}}|r_kx_k+s_{k-1}x_{k-1}|^{p_k/M}$.

Aydın and Altay [8] and Aydın and Başar [13] defined the spaces $\widehat{\lambda}(p)$ and $\widehat{\ell}(p)$ as the set of all sequences such that $B(r,s)-$transforms of them are in the spaces $\lambda(p)$ and $\ell(p)$, respectively, where

$\lambda \in \{\ell_\infty, c, c_0\}$, that is

$$\begin{aligned}
\widehat{\ell_\infty}(p) &:= \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |sx_{k-1} + rx_k|^{p_k} < \infty\right\}, \\
\widehat{c}(p) &:= \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |sx_{k-1} + rx_k - \ell|^{p_k} = 0 \text{ for some } \ell \in \mathbb{R}\right\}, \\
\widehat{c_0}(p) &:= \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |sx_{k-1} + rx_k|^{p_k} = 0\right\}, \\
\widehat{\ell}(p) &:= \left\{x = (x_k) \in \omega : \sum_k |sx_{k-1} + rx_k|^{p_k} < \infty\right\}, \quad (0 < p_k \le H < \infty).
\end{aligned}$$

In the case $p_k = p$ for all $k \in \mathbb{N}$ the sequence space $\widehat{\ell}(p)$ is reduced to the sequence space $\widehat{\ell_p}$ introduced by Kirişçi and Başar [31].

Theorem 4.5. *Aydın and Altay* [8] *and Aydın and Başar* [13] *obtained the following results:*

(i) [8, Theorem 2.1] *The spaces $\widehat{\lambda}(p)$ are the complete linear metric spaces paranormed by g, defined by*

$$g(x) = \sup_{k\in\mathbb{N}} |sx_{k-1} + rx_k|^{p_k/M}.$$

(ii) [8, Theorem 2.2] *The sequence spaces $\widehat{\ell_\infty}(p)$, $\widehat{c}(p)$ and $\widehat{c_0}(p)$ of non-absolute type are linearly isomorphic to the spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$, respectively, where $0 < p_k \le H < \infty$.*

(iii) [13, Theorem 2.1] *The space $\widehat{\ell}(p)$ is a complete linear metric spaces paranormed by g, defined by*

$$g(x) = \left(\sum_k |sx_{k-1} + rx_k|^{p_k}\right)^{1/M},$$

where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.

(iv) [13, Theorem 2.2] *The space $\widehat{\ell_p}$ is the linear space under the coordinatewise addition and scalar multiplication which is the BK-space with the norm*

$$\|x\| = \left(\sum_k |sx_{k-1} + rx_k|^p\right)^{1/p}, \quad 1 \le p < \infty.$$

(v) [13, Corollary 2.3] *The sequence space $\widehat{\ell}(p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$.*

Yeşilkayagil and Başar [60, 61] defined the *Nörlund sequence spaces* $N^t(p)$ and $\lambda(N^t, p)$ as the set of all sequences whose Nörlund transforms are in the spaces $\ell(p)$ and $\lambda(p)$, respectively, where $\lambda \in \{\ell_\infty, c, c_0\}$, as

$$\begin{aligned}
N^t(p) &:= \{x = (x_k) \in \omega : Nx \in \ell(p)\}, \\
\ell_\infty(N^t, p) &:= \{x = (x_k) \in \omega : Nx \in \ell_\infty(p)\}, \\
c(N^t, p) &:= \{x = (x_k) \in \omega : Nx \in c(p)\}, \\
c_0(N^t, p) &:= \{x = (x_k) \in \omega : Nx \in c_0(p)\}.
\end{aligned}$$

Theorem 4.6. *Yeşilkayagil and Başar* [60, 61] *obtained the following results:*

(i) [60, Theorem 1] *The space $N^t(p)$ is a complete linear metric spaces paranormed by g, defined by*

$$g(x) = \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k}\right)^{1/M} \quad \text{with } 0 < p_k \le H < \infty.$$

(ii) [60, Theorem 3] *The Nörlund sequence space $N^t(p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.*

(iii) [61, Theorem 2.1] *The spaces $\lambda(N^t,p)$ are the complete linear metric spaces paranormed by g, defined by*

$$g(x)=\sup_{k\in\mathbb{N}}\left|\frac{1}{T_k}\sum_{j=0}^{k}t_{k-j}x_j\right|^{p_k/M}.$$

(iv) [61, Theorem 2.2] *The spaces $\ell_\infty(N^t,p)$, $c(N^t,p)$ and $c_0(N^t,p)$ of non-absolute type are linearly isomorphic to the space $\ell_\infty(p)$, $c(p)$ and $c_0(p)$, respectively, where $0<p_k\leq H<\infty$ for all $k\in\mathbb{N}$.*

Çapan and Başar [23] have defined the domain space $\ell(F,p)$ of the band matrix F in the sequence space $\ell(p)$ as

$$\ell(F,p) \quad := \quad \left\{x=(x_k)\in\omega:\sum_k\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|^{p_k}<\infty\right\}.$$

If we take $p_k=p$ for all $k\in\mathbb{N}$, the space $\ell(F,p)$ is reduced to the space $\ell_p(F)$.

Theorem 4.7. *Çapan and Başar* [23] *have obtained the following results:*

(i) [23, Theorem 2.1] *$\ell(F,p)$ is a linear complete metric space paranormed by g defined by*

$$g(x)=\left(\sum_k\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|^{p_k}\right)^{1/M} \quad \text{with } 0<p_k\leq H<\infty.$$

(ii) [23, Theorem 2.2] *Convergence in $\ell(F,p)$ is strictly stronger than coordinatewise convergence, but the converse is not true, in general.*

(iii) [23, Theorem 2.4] *$\ell(F,p)$ is a $K-$space.*

(iv) [23, Theorem 2.5] *$\ell(F,p)$ is an $FK-$space.*

(v) [23, Theorem 2.6] *$\ell_p(F)$ is the linear space under the coordinatewise addition and scalar multiplication which is a $BK-$space with the norm $\|x\|=\left(\sum_k\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|^{p}\right)^{1/p}$, where $x\in\ell_p(F)$ and $1\leq p<\infty$.*

(vi) [23, Theorem 2.8] *$\ell_p(F)$ is a Fréchet space.*

(vii) [23, Corollary 2.1] *The sequence space $\ell_p(F)$ of non-absolute type is linearly paranorm isomorphic to the space $\ell(p)$, where $0<p_k\leq H<\infty$ for all $k\in\mathbb{N}$.*

Benefiting from Başar's book [15], we give the following table for the concerning literature about the domain λ_A of an infinite matrix A in a Maddox's space λ:

Table 1. The domains of some triangle matrices in Maddox's spaces.

λ	A	λ_A	refer to:
$\ell_\infty(p), c(p), c_0(p)$	Δ	$\Delta\ell_\infty(p), \Delta c(p), \Delta c_0(p)$	[1, 22, 44]
$\ell_\infty(p)$	S	$bs(p)$	[14, 16]
$\ell(p)$	R^t	$r^t(p)$	[2]
$\ell_\infty(p), c(p), c_0(p)$	R^t	$r^t_\infty(p), r^t_c(p), r^t_0(p)$	[4]
$\ell_\infty(p), c(p), c_0(p)$	$G(u,\nu)$	$\ell_\infty(u,\nu;p), c(u,\nu;p), c_0(u,\nu;p)$	[3]
$\ell(p)$	$G(u,\nu)$	$\ell(u,\nu;p)$	[5]
$c(p), c_0(p)$	A^r	$a^r_c(\upsilon;p), a^r_0(\upsilon;p)$	[9]
$\ell(p)$	A^r	$a^r(\upsilon;p)$	[10]
$\ell_\infty(p), c(p), c_0(p)$	A^u	$\ell_\infty(u,\Delta;p), c(u,\Delta;p), c_0(u,\Delta;p)$	[7]
$\ell_\infty(p), \ell(p)$	A^u	$bv_\infty(u,\Delta;p), bv(u,\Delta;p)$	[18]
$\ell(p)$	E^r	$e^r(p)$	[30]
$\ell_\infty(p), c(p), c_0(p)$	$B(r,s,t)$	$\ell_\infty(B,p), c(B,p), c_0(B,p)$	[19]
$\ell(p)$	$B(\widetilde{r},\widetilde{s})$	$\ell(\widetilde{B},p)$	[56]
$\ell_\infty(p), c(p), c_0(p)$	$B(\widetilde{r},\widetilde{s})$	$\ell_\infty(\widetilde{B},p), c(\widetilde{B},p), c_0(\widetilde{B},p)$	[59]
$\ell_\infty(p), c(p), c_0(p)$	$B(r,s)$	$\widehat{\ell_\infty}(p), \widehat{c}(p), \widehat{c}_0(p)$	[8]
$\ell(p)$	$B(r,s)$	$\widehat{\ell}(p)$	[13]
$\ell(p)$	N^t	$N^t(p)$	[60]
$\ell_\infty(p), c(p), c_0(p)$	N^t	$\ell_\infty(N^t,p), c(N^t,p), c_0(N^t,p)$	[61]
$\ell(p)$	F	$\ell(F,p)$	[23]

5. Dual spaces

For the sequence spaces λ and μ, the set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \ \text{ for all } \ x \in \lambda\}, \tag{12}$$

is called the *multiplier space* λ and μ. One can observe that for a sequence space η with $\mu \subset \eta \subset \lambda$ that the inclusions $S(\lambda,\mu) \subset S(\eta,\mu)$ and $S(\lambda,\mu) \subset S(\lambda,\eta)$ hold. With the notation of (12), the *alpha-*, *beta-* and *gamma*-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda,\ell_1), \quad \lambda^\beta = S(\lambda,cs) \ \text{ and } \ \lambda^\gamma = S(\lambda,bs).$$

Let $\eta \in \{\alpha,\beta,\gamma\}$ and let λ be a sequence space. λ is called a $\eta-$*space* if $\lambda = \lambda^{\eta\eta}$. Further, an $\alpha-$space is also called a *Köthe space* or *perfect sequence space*.

Define the sets $\mathcal{M}(p)$, $\mathcal{M}_\infty(p)$, $\mathcal{M}_0(p)$, $\mathcal{K}(p)$, $\mathcal{S}(p)$, $\mathcal{L}(p)$ and $\mathcal{Q}$ as:

$$\begin{aligned}
\mathcal{M}(p) &:= \bigcap_{B>1}\left\{a = (a_k) \in \omega : \sum_k |a_k|^{q_k} B^{-p_k/q_k} < \infty\right\},\\
\mathcal{M}_\infty(p) &:= \bigcap_{B>1}\left\{a = (a_k) \in \omega : \sum_k |a_k| B^{1/p_k} < \infty\right\},\\
\mathcal{M}_0(p) &:= \bigcup_{B>1}\left\{a = (a_k) \in \omega : \sum_k |a_k| B^{-1/p_k} < \infty\right\},\\
\mathcal{K}(p) &:= \bigcap_{B>1}\left\{a = (a_k) \in \omega : \sum_r \max_{2^r \le k \le 2^{r+1}} |2^{r/p_k} a_k| < \infty\right\},\\
\mathcal{S}(p) &:= \left\{a = (a_k) \in \omega : \sup_{r\in\mathbb{N}} 2^r \max_{2^r \le k \le 2^{r+1}} |a_k|^{p_k} < \infty\right\},
\end{aligned}$$

$$\mathcal{L}(p) := \bigcap_{B>1}\left\{a=(a_k)\in\omega : \sum_r \max_{2^r\le k\le 2^{r+1}}(2^rB^{-1})^{1/p_k}|a_k|<\infty\right\},$$

$$\mathcal{Q} := \left\{p=(p_k)\in\omega : \text{there exists a } B>1 \ni \sum_k B^{-1/p_k}<\infty\right\},$$

$$\mathcal{V} := \bigcap_{B>1}\left\{a\in\omega : \sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-1}B^{1/p_j} \text{ converges and } \sum_{k=1}^{n}B^{1/p_k}|G_k|<\infty\right\},$$

where $G_k=\sum_{v=k+1}^{\infty}a_v$ for all $k\in\mathbb{N}$.

