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NUMERICAL APPROXIMATION OF TIME FRACTIONAL ADVECTION-DISPERSION MODEL ARISING FROM SOLUTE TRANSPORT IN RIVERS

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Abstract. In this note, radial basis functions (RBFs) approximation is utilized for solving fractional advection-dispersion equation, which has been used in groundwater hydrology as a reliable approach for modeling transport of passive tracers during fluid flow in a porous media. In this method, we discretize time fractional derivative of order α in the range $(0, 1]$ and spatial derivative terms by using the finite difference scheme and the Kansa method, respectively. Moreover, the stability and convergence of time-discretized scheme are performed in detail throughout the paper. Two numerical examples are included to illustrate the validity and applicability of the approach.

Keywords: time fractional derivative, time fractional advection-dispersion equation, radial basis functions, collocation method.

AMS Subject Classification: 35R11,91G60,65M70.

1. Introduction

We consider the following general advection-dispersion model which is naturally used to describe the transient transport of solutes through the homogeneous soil :

$$R\frac{\partial C(\xi,\tau)}{\partial \tau} = \left[D_L\frac{\partial^2}{\partial \xi^2} - \nu\frac{\partial}{\partial \xi}\right] C(\xi,\tau), \tag{1}$$

where $D_L = D_e + \alpha_L \nu$, $D_L > 0$, and $\nu > 0$. A complete list of parameters and variables for this model are defined in Table 1.

Table 1. Parameters for advection-dispersion model.

Parameter	Description
R	The retardation factor
D_L	The longitudinal dispersion coefficient
D_e	The effective diffusion coefficient
α_L	The dynamic dispersivity
ν	The average flow velocity
C	The concentration of the tracer (The pore water velocity)
ξ	The spatial coordinate
τ	The temporal coordinate

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The relation (1) is derived following Ficks first law and in principle it holds after the initial mixing period or for the far field where the longitudinal shear flow dispersion becomes a dominant mechanism of pollutant mixing in rivers. For simplicity in the rest of this work, we introduce dimensionless space, time, and concentration variables via

$$x = \frac{\xi}{L}, \qquad t = \frac{\tau}{L/\nu}, \qquad u = \frac{C}{C_0}.$$

Then the dimensionless reaction-advection-dispersion model becomes

$$\frac{\partial u(x,t)}{\partial t} = \beta_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \beta_2 \frac{\partial u(x,t)}{\partial x}.$$

The advection-dispersion model is one of the most well-known fundamental equations which has commonly been utilized to describe the Brownian motion of particles. Advection is the process by which a conserved physical quantity is transported in a fluid in motion. Also, dispersion is defined as the combined effect of advection and diffusion acting in a flow field with velocity gradients. The advection-dispersion model plays a significant role in groundwater hydrology to model the passive tracers carried by fluid and flow in a porous medium. Also, this model is widely used to solve a range of problems in chemical, physical , and biological sciences, involving diffusion or dispersion , such as mixing in inland and coastal waters [11], transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, and evolution of populations [25]. The advection-dispersion model assumes instantaneous, reversible partitioning to an immobile phase (i.e. sorption) with complete lateral mixing between regions of different velocity [2]. This model explains the change of probability of a random function in space and temporal. The main goal of this paper is to describe one approach based on fractional order derivatives. A fractional advection-dispersion equation (FADE) is a generalization of the classical advection-dispersion equation (ADE) in which the first-order derivative is replaced with a fractional-order derivative. In this note, we present a computational method to approximate the solution of the time fractional advection- dispersion equation with reaction order α $(0 < \alpha \leq 1)$ of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \beta_1 \frac{\partial^2 u(x,t)}{\partial x^2} - \beta_2 \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad a \leq x \leq b, \quad 0 \leq t \leq T, \tag{2}$$

with intial condition

$$u(x,0) = g(x), \quad a \leq x \leq b, \tag{3}$$

and the boundary conditions

$$u(a,t) = g_1(t), \quad u(b,t) = g_2(t), \qquad t > 0, \tag{4}$$

where u is the unknown concentration, β_1 and β_2 are the dispersion and advection coefficients,$\alpha \in (0,1]$ is the temporal fractional order, the operator $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the Caputo fractional derivative which will be introduced as follow. Here a source term $f(x,t)$ is added for the purposes of validation in Section 4. To determine an analytical solution of this problems is extremely difficult thus many authors are searching ways to numerically solve these problems. Several papers have been written [32, 33, 40] to show the equivalence between the transport equations using fractional order derivatives and some heavy-tailed motions. There exist several methods to solve fractional advection-dispersion equation such as implicit and explicit difference method [1, 5, 6, 27, 43], Green function [23], variable transformation [27], Adomians decomposition method (ADM) [10] and optimal homotopy asymptotic method [21, 24, 36]. Now, we briefly describe the definition and preliminaries of fractional calculus which will be used further in current paper [37].

Definition 1.1. *A real function* $f(x), x > 0$ *is said to be in the space* $C_\mu,\ \mu \in \mathbb{R}$ *if there exists a real number* $p(> \mu)$*, such that* $f(x) = x^p f_1(x)$ *where* $f_1(x) \in C[0, \infty)$ *and it is said to be in the space* C_μ^m iff $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 1.2. *The left-sided Riemann-Liouville fractional integral of order* α *of function* $f(x)$ *is defined as:*

$$ {}_a^x J^\alpha f\,(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, $$

where Γ *is the Gamma function.*

Lemma 1.1. *Properties of the operator* J^α *can be found in* [37]*, we mention only the following:*

For $f \in C_\mu,\ \mu \geq -1$,

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \quad for\ all\ \alpha, \beta \geq 0.$

(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \quad for\ all\ \alpha, \beta \geq 0.$

(3) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}, \quad \alpha > 0,\ \gamma > -1,\ x > 0.$

Definition 1.3. *The Caputo's time fractional derivative operator of order* α *of function* $f(x,t)$ *is defined as:*

$$ D_C^\alpha f(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t (x-t)^{m-\alpha-1} \frac{\partial^m f(x,\eta)}{\partial \eta^m} d\eta, & m-1 \leq \alpha < m, \\ \frac{\partial^m f(x,t)}{\partial t^m}, & \alpha = m. \end{cases} $$

1.1. **An overview of the concept of the meshless methods.** A meshless (meshfree) method is a method that is used to launch system of algebraic equations for the entire problem domain regardless of defining a predefined mesh for the domain discretization. In the past three decades, for omitting mesh structure, scientists have employed meshless methods. In such approach an assortment of scattered data were used as instead of constructing a meshing paradigm. One of the most prominent meshless method is radial basis function (RBF) method that seems to be a very well-organized system while facing interpolation of multidimensional scattered data. In actuality, the utilization of such novel method for mathematical solution of partial differential equation is based on the collocation method. It is (conditionally) positive definite, rotationally and translationally invariant. The chief benefits of this method are straightforward programming process and probable spectral precision. On the other hand, ill-conditioning of the resulting linear system is considered to be the main difficulty. RBF approaches that use infinitely differentiable origin functions that include a free factor are theoretically spectrally accurate. The well utilization of such RBF methods comprises development of a linear arrangement that is extremely ill-conditioned when the parameters of the method are in situation that the best accurateness is ideally comprehended. Consequently, in several applications, RBF methods does not possess the potential to produce exact results as they are skilled theoretically . Just contrary to mesh based approaches such as finite element method (FEM), finite volume method (FVM) and finite difference method (FDM), meshless methods use a set of accidental or uniform points which are not interlinked in the