Theorem 5.1. *Let $\inf_{k\in\mathbb{N}}p_k=h$ and $\sup_{k\in\mathbb{N}}p_k=H$. Then, the following statements hold:*

(i) [58, Theorem 7] *The dual space of $\ell(p)$ was shown in Simons* [58] *to be $\ell_\infty(p)$ when $0<p_k\le 1$.*
(ii) [35, Theorem 6] *Let $0<h\le p_k\le 1$ for all $k\in\mathbb{N}$. Then, the set $\mathcal{K}(p)$ is the dual space of $\omega(p)$.*
(iii) [35, Remark of Theorem 6] *$f(x)=\sum_k a_kx_k$ defines an element of $\omega_0^*(p)$ without restriction $0<h\le p_k$, where $x\in\omega_0(p)$ and $a\in\mathcal{K}(p)$.*
(iv) [36, Theorem 3] *Let $p\in\mathcal{Q}$. Then, $\omega_0^*(p)$ is $\mathcal{S}(p)$.*
(v) [36, Theorem 4] *Let $0<h\le H<\infty$ for all $k\in\mathbb{N}$. Then, $\ell^*(p)$ is $\ell(q)$, where $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$.*
(vi) [36, Note of Theorem 4] *$c_0^*(p)=\ell_1$ when $h>0$ and $c_0^*(p)=\ell_\infty(p)$ when $p\in\mathcal{Q}$.*
(vii) [38, Theorem 1] *Let $1<p_k\le H$ for all $k\in\mathbb{N}$. Then, $\{\ell(p)\}^\beta=\mathcal{M}(p)$.*
(viii) [38, Theorem 2] *Let $1<p_k\le H$ for all $k\in\mathbb{N}$. Then, $\ell(p)^*$ is isomorphic to $\mathcal{M}(p)$.*
(ix) [38, Theorem 3] *If $1<h\le H<\infty$ for all $k\in\mathbb{N}$, then $\ell(p)$ and $\mathcal{M}(p)$ are linearly homeomorphic.*
(x) [38, Theorem 4] *If $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$ and $\ell(q)$ has its natural paranorm topology, then $\ell(p)^*$ is linearly homeomorphic to $\ell(q)$, where $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$.*
(xi) [38, Theorem 6] *Let $p_k>0$ for all $k\in\mathbb{N}$. Then, $\{c_0(p)\}^\beta=\mathcal{M}_0(p)$ when $H<\infty$, $c_0^*(p)$ is isomorphic to $\mathcal{M}_0(p)$ and when in addition, $h>0$, $c_0^*(p)$ is linearly isomorphic to ℓ_1.*
(xii) [34, Theorem 2] *Let $p_k>0$ for all $k\in\mathbb{N}$. Then, $\{\ell_\infty(p)\}^\beta=\mathcal{M}_\infty(p)$.*
(xiii) [34, Theorem 4] *Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $\{\omega(p)\}^\beta=\mathcal{L}(p)$.*
(xiv) [33, Theorem 1] *For every (p_k), $\{c(p)\}^\beta=\{c_0(p)\}^\beta\cap cs$.*
(xv) [33, Theorem 2] *For every (p_k), $\{c_0(p)\}^{\beta\beta}=\bigcap_{B>1}\left\{a\in\omega:\sup_k|a_k|B^{1/p_k}<\infty\right\}$.*
(xvi) [33, Theorem 3] *For every (p_k), $\{\ell_\infty(p)\}^{\beta\beta}=\bigcup_{B>1}\left\{a\in\omega:\sup_k|a_k|B^{-1/p_k}<\infty\right\}$.*
(xvii) [33, Theorem 6] *The following statements are equivalent:*
  (1) *$h>0$.*
  (2) *$\{\ell_\infty(p)\}^\beta=\ell_1$.*
  (3) *$\{\ell_\infty(p)\}^{\beta\beta}=\ell_\infty$.*
(xviii) [33, Theorem 7] *The following statements are equivalent:*
  (1) *$\{c(p)\}^\beta=\ell_\infty$.*
  (2) *$h>0$.*
  (3) *$c_0\subset c_0(p)$.*

Theorem 5.2. *The following statements hold:*

(i) [33, Theorem 4(i)] *Let $p_k>1$ for all $k\in\mathbb{N}$. Then, $\ell(p)$ is perfect if and only if $p\in\ell_\infty$.*
(ii) [33, Theorem 4(ii)] *Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $\ell(p)$ is perfect if and only if $\ell(p)=\ell_1$.*
(iii) ([33, Theorem 5] *and* [1, Theorem 2.3]) *$\ell_\infty(p)$ is perfect if and only if $p\in\ell_\infty$.*
(iv) [33, Theorem 8] *$c_0(p)$ is perfect if and only if $p\in c_0$.*

Theorem 5.3. *For every sequence (p_k), Ahmad and Mursaleen* [1] *gave the following results:*

(i) [1, Theorem 2.1] $\{\Delta\ell_\infty(p)\}^\alpha=\bigcap_{B>1}\left\{a\in\omega:\sum_k k|a_k|B^{1/p_k}<\infty\right\}$.
(ii) [1, Theorem 2.2] $\{\Delta\ell_\infty(p)\}^{\alpha\alpha}=\bigcup_{B>1}\left\{a\in\omega:\sup_k(k^{-1}|a_k|)B^{-1/p_k}<\infty\right\}$.

(iii) [1, Remark of Theorem 2.2] (p_k), $\{\Delta c_0(p)\}^{\alpha\alpha} = \bigcap_{B>1} \left\{a \in \omega : \sup_k (k^{-1}|a_k|) B^{1/p_k} < \infty\right\}$.

Theorem 5.4. *For every strictly positive sequence (p_k) and for every $u \in \mathcal{U}$, Malkowsky* [44], *Asma and Çolak* [7] *and Başar and Altay* [16] *gave the following results:*

(i) ([44, Theorem 2.1(a)] and [22, Theorem 1]) $\{\Delta \ell_\infty(p)\}^\alpha = \bigcap_{B>1} \left\{a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} B^{1/p_j} < \infty\right\}$.

(ii) [44, Theorem 2.1(b)] $\{\Delta \ell_\infty(p)\}^{\beta\beta} = \bigcup_{B>1} \left\{a \in \omega : \sup_{k\geq 2} |a_k| \left[\sum_{j=1}^{k-1} B^{1/p_j}\right]^{-1} < \infty\right\}$.

(iii) [44, Theorem 2.1(c)] $\{\Delta c_0(p)\}^\alpha = \mathcal{D}_0 = \bigcup_{B>1} \left\{a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} B^{-1/p_j} < \infty\right\}$.

(iv) [44, Theorem 2.1(d)] $\{\Delta c_0(p)\}^{\alpha\alpha} = \bigcap_{B>1} \left\{a \in \omega : \sup_{k\geq 2} |a_k| \left[\sum_{j=1}^{k-1} B^{-1/p_j}\right]^{-1} < \infty\right\}$.

(v) [44, Theorem 2.2(a)] $\{\Delta c(p)\}^\alpha = \mathcal{D}_0 \cap \left\{a \in \omega : \sum_{k=1}^{\infty} k|a_k| < \infty\right\}$.

(vi) [44, Theorem 2.2(b)] $\{\Delta \ell_\infty(p)\}^\beta = \mathcal{V}$.

(vii) [7, Theorem 2.1(i)] $\{\ell_\infty(u, \Delta, p)\}^\alpha = \bigcap_{B>1} \left\{a \in \omega : \sum_k |a_k| \sum_{j=1}^{k-1} B^{1/p_j}/u_j < \infty\right\}$.

(viii) [7, Theorem 2.1(ii)] $\{c_0(u, \Delta, p)\}^\alpha = \mathcal{D} = \bigcup_{B>1} \left\{a \in \omega : \sum_k |a_k| \sum_{j=1}^{k-1} B^{1/p_j}/u_j < \infty\right\}$.

(ix) [7, Theorem 2.1(iii)] $\{c(u, \Delta, p)\}^\alpha = \mathcal{D} \cup \left\{a \in \omega : \sum_k |a_k| \sum_{j=1}^{k-1} 1/u_j < \infty\right\}$.

(x) [7, Theorem 2.4] $\{\ell_\infty(u, \Delta, p)\}^\beta = \mathcal{V}$ *with* $R_k = \frac{1}{u_k} \sum_{v=k+1}^{\infty} a_v$ *for all* $k \in \mathbb{N}$ *instead of* G_k.

(xi) [16, Theorem 2.3] $\{bs(p)\}^\alpha = \mathcal{M}_\infty(p) \cap \bigcap_{B>1} \left\{a \in \omega : \sum_k |\Delta a_k| B^{1/p_k} < \infty\right\}$.

(xii) [16, Theorem 2.3] $\{bs(p)\}^\beta = \bigcap_{B>1} \left\{a \in \omega : \sum_k |\Delta a_k| B^{1/p_k} < \infty \text{ and } \left\{a_k B^{1/p_k}\right\} \in c_0\right\}$.

(xiii) [16, Theorem 2.3] $\{bs(p)\}^\gamma = \bigcap_{B>1} \left\{a \in \omega : \sum_k |\Delta a_k| B^{1/p_k} < \infty \text{ and } \left\{a_k B^{1/p_k}\right\} \in \ell_\infty\right\}$.

Lemma 5.1. [6, Theorem 3.1] *Let $E = (e_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $Q = (q_{nk})$ by*

$$e_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk} & , \quad 0 \leq k \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k \in \mathbb{N}$. Then,

$$\{\lambda_Q\}^\beta = \{a = (a_k) \in \omega : E \in (\lambda : c)\},$$
$$\{\lambda_Q\}^\gamma = \{a = (a_k) \in \omega : E \in (\lambda : \ell_\infty)\}.$$

Following Altay and Başar [6], *we can say that*

$$\{\lambda_Q\}^\alpha = \{a = (a_k) \in \omega : E \in (\lambda : \ell_1)\},$$

under same conditions.

Define the inverses of the matrices given in (9), respectively, $\{R^t\}^{-1} = (r_{nk})$, $\{N^t\}^{-1} = (u_{nk})$, $\{G(u,\nu)\}^{-1} = (h_{nk})$, $\{B(r,s)\}^{-1} = (b_{nk})$, $\{B(\widetilde{r},\widetilde{s})\}^{-1} = (\varsigma_{nk})$, $\{A^r\}^{-1} = (\zeta_{nk})$, $F^{-1} = (z_{nk})$, $\{A^u\}^{-1} =$

(ϱ_{nk}), $\{B(r,s,t)\}^{-1}=(\xi_{nk})$ and $\{E^r\}^{-1}=(\delta_{nk})$ by

$$r_{nk}=\begin{cases}\frac{(-1)^{n-k}T_k}{t_n} & , \ n-1\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases}, \qquad u_{nk}=\begin{cases}(-1)^{n-k}D_{n-k}T_k & , \ 0\le k\le n,\\ 0 & , \ k>n\end{cases}$$

$$h_{nk}=\begin{cases}\frac{(-1)^{n-k}}{u_k\nu_n} & , \ n-1\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases}, \qquad b_{nk}=\begin{cases}\frac{1}{r}\left(-\frac{s}{r}\right)^{n-k} & , \ 0\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases},$$

$$\varsigma_{nk}=\begin{cases}\frac{(-1)^{n-k}}{r_n}\prod_{i=k}^{n-1}\frac{s_i}{r_i} & , \ 0\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases}, \qquad \zeta_{nk}=\begin{cases}(-1)^{n-k}\frac{(1+k)}{(1+r^n)u_n} & , \ n-1\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases}$$

$$\varrho_{nk}=\begin{cases}1/u_k & , \ 0\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases}, \qquad z_{nk}=\begin{cases}\frac{f_{n+1}^2}{f_kf_{k+1}} & , \ 0\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases}$$

$$\xi_{nk}=\begin{cases}\frac{1}{r}\sum_{j=0}^{n-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{n-k-j}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{j} & , \ 0\le k\le n,\\ 0 & , \ \text{otherwise}\end{cases}$$

$$\delta_{nk}=\begin{cases}\binom{n}{k}(r-1)^{n-k}r^{-k} & , \ 0\le k\le n\\ 0 & , \ \text{otherwise}\end{cases}$$

for all $k,n\in\mathbb{N}$, where $D_0=1$ and

$$D_n=\begin{vmatrix} t_1 & 1 & 0 & 0 & \dots & 0\\ t_2 & t_1 & 1 & 0 & \dots & 0\\ t_3 & t_2 & t_1 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \dots & 1\\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \dots & t_1\end{vmatrix}.$$

for $n\in\{1,2,3,\ldots\}$. Also, $\Delta^{-1}=(s_{nk})$ is as in (3).

Define the sets $d_1(p)-d_{14}(p)$ as:

$$d_1(p):=\bigcup_{B>1}\left\{a\in\omega:\sup_{N\in\mathcal{F}}\sum_k\left|\sum_{n\in N}a_nv_{nk}B^{-1}\right|^{q_k}<\infty\right\},$$

$$d_2(p):=\bigcup_{B>1}\left\{a\in\omega:\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}a_jv_{jk}B^{-1}\right|^{q_k}<\infty\right\},$$

$$d_3(p):=\left\{a)\in\omega:\sup_{N\in\mathcal{F}}\sup_{k\in\mathbb{N}}\left|\sum_{n\in N}a_nv_{nk}\right|^{p_k}<\infty\right\},$$

$$d_4(p):=\left\{a\in\omega:\sup_{k,n\in\mathbb{N}}\left|\sum_{j=k}^{n}a_jv_{jk}\right|^{p_k}<\infty\right\},$$

$$d_5(p):=\bigcap_{B>1}\left\{a\in\omega:\sup_{N\in\mathcal{F}}\sum_k\left|\sum_{n\in N}a_nv_{nk}B^{1/p_k}\right|<\infty\right\},$$

$$d_6(p):=\bigcap_{B>1}\left\{a\in\omega:\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}a_jv_{jk}\right|B^{1/p_k}<\infty\right\},$$

$$d_7(p):=\bigcup_{B>1}\left\{a\in\omega:\sup_{N\in\mathcal{F}}\sum_n\left|\sum_{k\in N}a_nv_{nk}B^{-1/p_k}\right|<\infty\right\},$$

$$d_8(p):=\left\{a\in\omega:\sum_n\left|\sum_k a_nv_{nk}\right|<\infty\right\},$$

$$d_9(p):=\bigcup_{B>1}\left\{a\in\omega:\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}a_jv_{jk}\right|B^{-1/p_k}<\infty\right\},$$

$$d_{10}(p):=\bigcap_{B>1}\left\{a\in\omega:\exists(\alpha_k)\in\omega\ni\lim_{n\to\infty}\sum_k\left|\sum_{j=k}^{n}a_jv_{jk}-\alpha_k\right|B^{1/p_k}=0\right\},$$

$$d_{11}(p) := \bigcup_{B>1}\left\{a\in\omega : \exists(\alpha_k)\in\omega \ni \sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n} a_j v_{jk} - \alpha_k\right| B^{-1/p_k} < \infty\right\},$$

$$d_{12}(p) := \left\{a\in\omega : \lim_{n\to\infty}\sum_k\left|\sum_{j=k}^{n} a_j v_{jk} - \alpha\right| = 0\right\},$$

$$d_{13}(p) := \left\{a\in\omega : \exists(\alpha_k)\in\omega \ni \lim_{n\to\infty}\left|\sum_{j=k}^{n} a_j v_{jk} - \alpha_k\right| = 0\right\},$$

$$d_{14}(p) := \left\{a\in\omega : \sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n} a_j v_{jk}\right| < \infty\right\}.$$

Theorem 5.5. *Taking r_{nk}, ζ_{nk}, ϱ_{nk}, δ_{nk}, ξ_{nk}, b_{nk}, ς_{nk}, z_{nk} and u_{nk} instead of v_{nk}, respectively, Altay and Başar* [2, 4]*, Aydn and Başar* [9, 10]*, Başar et al.* [18]*, Kara et al.* [30]*, Başar and Çakmak* [19]*, Aydın and Altay* [8] *and Aydn and Başar* [13]*, Nergiz and Başar* [56] *and Özger and Başar* [59]*, Çapan and Başar* [23]*, Yeşilkayagil and Başar* [60, 61] *obtained the following results:*

(i) [2, Theorem 2.7] *Let $1 < p_k \le H < \infty$ for all $k\in\mathbb{N}$. Then,*
  (a) $\{r^t(p)\}^{\alpha} = d_1(p)$.
  (b) $\{r^t(p)\}^{\beta} = \{r^t(p)\}^{\gamma} = d_2(p)\bigcap \bigcup_{B>1}\{a\in\omega : \{(a_kT_kB^{-1}/t_k)^{q_k}\}\in\ell_\infty\}$.
(ii) [2, Theorem 2.8] *Let $0 < p_k \le 1$ for all $k\in\mathbb{N}$. Then,*
  (a) $\{r^t(p)\}^{\alpha} = d_3(p)$.
  (b) $\{r^t(p)\}^{\beta} = \{r^t(p)\}^{\gamma} = \{a\in\omega : d_4(p)\cap\{(a_kT_k/t_k)^{p_k}\}\in\ell_\infty\}$.
(iii) [4, Theorem 2.6] $\{r^t_\infty(p)\}^{\alpha} = d_5(p)$, $\{r^t_\infty(p)\}^{\beta} = d_6(p)\bigcap\cap_{B>1}\{a\in\omega : \{a_kT_kB^{1/p_k}/t_k\}\in c_0\}$ *and* $\{r^t_\infty(p)\}^{\gamma} = d_6(p)\bigcap\cap_{B>1}\{a\in\omega : \{\Delta(a_k/t_k)T_kB^{1/p_k}\}\in\ell_\infty\}$.
(iv) [4, Theorem 2.6] $\{r^t_c(p)\}^{\alpha} = d_7(p)\cap d_8(p)$, $\{r^t_c(p)\}^{\beta} = d_9(p)\cap cs$ *and* $\{r^t_c(p)\}^{\gamma} = d_9(p)\cap bs$.
(v) [4, Theorem 2.6] $\{r^t_0(p)\}^{\alpha} = d_7(p)$ *and* $\{r^t_0(p)\}^{\beta} = \{r^t_c(p)\}^{\gamma} = d_9(p)$.
(vi) [9, Theorem 4.5] $\{a^r_0(p)\}^{\beta} = \{a^r_0(p)\}^{\gamma} = d_9(p)\bigcap\bigcup_{B>1}\left\{a\in\omega : \left\{\frac{k+1}{(1+r^k)u_k}a_kB^{-1/p_k}\right\}_{k\in\mathbb{N}}\in\ell_\infty\right\}$ *and* $\{a^r_0(p)\}^{\alpha} = d_7(p)$.
(vii) [9, Theorem 4.5] $\{a^r_c(p)\}^{\alpha} = d_7(p)\cap d_3(p)$, $\{a^r_c(p)\}^{\beta} = \{a^r_0(p)\}^{\beta}\cap\left\{a\in\omega : \left\{\frac{a_k}{(1+r^k)u_k}\right\}_{k\in\mathbb{N}}\in cs\right\}$ *and* $\{a^r_c(p)\}^{\gamma} = \{a^r_0(p)\}^{\gamma}\cap\left\{a\in\omega : \left\{\frac{a_k}{(1+r^k)u_k}\right\}_{k\in\mathbb{N}}\in bs\right\}$.
(viii) *Let $1 < p_k \le H < \infty$ for all $k\in\mathbb{N}$. Then,*
  (a) [10, Theorem 3.4(ii)] $\{a^r(u,p)\}^{\alpha} = d_2(p)$.
  (b) [10, Theorem 3.5(ii)] $\{a^r(u,p)\}^{\beta} = d_2(p)\bigcap\bigcup_{B>1}\left\{a\in\omega : \left\{\left(\frac{k+1}{(1+r^k)u_k}a_kB^{-1}\right)^{q_k}\right\}_{k\in\mathbb{N}}\in\ell_\infty\right\}$.
  (c) [10, Theorem 3.6(ii)] $\{a^r(u,p)\}^{\gamma} = \{a^r(u,p)\}^{\beta}$.
(ix) *Let $0 < p_k \le 1$ for all $k\in\mathbb{N}$. Then,*
  (a) [10, Theorem 3.4(i)] $\{a^r(u,p)\}^{\alpha} = d_3(p)$.
  (b) [10, Theorem 3.5(i)] $\{a^r(u,p)\}^{\beta} = d_4(p)\bigcap\left\{a\in\omega : \left\{\left(\frac{k+1}{(1+r^k)u_k}a_k\right)^{p_k}\right\}_{k\in\mathbb{N}}\in\ell_\infty\right\}$.
  (c) [10, Theorem 3.6(i)] $\{a^r(u,p)\}^{\gamma} = \{a^r(u,p)\}^{\beta}$.
(x) [18, Theorems 3.4-3.5(i)] $\{bv(u,p)\}^{\alpha} = d_3(p)$, $\{bv(u,p)\}^{\beta} = d_4(p)\cap cs$, $\{bv(u,p)\}^{\gamma} = d_4(p)$, *where $0 < p_k \le 1$ for all $k\in\mathbb{N}$.*
(xi) [18, Theorems 3.4-3.5(ii)] $\{bv(u,p)\}^{\alpha} = d_1(p)$, $\{bv(u,p)\}^{\beta} = d_2(p)\cap cs$, $\{bv(u,p)\}^{\gamma} = d_2(p)$, *where $1 < p_k \le H < \infty$ for all $k\in\mathbb{N}$.*
(xii) [18, Theorem 3.6] $\{bv_\infty(u,p)\}^{\alpha} = d_5(p)$, $\{bv_\infty(u,p)\}^{\beta} = d_6(p)\cap d_{10}(p)$, $\{bv_\infty(u,p)\}^{\gamma} = d_6(p)$.
(xiii) [30, Theorem 3] *Let $1 < p_k \le H < \infty$ for all $k\in\mathbb{N}$. Then,* $\{e^r(p)\}^{\alpha} = d_1(p)$.
(xiv) [30, Theorem 4] *Let $1 < p_k \le H < \infty$ for all $k\in\mathbb{N}$. Then,* $\{e^r(p)\}^{\gamma} = d_2(p)$ *and* $\{e^r(p)\}^{\beta} = d_2(p)\bigcap\left\{a\in\omega : \sum_{j=k}^{\infty}\binom{j}{k}(r-1)^{j-k}r^{-j}a_j \text{ exists for each } k\in\mathbb{N}\right\}$.

(xv) [30, Theorem 5] *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $\{e^r(p)\}^\alpha = d_3(p)$, $\{e^r(p)\}^\gamma = d_4(p)$ *and* $\{e^r(p)\}^\beta = d_4(p) \bigcap \left\{a \in \omega : \sum_{j=k}^{\infty} \binom{j}{k}(r-1)^{j-k}r^{-j}a_j \text{ exists for each } k \in \mathbb{N}\right\}$.

(xvi) [19, Theorems 2.9-2.11] $\{\ell_\infty(B,p)\}^\alpha = d_5(p)$, $\{\ell_\infty(B,p)\}^\beta = d_6(p) \cap d_{10}(p)$, $\{\ell_\infty(B,p)\}^\gamma = d_6(p)$.

(xvii) [8, Corollary 2.11] $\{\widehat{\ell}_\infty(p)\}^\beta = d_6(p) \cap d_{10}(p)$, $\{\widehat{\ell}_\infty(p)\}^\gamma = d_6(p)$, $\{\widehat{c}_0(p)\}^\beta = d_9(p) \cap d_{11}(p) \cap d_{13}(p)$, $\{\widehat{c}_0(p)\}^\gamma = d_9(p)$, $\{\widehat{c}(p)\}^\beta = d_9(p) \cap d_{11}(p) \cap d_{12}(p) \cap d_{13}(p)$, $\{\widehat{c}(p)\}^\gamma = d_{14}(p)$.

(xviii) *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,*

(a) [13, Theorem 3.4] $\{\widehat{\ell}(p)\}^\alpha = d_1(p)$.

(b) [13, Theorem 3.5] $\{\widehat{\ell}(p)\}^\beta = d_2(p) \bigcap \bigcup_{B>1} \left\{a \in \omega : \left\{\sum_{j=k}^{n} \left(-\frac{s}{r}\right)^{n-k} a_j\right\}_{n\in\mathbb{N}} \in c\right\}$.

(c) [13, Theorem 3.6] $\{\widehat{\ell}(p)\}^\gamma = d_2(p)$.

(xix) *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,*

(a) [13, Theorem 3.4] $\{\widehat{\ell}(p)\}^\alpha = d_3(p)$.

(b) [13, Theorem 3.5] $\{\widehat{\ell}(p)\}^\beta = \left\{a \in \omega : d_4(p) \cap \left\{\sum_{j=k}^{n} \left(-\frac{s}{r}\right)^{n-k} a_j\right\}_{n\in\mathbb{N}} \in c\right\}$.

(c) [13, Theorem 3.6] $\{\widehat{\ell}(p)\}^\gamma = d_4(p)$.

(xx) [56, Theorems 10-12] *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $\{\ell(\widetilde{B},p)\}^\alpha = d_3(p)$, $\{\ell(\widetilde{B},p)\}^\gamma = d_2(p)$, $\{\ell(\widetilde{B},p)\}^\beta = d_4(p) \cap \mathcal{Z}$, *where* $\mathcal{Z} = \left\{a \in \omega : \sum_{i=k}^{\infty} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} < \infty\right\}$.

(xxi) [56, Theorems 10-12] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $\{\ell(\widetilde{B},p)\}^\alpha = d_1(p)$, $\{\ell(\widetilde{B},p)\}^\gamma = d_2(p)$, $\{\ell(\widetilde{B},p)\}^\beta = d_2(p) \cap \mathcal{Z}$.

(xxii) [59, Theorem 4.1] $\{c_0(\widetilde{B},p)\}^\alpha = d_7(p)$, $\{c_0(\widetilde{B},p)\}^\gamma = d_9(p)$, $\{c_0(\widetilde{B},p)\}^\beta = d_9(p) \cap d_{11}(p)$, $\{c(\widetilde{B},p)\}^\alpha = d_7(p) \cap d_8(p)$, $\{c(\widetilde{B},p)\}^\beta = d_9(p) \cap d_{11}(p) \cap cs$, $\{c(\widetilde{B},p)\}^\gamma = d_9(p) \cap bs$, $\{\ell_\infty(\widetilde{B},p)\}^\alpha = d_5(p)$, $\{\ell_\infty(\widetilde{B},p)\}^\beta = d_6(p) \cap cs$, $\{\ell_\infty(\widetilde{B},p)\}^\gamma = d_6(p)$.

(xxiii) [23, Theorems 3.4-3.6] *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $\{\ell(F,p)\}^\alpha = d_3(p)$, $\{\ell(F,p)\}^\gamma = d_4(p)$, $\{\ell(F,p)\}^\beta = d_4(p) \cap d_{13}(p)$.

(xxiv) [23, Theorems 3.4-3.6] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $\{\ell(F,p)\}^\alpha = d_2(p)$, $\{\ell(F,p)\}^\gamma = d_4(p)$, $\{\ell(F,p)\}^\beta = d_2(p) \cap d_{13}(p)$.

(xxv) [60, Theorem 8] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $\{N^t(p)\}^\alpha = d_1(p)$, $\{N^t(p)\}^\alpha\}^\gamma = d_2(p)$, $\{N^t(p)\}^\alpha\}^\beta = d_2(p) \cap cs$.

(xxvi) [60, Theorem 9] *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $\{N^t(p)\}^\alpha = d_3(p)$, $\{N^t(p)\}^\gamma = d_4(p)$, $\{N^t(p)\}^\beta = d_4(p) \cap \{a \in \omega : \{(a_nT_n)^{p_k}\} \in \ell_\infty\}$.

(xxvii) [61, Theorem 3.4] $\{\ell_\infty(N^t,p)\}^\alpha = d_5(p)$, $\{\ell_\infty(N^t,p)\}^\gamma = d_6(p)$, $\{\ell_\infty(N^t,p)\}^\beta = d_6(p) \cap d_{10}(p)$, $\{c_0(N^t,p)\}^\alpha = d_7(p)$, $\{c_0(N^t,p)\}^\gamma = d_9(p)$, $\{c_0(N^t,p)\}^\beta = d_9(p) \cap d_{11}(p) \cap cs$, $\{c(N^t,p)\}^\alpha = d_7(p) \cap d_8(p)$, $\{c(N^t,p)\}^\gamma = d_9(p) \cap d_{14}(p)$, $\{c(N^t,p)\}^\beta = d_9(p) \cap d_{11}(p) \cap d_{14}(p) \cap cs$.

It is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is triangle, [29]. Let $\lambda(p)$ be any Maddox's space, $A = (a_{nk})$ be an infinite matrix and denote $A^{-1} = (a_{nk}^{-1})$ with the inverse of A, where $\lambda \in \{\ell_p, c_0, c\}$. Then, the following Theorem holds:

Theorem 5.6. *Define the sequence* $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ *of the elements of the space* $(\lambda(p))_A$ *for every fixed* $k \in \mathbb{N}$ *by*

$$b_n^{(k)} = a_{nk}^{-1}. \tag{13}$$

Then,

(i) *the sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $(\lambda(p))_A$ *and any* $x \in (\lambda(p))_A$ *has a unique representation of the form*

$$x = \sum_k \alpha_k b^{(k)},$$

where $\alpha_k = (Ax)_k$ *for all* $k \in \mathbb{N}$, $0 < p_k \leq H < \infty$ *and* $\lambda \in \{\ell_p, c_0\}$.

(ii) *the set* $\{\vartheta, b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $(c(p))_A$ *and any* $x \in (c(p))_A$ *has a unique representation of the form*

$$x = \ell\vartheta + \sum_k [\alpha_k - \ell\vartheta_k] b^{(k)},$$

where $\vartheta = (\vartheta_k)$ *with* $\vartheta_k = (A^{-1}e)_k$ *for all* $k \in \mathbb{N}$ *and* $\ell = \lim_{k\to\infty}(Ax)_k$.

Using Theorem 5.6 and taking r_{nk}, h_{nk}, ζ_{nk}, ϱ_{nk}, δ_{nk}, ξ_{nk}, b_{nk}, ς_{nk}, z_{nk} and u_{nk} instead of a_{nk} in (13), respectively, Altay and Başar [2, 4] ,Altay and Başar [3, 5], Aydn and Başar [9, 10], Başar et al. [18], Kara et al. [30], Başar and Çakmak [19], Aydın and Altay [8] and Aydn and Başar [13], Nergiz and Başar [56] and Özger and Başar [59], Çapan and Başar [23], Yeşilkayagil and Başar [60, 61] obtained the basis of the spaces $r^t(p)$, $r_0^t(p)$, $r_c^t(p)$; $c_0(u,\nu,p)$, $c(u,\nu,p)$, $\ell(u,\nu,p)$; $a^r(u,p)$, $a_0^r(u,p)$, $a_c^r(u,p)$; $bv(u,p)$; $e^r(p)$; $c_0(B,p)$, $c(B,p)$; $\widehat{\ell}(p)$, $\widehat{c}_0(p)$, $\widehat{c}(p)$; $\ell(\widetilde{B},p)$, $c_0(\widetilde{B},p)$, $c(\widetilde{B},p)$; $\ell(F,p)$; $N^t(p)$, $\ell_\infty(N^t,p)$, respectively.

6. Matrix transformations

In this section, we give a list of characterizations of matrix transformations between Maddox's sequence spaces.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk}x_k \quad \text{for each} \quad n \in \mathbb{N}. \tag{14}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (14) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$.

Let B and M denote the natural numbers and define the sets K_1 and K_2 by $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$ and $K_2 = \{k \in \mathbb{N} : p_k > 1\}$. We suppose that $p = (p_k), q = (q_k) \in \ell_\infty$ and $q_k > 0$ with $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Consider the following conditions:

$$\sup_{n\in\mathbb{N}} \left(\sup_{k\in\mathbb{N}} |a_{nk}|B^{-1/p_k}\right)^{q_n} < \infty \quad \text{for some} \quad B > 1, \tag{15}$$

$$\lim_{B\to\infty} \limsup_{n\to\infty} \left(\sup_{k\in\mathbb{N}} |a_{nk}|B^{-1/p_k}\right)^{q_n} = 0, \tag{16}$$

$$\exists(\alpha_k) \in \omega \quad \text{such that} \quad \lim_{B\to\infty} \limsup_{n\to\infty} \left(\sup_{k\in\mathbb{N}} |a_{nk} - \alpha_k|B^{-1/p_k}\right)^{q_n} = 0, \tag{17}$$

$$\sup_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}} |a_{nk}|B^{-1/p_k} < \infty \quad \text{for some} \quad B > 1, \tag{18}$$

$$\sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}|B^{1/p_k}\right)^{q_n} < \infty \quad \text{for all} \quad B > 1, \tag{19}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|B^{1/p_k} < \infty \quad \text{for all} \quad B > 1, \tag{20}$$

$$\exists(\alpha_k) \in \omega \text{ such that} \quad \lim_{n\to\infty} \left(\sum_k |a_{nk} - \alpha_k|B^{1/p_k}\right)^{q_n} = 0 \quad \text{for all} \quad B > 1, \tag{21}$$

$$\lim_{n\to\infty}\left(\sum_k |a_{nk}|B^{1/p_k}\right)^{q_n} = 0 \text{ for all } B>1, \tag{22}$$

$$q_n \geq 1 \text{ for all } n \text{ and for all } B>1 \sup_{N\in\mathcal{F}}\sum_n\left|\sum_{k\in N} a_{nk}B^{1/p_k}\right|^{q_n} < \infty, \tag{23}$$

$$\exists B>1 \text{ such that } \sup_{n\in\mathbb{N}}\left(\sum_k |a_{nk}|B^{-1/p_k}\right)^{q_n} < \infty, \tag{24}$$

$$\exists B>1 \text{ such that } \sup_{n\in\mathbb{N}}\sum_k |a_{nk}|B^{-1/p_k} < \infty, \tag{25}$$

$$\forall M, \exists B>1 \text{ and } \exists(\alpha_k)\in\omega \text{ such that } \sup_{n\in\mathbb{N}}\sum_{k\in K_2}|a_{nk}-\alpha_k|M^{1/q_n}B^{-1/p_k} < \infty, \tag{26}$$

$$\exists(\alpha_k)\in\omega \text{ such that } \lim_{n\to\infty}|a_{nk}-\alpha_k|^{q_n} = 0 \text{ for all } k\in\mathbb{N}, \tag{27}$$

$$\forall M, \exists B>1 \text{ such that } \sup_{n\in\mathbb{N}}\sum_k |a_{nk}|M^{1/q_n}B^{-1/p_k} < \infty, \tag{28}$$

$$\lim_{n\to\infty}|a_{nk}|^{q_n} = 0 \text{ for all } k\in\mathbb{N}, \tag{29}$$

$$\exists B>1 \text{ such that } \sup_{N\in\mathcal{F}}\sum_n\left|\sum_{k\in N} a_{nk}B^{-1/p_k}\right|^{q_n} < \infty \text{ for all } q_n\geq 1, \tag{30}$$

$$\sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}\right|^{q_n} < \infty, \tag{31}$$

$$\exists\alpha\in\mathbb{C} \text{ such that } \lim_{n\to\infty}\left|\sum_k a_{nk}-\alpha\right|^{q_n} = 0, \tag{32}$$

$$\lim_{n\to\infty}\left|\sum_k a_{nk}\right|^{q_n} = 0, \tag{33}$$

$$\sum_n\left|\sum_k a_{nk}\right|^{q_n} < \infty \text{ for all } q_n\geq 1, \tag{34}$$

$$\exists B>1 \text{ such that } \sup_{N\in\mathcal{F}}\sum_{k\in K_2}\left|\sum_{n\in N} a_{nk}B^{-1}\right|^{q_k} < \infty, \tag{35}$$

$$\sup_{N\in\mathcal{F}}\sup_{k\in K_1}\left|\sum_{n\in N} a_{nk}\right|^{p_k} < \infty, \tag{36}$$

$$\exists B>1 \text{ such that } \sup_{n\in\mathbb{N}}\sum_{k\in K_2}\left|a_{nk}B^{-1}\right|^{q_k} < \infty, \tag{37}$$

$$\sup_{n\in\mathbb{N}}\sup_{k\in K_1}|a_{nk}|^{p_k} < \infty, \tag{38}$$

$$\sum_k |a_{nk}|B^{1/p_k} < \infty \text{ converges uniformly in } n \text{ for all } B>1, \tag{39}$$

$$\exists(\alpha_k)\in\omega \text{ such that } \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for all } k\in\mathbb{N}, \tag{40}$$

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for all } k \in \mathbb{N}, \tag{41}$$

$$\lim_{k\to\infty} a_{nk} B^{1/p_k} = 0 \text{ for all } n \in \mathbb{N}, \tag{42}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha \text{ exists }, \tag{43}$$

$$\sup_{n\in\mathbb{N}} \sup_{k\in K_1} |a_{nk} B^{1/q_n}|^{p_k} < \infty, \tag{44}$$

$$\forall M, \exists B > 1 \text{ such that } \sup_{n\in\mathbb{N}} \sum_{k\in K_2} |a_{nk} M^{1/q_n} B^{-1}|^{q_k} < \infty, \tag{45}$$

$$\exists(\alpha_k) \in \omega \text{ such that } \sup_{n\in\mathbb{N}} \sup_{k\in K_1} (|a_{nk} - \alpha_k| B^{1/q_n})^{p_k} < \infty \text{ for all } B > 1, \tag{46}$$

$$\forall M, \exists B > 1 \text{ and } \exists(\alpha_k) \in \omega \text{ such that } \sup_{n\in\mathbb{N}} \sum_{k\in K_2} (|a_{nk} - \alpha_k| M^{1/q_n} B^{-1})^{q_k} < \infty, \tag{47}$$

$$\sup_{n\in\mathbb{N}} \sup_{k\in K_1} |a_{nk} B^{-1/q_n}|^{p_k} < \infty, \tag{48}$$

$$\sup_{n\in\mathbb{N}} \sum_{k\in K_2} |a_{nk} B^{-1/q_n}|^{q_k} < \infty. \tag{49}$$

Lemma 6.1. *Let $A = (a_{nk})$ be an infinite matrix and $0 < p_k \leq 1$ for all $k \in \mathbb{N}$ and $q = (q_k)$ be bounded. Then, the following statements hold:*

(i) [42, Theorem 5(i)] $A \in (\ell(p) : \ell_\infty(q))$ if and only if (15) holds.
(ii) [42, Theorem 5(ii)] $A \in (\ell(p) : c_0(q))$ if and only if (16) and (29) hold.
(iii) [42, Theorem 5(iii)] $A \in (\ell(p) : c(q))$ if and only if (17), (18) and (27) hold.
(iv) [42, Theorem 6] Let $q = (q_k) \in c_0$. Then, $A \in (\ell(p) : c_0(q))$ if and only if (17) holds.

Lemma 6.2. *Let $A = (a_{nk})$ be an infinite matrix and $1 < p_k \leq H$ for all $k \in \mathbb{N}$ and $1/p_k + 1/s_k = 1$ and let $q = (q_k)$ be bounded. Then, the following statements hold:*

(i) [42, Theorem 7] $A \in (\ell(p) : \ell_\infty(q))$ if and only if

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|^{s_k} B^{-s_k/q_n} < \infty \text{ for some } B > 1.$$

(ii) [42, Theorem 8] $A \in (\ell(p) : c_0(q))$ if and only if (29) holds and for every $D \geq 1$

$$\lim_{B\to\infty} \limsup_{n\to\infty} \left(\sum_k |a_{nk}|^{s_k} D^{s_k/q_n} B^{-s_k} \right)^{q_n} = 0 \text{ for some } B > 1.$$

(iii) [42, Theorem 9] $A \in (\ell(p) : c(q))$ if and only if (27) holds and

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|^{s_k} B^{-s_k} < \infty \text{ for some } B > 1,$$

$$\exists(\alpha_k) \in \omega \text{ such that } \lim_{B\to\infty} \limsup_{n\to\infty} \left(\sum_k |a_{nk} - \alpha_k|^{s_k} D^{s_k/q_n} B^{-s_k} \right)^{q_n} = 0 \text{ for all } D \geq 1.$$

Following Maddox and Willey [42], Grosse-Erdmann [26] redefined the matrix classes $(\ell(p) : \lambda(q))$, where $\lambda \in \{\ell_\infty, c_0, c)$ and gave the following results:

Lemma 6.3. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) [26, Theorem 5.1.15] $A \in (\ell_\infty(p) : \ell_\infty(q))$ if and only if (19) holds.
(ii) [26, Theorem 5.1.11] $A \in (\ell_\infty(p) : c(q))$ if and only if (20) and (21) hold.
(iii) [26, Theorem 5.1.7] $A \in (\ell_\infty(p) : c_0(q))$ if and only if (22) holds.
(iv) [26, Theorem 5.1.3] $A \in (\ell_\infty(p) : \ell(q))$ if and only if (23) holds.
(v) [26, Theorem 5.1.13] $A \in (c_0(p) : \ell_\infty(q))$ if and only if (24) holds.
(vi) [26, Theorem 5.1.9] $A \in (c_0(p) : c(q))$ if and only if (25)-(27) hold.
(vii) [26, Theorem 5.1.5] $A \in (c_0(p) : c_0(q))$ if and only if (28) and (29) hold.
(viii) [26, Theorem 5.1.1] $A \in (c_0(p) : \ell(q))$ if and only if (30) holds.

(ix) [26, Theorem 5.1.14] $A \in (c(p) : \ell_\infty(q))$ if and only if (24) and (31) hold.
(xx) [26, Theorem 5.1.10] $A \in (c(p) : c(q))$ if and only if (25)-(27) and (32) hold.
(xi) [26, Theorem 5.1.6] $A \in (c(p) : c_0(q))$ if and only if (28), (29) and (33) hold.
(xii) [26, Theorem 5.1.2] $A \in (c(p) : \ell(q))$ if and only if (30) and (34) hold.
(xiii) [26, Theorem 5.1.4] $A \in (\ell(p) : c_0(q))$ if and only if (29), (44) and (45) hold.
(xiv) [26, Theorem 5.1.8] $A \in (\ell(p) : c(q))$ if and only if (27), (37), (38), (46) and (47) hold.
(xv) [26, Theorem 5.1.8] $A \in (\ell(p) : \ell_\infty(q))$ if and only if (48) and (49) hold.

Lemma 6.4. *The following statements hold:*

(i) [26, Theorem 5.1.0 with $q_n = 1$] $A \in (\ell(p) : \ell_1)$ if and only if (35) holds, where $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.
(ii) [26, Theorem 5.1.0] $A \in (\ell(p) : \ell_1)$ if and only if if and only if (36) holds, where $0 < p_k \leq 1$ for all $k \in \mathbb{N}$.
(iii) *(*[34, Theorem 1(i)] *and* [26, Proposition 3.2(i)]*)* $A \in (\ell(p) : \ell_\infty)$ if and only if (37) holds, where $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.
(iv) *(*[34, Theorem 1(ii)] *and* [26, Proposition 3.2(i)]*)* $A \in (\ell(p) : \ell_\infty)$ if and only if (38) holds, where $0 < p_k \leq 1$ for all $k \in \mathbb{N}$.
(v) [34, Corollary of Theorem 1] $A \in (\ell(p) : c)$ if and only if (37), (38) and (40) hold, where $0 < p_k \leq H$ for all $k \in \mathbb{N}$.
(vi) [34, Theorem 3] $A \in (\ell_\infty(p) : \ell_\infty)$ if and only if (20) holds.
(vii) [34, Corollary of Theorem 3] $A \in (\ell_\infty(p) : c)$ if and only if (39) and (40) hold, where $0 < p_k \leq H$ for all $k \in \mathbb{N}$.
(viii) [33, Theorem 9] $A \in (c(p) : c)$ if and only if (25), (40) and (43) hold, where $p \in \ell_\infty$.
(ix) [33, Theorem 9] $A \in (c_0(p) : c)$ if and only if (25) and (40) hold, where $p \in \ell_\infty$.
(x) [34, Theorem 5] Let $0 < p_k \leq 1$. Then, $A \in (\omega(p) : c)$ if and only if (42) and (43) hold and

$$\exists B > 1 \text{ such that } \sup_{n\in\mathbb{N}} \sum_r \max_{r\in\mathbb{N}} \left((2^r B^{-1})^{1/p_k} |a_{nk}| \right) < \infty.$$

Theorem 6.1. *Let $0 < p_k \leq \sup_k p_k < \infty$ for all $k \in \mathbb{N}$. Then, Nanda* [53, 54, 55] *gave the following results:*

(i) $A \in (c_0(p) : f_0(p))$ *if and only if*

$$\exists B > 1 \ni \sup_{m\in\mathbb{N}} \left(\sum_k |a(n,k,m)| B^{-1/p_k} \right)^{p_m} < \infty \text{ for all } n \in \mathbb{N}, \tag{50}$$
$$\exists \alpha_k \in \mathbb{C} \text{ for all } k \in \mathbb{N} \ni \lim_{m\to\infty} |a(n,k,m)|^{p_m} = \alpha_k \text{ uniformly in } n.$$

(ii) $A \in (c(p) : f)$ *if and only if*

$$\exists B > 1 \ni \sup_{m\in\mathbb{N}} \sum_k |a(n,k,m)| B^{-1/p_k} < \infty \text{ for all } n \in \mathbb{N}, \tag{51}$$
$$\exists \alpha_k \in \mathbb{C} \text{ for all } k \in \mathbb{N} \ni \lim_{m\to\infty} a(n,k,m) = \alpha_k \text{ uniformly in } n, \tag{52}$$
$$\exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_k a(n,k,m) = \alpha \text{ uniformly in } n. \tag{53}$$

(iii) $A \in (\ell_\infty(p) : f)$ *if and only if (52) holds, and*

$$\exists B > 1 \ni \lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| B^{1/p_k} = 0 \text{ uniformly in } n \tag{54}$$
$$\sup_{m\in\mathbb{N}} \sum_k |a(n,k,m)| < \infty.$$

(iv) $A \in (\ell(p) : f)$ *if and only if (52) holds and*

$$\exists B > 1 \ni \sup_{m\in\mathbb{N}} \sum_k |a(n,k,m)|^{q_k} B^{-q_k} < \infty, \quad \text{if } p_k \geq 1, \tag{55}$$

$$\sup_{m,k\in\mathbb{N}} |a(n,k,m)|^{p_k} < \infty, \quad \text{if } 0 < p_k \leq 1. \tag{56}$$

(v) $A \in (\ell(p) : f_0)$ *if and only if (52) is satisfied with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and (55), (56) hold.*

(vi) *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\omega(p) : f)$ *if and only if (52) and (53) are satisfied and*

$$\sup_{m\in\mathbb{N}} \sum_r \max_{r\in\mathbb{N}} (2^2 B^{-1})^{1/p_k} |a(n,k,m)| < \infty.$$

(vii) $A \in (\ell_\infty(p) : \widehat{f})$ *if and only if*

$$\sup_{m,n\in\mathbb{N}} \sum_k |a(n,k,m)| B^{1/p_k} < \infty \quad \text{for all} \quad B > 1.$$

(viii) $A \in (c_0(p) : \widehat{f}(p))$ *if and only if (50) holds,*

where

$$a(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} a_{n+i,k}$$

for all $k, m, n \in \mathbb{N}$.

Theorem 6.2. *Let* $A = (a_{nk})$ *be an infinite matrix, let* $r = (r_n)$ *be bounded and denote* $a(n,k) = \sum_{i=0}^{n} a_{ik}$ *for all* $n, k \in \mathbb{N}$. *Başar [14] gave the following matrix classes:*

(i) [14, Theorem 1(i)] *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(p) : \widehat{f}(r))$ *if and only if*

$$\exists B > 1 \quad \text{such that} \quad \sup_{n,k,m\in\mathbb{N}} (|a(n,k,m)| B^{-1/p_k})^{r_n} < \infty.$$

(ii) [14, Theorem 1(ii)] *Let* $1 < p_k < \infty$ *for all* $k \in \mathbb{N}$ *and* $1/p_k + 1/q_k = 1$. *Then,* $A \in (\ell(p) : \widehat{f}(r))$ *if and only if*

$$\exists B > 1 \quad \text{such that} \quad \sup_{n,m\in\mathbb{N}} \sum_k |a(n,k,m)|^{q_k} B^{-q_k/r_n} < \infty.$$

(iii) [14, Theorem 2(i)] *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(p) : \widehat{bs}(r))$ *if and only if*

$$\exists B > 1 \quad \text{such that} \quad \sup_{n,k,m\in\mathbb{N}} \left(\frac{1}{m+1} \left| \sum_{i=0}^{m} a(n+i,k) \right| B^{-1/p_k} \right)^{r_n} < \infty.$$

(iv) [14, Theorem 2(ii)] *Let* $1 < p_k < \infty$ *for all* $k \in \mathbb{N}$ *and* $1/p_k + 1/q_k = 1$. *Then,* $A \in (\ell(p) : \widehat{bs}(r))$ *if and only if*

$$\exists B > 1 \quad \text{such that} \quad \sup_{n,m\in\mathbb{N}} \sum_k \left| \frac{1}{m+1} \sum_{i=0}^{m} a(n+i,k) \right|^{q_k} B^{-q_k/r_n} < \infty.$$

(v) [14, Theorem 4] $A \in (c_0(p) : \widehat{f}(r))$ *if and only if*

$$\exists B > 1 \quad \text{such that} \quad \sup_{n,m\in\mathbb{N}} \left(\sum_k |a(n,k,m)| B^{-1/p_k} \right)^{r_n} < \infty.$$

(vi) [14, Theorem 5] $A \in (c_0(p) : \widehat{bs}(r))$ *if and only if*

$$\exists B > 1 \quad \text{such that} \quad \sup_{n,m\in\mathbb{N}} \left(\sum_k \left| \frac{1}{m+1} \sum_{i=0}^{m} a(n+i,k) \right| B^{-1/p_k} \right)^{r_n} < \infty.$$

(vii) [14, Theorem 5] $A \in (c_0(p) : bs(r))$ *if and only if*

$$\exists B > 1 \quad \textit{such that} \quad \sup_{n\in\mathbb{N}} \left(\sum_k |a(n,k)| B^{-1/p_k} \right)^{r_n} < \infty.$$

Theorem 6.3. *Let $A = (a_{nk})$ be an infinite matrix. Başar and Altay* [16] *gave the following results:*

(i) [16, Theorem 3.1] $A \in (bs(p) : \ell_\infty(q))$ *if and only if (19) holds with $j_{nk} = \Delta a_{nk}$ instead of a_{nk} and (42) is satisfied.*

(ii) [16, Theorem 3.2] $A \in (bs(p) : bs(q))$ *if and only if (19) and (42) hold with $j_{nk} = \Delta a(n,k)$ instead of a_{nk}, where $a(n,k) = \sum_{i=0}^{n} a_{ik}$.*

(iii) [16, Corollary 3.3] $A \in (bs(p) : \ell_\infty)$ *if and only if (42) is satisfied and (20) holds with $j_{nk} = \Delta a_{nk}$ instead of a_{nk}.*

(iv) [16, Corollary 3.4] $A \in (bs(p) : bs)$ *if and only if (20) and (42) hold with $j_{nk} = \Delta a(n,k)$ instead of a_{nk}.*

(v) [16, Theorem 3.5] $A \in (bs(p) : f)$ *if and only if if and only if (20) is satisfied with $j_{nk} = \Delta a_{nk}$ instead of a_{nk}, and (52) and (54) hold with $\Delta a(n,k,m)$ instead of $a(n,k,m)$.*

(vi) [16, Theorem 3.7] $A \in (bs(p) : c)$ *if and only if if and only if (39), (40) and (42) hold with $j_{nk} = \Delta a_{nk}$ instead of a_{nk}.*

Lemma 6.5. [31, Theorem 4.1] *Let λ be an $FK-$space, $E = (e_{nk})$ be triangle, $V = (v_{nk})$ be its inverse and μ be arbitrary subset of ω. Then, we have $A \in (\lambda_E : \mu)$ if and only if*

$$Q^{(n)} = (q_{mk}^{(n)}) \in (\lambda : c) \text{ for all } n \in \mathbb{N}$$

and

$$Q = (q_{nk}) \in (\lambda : \mu),$$

where

$$q_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} a_{nj} v_{jk} & , \quad 0 \le k \le m, \\ 0 & , \quad k > m. \end{cases} \quad \text{and} \quad q_{mk} = \sum_{j=k}^{\infty} a_{nj} v_{jk}, \tag{57}$$

$k, m, n \in \mathbb{N}$.

Theorem 6.4. *Let $p_k > 0$ for all $k \in \mathbb{N}$. Then, Ahmad and Mursaleen* [1] *gave results:*

(i) [1, Theorem 3.3] $A \in (\Delta\ell_\infty(p) : \ell_\infty)$ *if and only if (20) holds with $q_{nk} = k|a_{nk}|$ instead of a_{nk}.*

(ii) [1, Theorem 3.4] $A \in (\Delta\ell_\infty(p) : c)$ *if and only if (40) holds and (39) holds with $q_{nk} = k|a_{nk}|$ instead of a_{nk}.*

Using Lemma 6.5., we give following results:

Theorem 6.5. *The following statements hold:*

(i) [2, Theorem 3.1(i)] *Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (r^t(p) : \ell_\infty)$ if and only if*

(1*) $\left\{ \left(\frac{a_{nk}}{q_k} Q_k B^{-1} \right)^{q_k} \right\}_{k\in\mathbb{N}} \in \ell_\infty$ *for all $n \in \mathbb{N}$.*

(2*) *(37) holds with $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ instead of a_{nk}.*

(ii) [2, Theorem 3.1(ii)] *Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then, $A \in (r^t(p) : \ell_\infty)$ if and only if*

(3*) *(38) holds with $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ instead of a_{nk}.*

(iii) [2, Theorem 3.4] *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (r^t(p) : c)$ if and only if (1*)-(3*) hold and there exists a sequence (α_k) of scalars such that*

(4*) $\lim_{n\to\infty} \Delta \left(\frac{a_{nk} - \alpha_k}{t_k} \right) T_k = 0$ *for all $k \in \mathbb{N}$.*

(iv) [2, Theorem 3.5] *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (r^t(p) : c_0)$ if and only if (1*)-(4*) hold.*

(v) [2, Theorem 3.2(i)] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (r^t(p) : bs)$ *if and only if (1*) is satisfied with* $a(n,k)$ *instead of* a_{nk} *and (37) holds with* $j_{nk} = \Delta\left[\frac{a(n,k)}{q_k}\right] Q_k$ *instead of* a_{nk}.

(vi) [2, Theorem 3.2(ii)] *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (r^t(p) : bs)$ *if and only if (38) holds with* $j_{nk} = \Delta\left[\frac{a(n,k)}{q_k}\right] Q_k$ *instead of* a_{nk}.

(vii) [2, Theorem 3.4(i)] *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (r^t(p) : cs)$ *if and only if (37) and (38) are satisfied with* $j_{nk} = \Delta\left[\frac{a(n,k)}{q_k}\right] Q_k$ *instead of* a_{nk} *and (1*) and (4*) hold with* $a(n,k)$ *instead of* a_{nk}.

(viii) [2, Theorem 3.4(ii)] *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (r^t(p) : cs_0)$ *if and only if (37) and (38) are satisfied with* $j_{nk} = \Delta\left[\frac{a(n,k)}{q_k}\right] Q_k$ *instead of* a_{nk} *and (1*) and (4*) hold with* $a(n,k)$ *instead of* a_{nk} *and with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(ix) [4, Theorem 4.3(i)] $A \in (r^t_\infty(p) : \ell_\infty(q))$ *if and only if (19) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk} *and*
(5*) *(42) holds with* r_{nk} *instead of* a_{nk}.

(x) [4, Theorem 4.3(iv)] $A \in (r^t_\infty(p) : \ell(q))$ *if and only if (5*) holds and (23) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xi) [4, Theorem 4.3(vii)] $A \in (r^t_\infty(p) : c(q))$ *if and only if (5*) holds and (20)-(21) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xii) [4, Theorem 4.3(x)] $A \in (r^t_\infty(p) : c_0(q))$ *if and only if (5*) holds and (21) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk} *and with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(xiii) [4, Theorem 4.4(i)] $A \in (r^t_c(p) : \ell_\infty(q))$ *if and only if (5*) holds and (24), (31) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xiv) [4, Theorem 4.4(iv)] $A \in (r^t_c(p) : \ell(q))$ *if and only if (5*) holds and (30), (34) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xv) [4, Theorem 4.4(vii)] $A \in (r^t_c(p) : c(q))$ *if and only if (5*) holds and (25)-(27) and (32) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xvi) [4, Theorem 4.4(x)] $A \in (r^t_c(p) : c_0(q))$ *if and only if (5*) holds and (26), (27) and (32) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk} *and with* $\alpha = 0$, $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(xvii) [4, Theorem 4.5(i)] $A \in (r^t_0(p) : \ell_\infty(q))$ *if and only if (5*) holds and (24) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xviii) [4, Theorem 4.5(iv)] $A \in (r^t_0(p) : \ell(q))$ *if and only if (5*) holds and (30) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xix) [4, Theorem 4.5(vii)] $A \in (r^t_0(p) : c(q))$ *if and only if (5*) holds and (25)-(27) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk}.

(xx) [4, Theorem 4.5(x)] $A \in (r^t_0(p) : c_0(q))$ *if and only if (5*) holds and (26) and (27) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} r_{jk}$ *instead of* a_{nk} *and with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(xxi) [9, Corollary 5.2] $A \in (a^r_0(u,p) : \ell_\infty(q))$ *if and only if (24) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj} \zeta_{jk}$ *instead of* a_{nk} *and*
(6*) $\left\{\frac{k+1}{(1+r^k)u_k} a_{nk} B^{-1/p_k}\right\}_{k \in \mathbb{N}} \in c$ *for all* $n \in \mathbb{N}$.

(xxii) [9, Corollary 5.3] *$A \in (a_0^r(u,p) : c(q))$ if and only if (6*) holds and (25)-(27) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk}.

(xxiii) [9, Corollary 5.4] *$A \in (a_0^r(u,p) : c_0(q))$ if and only if (6*) holds and (27), (28) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk}.

(xxiv) [9, Corollary 5.5] *$A \in (a_0^r(u,p) : \ell(q))$ if and only if (6*) holds and (30) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk}.

(xxv) [10, Theorem 4.1(i)] *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (a^r(u,p) : \ell_\infty)$ if and only if (37) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk} *and*

(7*) $\left\{\left(\frac{(k+1)}{(1+r^k)u_k}a_{nk}B^{-1}\right)^{q_k}\right\}_{k\in\mathbb{N}} \in \ell_\infty$ *for all* $n \in \mathbb{N}$.

(xxvi) [10, Theorem 4.1(ii)] *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (a^r(u,p) : \ell_\infty)$ if and only if (38) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk} *and*

(8*) $\left\{\left(\frac{(k+1)}{(1+r^k)u_k}a_{nk}\right)^{p_k}\right\}_{k\in\mathbb{N}} \in \ell_\infty$ *for all* $n \in \mathbb{N}$.

(xxvii) [10, Theorem 4.2] *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (a^r(u,p) : c)$ if and only if (7*), (8*) hold and (37), (38) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk} *and (27) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk} *and with* $q_n = 1$ *for all* $n \in \mathbb{N}$.

(xxviii) [10, Corollary 4.3] *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (a^r(u,p) : c_0)$ if and only if (7*), (8*) hold and (37), (38) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk} *and (33) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\zeta_{jk}$ *instead of* a_{nk} *and with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

(xxix) [3, Theorem 3.1] *Let μ be any given sequence space. Then, $A \in (\lambda(u,\nu,p) : \mu)$ if and only if $Q \in (\lambda(p) : \mu)$ and $Q^{(n)} \in (\lambda(p) : c)$, where* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}h_{jk}$ *and* $Q^{(n)} = (q_{mk}^{(n)})$ *is as in (57).*

(xxx) [18, Theorem 4.1(i)] *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (bv(u,p) : \ell_\infty)$ if and only if (37) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varrho_{jk}$ *instead of* a_{nk} *and*

(9*) $\{a_{nk}\}_{k\in\mathbb{N}} \in d_2(p) \cap cs$ *for all* $n \in \mathbb{N}$.

(xxxi) [18, Theorem 4.1(ii)] *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (bv(u,p) : \ell_\infty)$ if and only if (38) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varrho_{jk}$ *instead of* a_{nk} *and*

(10*) $\{a_{nk}\}_{k\in\mathbb{N}} \in d_4(p) \cap cs$ *for all* $n \in \mathbb{N}$.

(xxxii) [18, Theorem 4.2] *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (bv(u,p) : c)$ if and only if (9*), (10*) hold and (37), (38), (40) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varrho_{jk}$ *instead of* a_{nk}.

(xxxiii) [18, Corollary 4.3] *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (bv(u,p) : c_0)$ if and only if (9*), (10*) hold and (37), (38) and (41) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varrho_{jk}$ *instead of* a_{nk}.

(xxxiv) [19, Theorem 3.1] *Let μ be any given sequence space. Then, $A \in (\lambda(B,p) : \mu)$ if and only if $Q \in (\lambda(p) : \mu)$ and $Q^{(n)} \in (\lambda(p) : c)$, where* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\xi_{jk}$ *and* $Q^{(n)} = (q_{mk}^{(n)})$ *is as in (57).*

(xxxv) [8, Theorem 3.2(i)] *$A \in (\widehat{\ell}_\infty(p) : \ell_\infty)$ if and only if (20) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ *instead of* a_{nk}.

(xxxvi) [8, Theorem 3.2(ii)] *$A \in (\widehat{\ell}_\infty(p) : c)$ if and only if (39) and (40) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xxxvii) [8, Theorem 3.2(ii)] *$A \in (\widehat{\ell}_\infty(p) : c_0)$ if and only if (22) holds with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk} and with $q_n = 1$ for all $n \in \mathbb{N}$.*

(xxxviii) [8, Theorem 3.3(i)] *$A \in (\widehat{c}_0(p) : \ell_\infty(q))$ if and only if (24), (26) and (30) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xxxix) [8, Theorem 3.3(ii)] *$A \in (\widehat{c}_0(p) : c_0(q))$ if and only if (24), (26), (29) and (28) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xl) [8, Theorem 3.3(iii)] *$A \in (\widehat{c}_0(p) : c(q))$ if and only if (24)-(27) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xli) [8, Theorem 3.4(i)] *$A \in (\widehat{c}(p) : \ell_\infty(q))$ if and only if (24), (26), (30), (31) and (43) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}..*

(xlii) [8, Theorem 3.4(ii)] *$A \in (\widehat{c}(p) : c_0(q))$ if and only if (24), (26), (29), (28), (33) and (43) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xliii) [8, Theorem 3.4(iii)] *$A \in (\widehat{c}(p) : c(q))$ if and only if (24)-(27), (32) and (34) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xliv) [13, Theorem 4.1] *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\widehat{\ell}(p) : \ell_\infty)$ if and only if (38) holds with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk} and $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{\ell}(p)\right\}^\beta$.*

(xlv) [13, Theorem 4.1] *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\widehat{\ell}(p) : \ell_\infty)$ if and only if (37) holds with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk} and $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{\ell}(p)\right\}^\beta$.*

(xlvi) [13, Theorem 4.2] *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\widehat{\ell}(p) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{\ell}(p)\right\}^\beta$ and (37), (38) and (40) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xlvii) [13, Corollary4.3] *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\widehat{\ell}(p) : c_0)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{\ell}(p)\right\}^\beta$ and (37), (38) and (41) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ instead of a_{nk}.*

(xlviii) [56, Theorem 13(i)] *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widehat{B}, p) : \ell_\infty)$ if and only if (37) holds with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varsigma_{jk}$ instead of a_{nk} and*

(11*) $\sum_{i=k}^{\infty} \frac{(-1)^{i-k}}{r_i} \prod_{j=k}^{i-1} \frac{s_j}{r_j} a_{ni} < \infty$ *for all* $n \in \mathbb{N}$.

(xlix) [56, Theorem 13(ii)] *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widehat{B}, p) : \ell_\infty)$ if and only if (11*) holds and (38) holds with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varsigma_{jk}$ instead of a_{nk}.*

(l) [56, Theorem 15] *$A \in (\ell(\widehat{B}, p) : f)$ if and only if $Q \in (\ell(p) : f)$ and $Q^{(n)} \in (\ell(p) : c)$, where $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varsigma_{jk}$ and $Q^{(n)} = (q_{mk}^{(n)})$ is as in (57).*

(li) [56, Theorem 16] *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(\widehat{B}, p) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\ell(\widehat{B}, p)\right\}^\beta$ and (37), (38) and (40) hold with $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varsigma_{jk}$ instead of a_{nk}.*

(lii) [56, Corollary 17] *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(\widehat{B}, p) : c_0)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\ell(\widehat{B}, p)\right\}^{\beta}$ *and (37), (38) and (41) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\varsigma_{jk}$ *instead of* a_{nk}.

(liii) [23, Theorem 4.1(i)] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(F, p) : \ell_\infty)$ *if and only if (38) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}z_{jk}$ *instead of* a_{nk} *and*

(12*) $\sum_{i=k}^{\infty} \frac{f_{i+1}^2}{f_k f_{k+1}} a_{ni} < \infty$ *for all* $n \in \mathbb{N}$.

(liv) [23, Theorem 4.1(i)] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(F, p) : \ell_\infty)$ *if and only if (12*) holds and (37) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}z_{jk}$ *instead of* a_{nk}.

(lv) [23, Theorem 4.2(i)] *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(F, p) : c)$ *if and only if (12*) holds and (38), (40) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}z_{jk}$ *instead of* a_{nk}.

(lvi) [23, Theorem 4.2(ii)] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(F, p) : c)$ *if and only if (12*) holds and (37), (40) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}z_{jk}$ *instead of* a_{nk}.

(lvii) [23, Corollary 4.3(i)] *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(F, p) : c_0)$ *if and only if (12*) holds and (38), (40) and (41) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}z_{jk}$ *instead of* a_{nk}.

(lviii) [23, Corollary 4.3(ii)] *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A \in (\ell(F, p) : c_0)$ *if and only if (12*) holds and (37), (40) and (41) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}z_{jk}$ *instead of* a_{nk}.

(lix) [60, Theorem 10] *Let* μ *be any given sequence space. Then,* $A \in (N^t(p) : \mu)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{N^t(p)\}^{\beta}$ *and* $Q \in (\ell(p) : \mu)$, *where* $Q = (q_{nk})$ *is* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}\xi_{jk}$ *for all* $n, k \in \mathbb{N}$.

(lx) [61, Theorem 4.1] $A \in (\ell_\infty(N^t, p) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell_\infty(N^t, p)\}^{\beta}$ *and (20) holds with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}u_{jk}$ *instead of* a_{nk}.

(lxi) [61, Theorem 4.4] $A \in (\ell_\infty(N^t, p) : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell_\infty(N^t, p)\}^{\beta}$ *and (39), (40) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}u_{jk}$ *instead of* a_{nk}.

(lxii) [61, Theorem 4.4] $A \in (\ell_\infty(N^t, p) : c_0)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell_\infty(N^t, p)\}^{\beta}$ *and (39), (40) and (41) hold with* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}u_{jk}$ *instead of* a_{nk}.

Theorem 6.6. *Let* $\widetilde{a}(n, k, m) = \frac{1}{m+1}\sum_{i=0}^{m} q_{n+i,k}$, *where* $q_{nk} = \sum_{j=k}^{\infty} a_{nj}b_{jk}$ *for all* $n, k \in \mathbb{N}$. *Then, the following statements hold:*

(i) [59, Theorem 5.8(i)] $A \in (c(\widehat{B}, p) : f)$ *if and only if (51)-(53) hold with* $\widetilde{a}(n, k, m)$ *instead of* $a(n, k, m)$.

(ii) [59, Theorem 5.8(ii)] $A \in (c_0(\widehat{B}, p) : f)$ *if and only if (51) and (52) hold with* $\widetilde{a}(n, k, m)$ *instead of* $a(n, k, m)$ *and* $Q^{(n)} \in (c_0(p) : c)$, *where* $Q^{(n)} = (q_{mk}^{(n)})$ *is as in (57).*

(iii) [59, Theorem 5.8(iii)] $A \in (\ell_\infty(\widehat{B}, p) : f)$ *if and only if (51), (52) and (54) hold with* $\widetilde{a}(n, k, m)$ *instead of* $a(n, k, m)$ *and* $Q^{(n)} \in (\ell_\infty(p) : c)$, *where* $Q^{(n)} = (q_{mk}^{(n)})$ *is as in (57).*

(iv) [59, Theorem 5.8(iv)] $A \in (\ell_\infty(\widehat{B}, p) : f_0)$ *if and only if (52) and (54) hold with* $\widetilde{a}(n, k, m)$ *instead of* $a(n, k, m)$ *and with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* $Q^{(n)} \in (\ell_\infty(p) : c)$, *where* $Q^{(n)} = (q_{mk}^{(n)})$ *is as in (57).*

Lemma 6.6. [17, Lemma 5.3] *Let* λ *and* μ *be any two sequence spaces,* A *be an infinite matrix and* B *be a triangle matrix. Then,* $A \in (\lambda : \mu_A)$ *if and only if* $BA \in (\lambda : \mu)$. Using Lemma 6.6., the authors mentioned above gave comprehensive matrix classes. Also, we have benefited from Malkowsky and Başar [47] in this section.

7. Some geometric properties of the space $(\lambda(p))_A$

In Functional Analysis, the rotundity of Banach spaces is one of the most important geometric property. For details, the reader may refer to [21, 24, 43]. In this section, we give the necessary and sufficient condition in order to the space $(\lambda(p))_A$ be rotund and present some results related to this concept, where $\lambda(p)$ is any Maddox's space and $A = (a_{nk})$ is an infinite matrix.

Definition 7.1. *Let $S(X)$ be the unit sphere of a Banach space X. Then, a point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space X is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point.*

Definition 7.2. *A Banach space X is said to have Kadec-Klee property (or propert (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.*

Definition 7.3. *A Banach space X is said to have*

(i) the *Opial property* if every sequence (x_n) weakly convergent to $x_0 \in X$ satisfies

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n + x\|$$

for every $x \in X$ with $x \neq x_0$.

(ii) the *uniform Opial property* if for each $\epsilon > 0$, there exists an $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x_n + x\|$$

for each $x \in X$ with $\|x\| \geq \epsilon$ and each sequence (x_n) in X such that $x_n \xrightarrow{w} 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$.

Definition 7.4. *Let X be a real vector space. A functional $\sigma : X \to [0, \infty)$* is called a modular if

(i) $\sigma(x) = 0$ if and only if $x = \theta$;
(ii) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$;
(iii) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
(iv) the modular σ is called convex if $\sigma(\alpha x + \beta y) \leq \alpha\sigma(x) + \beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$;

A modular σ on X is called

(a) right continuous if $\lim_{\alpha\to 1^+} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.
(b) left continuous if $\lim_{\alpha\to 1^-} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.
(c) continuous if it is both right and left continuous, where

$$X_\sigma = \left\{ x \in X : \lim_{\alpha\to 0^+} \sigma(\alpha x) = 0 \right\}.$$

Let $\lambda(p)$ be any Maddox's space and $A = (a_{nk})$ be an infinite matrix. Define σ_p on a sequence space $(\lambda(p))_A$ by

$$\sigma_p(x) = \sum_k |(Ax)_k|^{p_k}. \tag{58}$$

If $p_k \geq 1$ for all $k \in \mathbb{N} = \{1, 2, \ldots\}$, by the convexity of the function $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, σ_p is a convex modular on $(\lambda(p))_A$. Consider $(\lambda(p))_A$ equipped with Luxemburg norm given by

$$\|x\| = \inf\{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\}. \tag{59}$$

$(\lambda(p))_A$ is a Banach space with this norm.

Taking A^r, A^u, E^r, $B(r, s)$, $B(\widetilde{r}, \widetilde{s})$ and N^t instead of A in (58), respectively, Aydn and Başar [10], Başar et al. [18], Kara et al. [30], Aydın and Altay [8] and Aydn and Başar [13], Nergiz and Başar [56], Yeşilkayagil and Başar [60] gave the following results:

Proposition 1. ([10, Proposition 5.1], [18, Proposition 5.1], [30, Proposition 2], [8, Theorem 4.1], [13, Theorem 5.1], [56, Proposition 5], [60, Proposition 16]) *The modular σ_p on $a^r(u, p)$ $[bv(u, p)$, $e^r(p)$, $\widehat{\ell}(p)$,* $\widehat{\ell}_\infty(p)$, $\ell(\widetilde{B}, p)$, $N^t(p)$, respectively] *satisfies the following properties with $p_k \geq 1$ for all $k \in \mathbb{N}$:*

(i) *If $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha\sigma_p(x)$.*
(ii) *If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$.*

(iii) *If* $\alpha \geq 1$, *then* $\alpha\sigma_p(x/\alpha) \leq \sigma_p(x)$.
(iv) *The modular* σ_p *is continuous.*

Proposition 2. ([10, Proposition 5.2], [18, Proposition 5.3], [30, Proposition 3], [8, Theorem 4.2], [13, Theorem 5.2], [56, Proposition 6], [60, Proposition 17]) *For any* $x \in a^r(u,p)$ $[bv(u,p)$, $e^r(p)$, $\widehat{\ell}(p)$, $\widehat{\ell}_\infty(p)$, $\ell(\widetilde{B},p)$, $N^t(p)$, respectively], *the following statements hold:*

(i) *If* $\|x\| < 1$, *then* $\sigma_p(x) \leq \|x\|$.
(ii) *If* $\|x\| > 1$, *then* $\sigma_p(x) \geq \|x\|$.
(iii) $\|x\| = 1$ *if and only if* $\sigma_p(x) = 1$.
(iv) $\|x\| < 1$ *if and only if* $\sigma_p(x) < 1$.
(v) $\|x\| > 1$ *if and only if* $\sigma_p(x) > 1$.
(vi) *If* $0 < \alpha < 1$ *and* $\|x\| > \alpha$, *then* $\sigma_p(x) > \alpha^M$.
(vii) *If* $\alpha \geq 1$ *and* $\|x\| < \alpha$, *then* $\sigma_p(x) < \alpha^M$.

Theorem 7.1. *The following statements hold:*

(i) [10, Theorem 5.1] *The space* $a^r(u,p)$ *is rotund if only if* $p_k > 1$ *for all* $k \in \mathbb{N}$.
(ii) [18, Theorem 5.4] *The space* $bv(u,p)$ *is rotund if only if* $p_k > 1$ *for all* $k \in \mathbb{N}$.
(iii) [56, Theorem 8] *The space* $\ell(\widetilde{B},p)$ *is rotund if only if* $p_k > 1$ *for all* $k \in \mathbb{N}$.
(iv) [60, Theorem 18] *The space* $N^t(p)$ *is rotund if only if* $p_k > 1$ *for all* $k \in \mathbb{N}$.

Theorem 7.2. ([56, Theorem 9] and [60, Theorem 19])
Let (x_n) *be a sequence in* $\ell(\widetilde{B},p)$ [or $N^t(p)$]. *Then, the following statements hold:*

(i) $\lim_{n\to\infty} \|x_n\| = 1$ *implies* $\lim_{n\to\infty} \sigma_p(x_n) = 1$.
(ii) $\lim_{n\to\infty} \sigma_p(x_n) = 0$ *implies* $\lim_{n\to\infty} \|x_n\| = 0$.

Theorem 7.3. *The sequence space* $N^t(p)$ *has the Kadec-Klee property.*

(i) ([56, Theorem 12] and [60, Theorem 21]) *The sequence space* $\ell(\widetilde{B},p)$ $[N^t(p)]$ *has the Kadec-Klee property.*
(ii) ([56, Theorem 12] and [60, Theorem 21]) *For any* $1 < p < \infty$, *the space* $(\ell_p)_{\widetilde{B}}$ $[(\ell_p)^t_N]$ *has the uniform Opial property.*

8. Some problems for researchers

1. Investigate the domain of the Cesàro matrix C_1 of order 1 in the following spaces;
   (i) $\omega(p)$,
   (ii) $\omega_0(p)$,
   (iii) $\omega_\infty(p)$,
   (iv) $f_0(p)$,
   (v) $f(p)$,
   (vi) $\widehat{f}(p)$.
2. Define the matrix $\widetilde{B} = (\widetilde{b}_{nk})$ by the composition of the matrices E_1, C_1 and Δ as

$$\widetilde{b}_{nk} := \begin{cases} \frac{\binom{n}{k}}{2^n(k+1)} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

   for all $k, n \in \mathbb{N}$. Investigate the domain of the matrix $\widetilde{B}$ in the paranormed spaces listed in Problem 1.
3. Investigate the domain of the Riesz matrix R^t in the paranormed spaces listed in Problem 1.
4. Investigate the domain of the Nrlund matrix N^t in the paranormed spaces listed in Problem 1.
5. Investigate the domains $A(\ell_\infty(p))$, $A(c(p))$, $A(c_0(p))$ and $A(\ell(p))$ of Abel method in the Maddox's spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$ and $\ell(p)$, respectively.
6. Investigate the domains $S(\ell(p))$, $S(c(p))$ and $S(c_0(p))$ of the summation matrix S in the Maddox's spaces $\ell(p)$, $c(p)$ and $c_0(p)$, respectively.

7. Investigate the domains $F(\ell(p))$, $F(c(p))$ and $F(c_0(p))$ of double band matrix F in the Maddox's spaces $\ell(p)$, $c(p)$ and $c_0(p)$, respectively.
8. Investigate the domains $\Delta(\ell(p))$ and $A^u(\ell(p))$ of the matrices Δ and A^u in the Maddox's space $\ell(p)$, respectively.
9. Investigate the domains $E^r(\ell_\infty(p))$, $E^r(c(p))$ and $E^r(c_0(p))$ of the Euler mean in the Maddox's spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$, respectively.

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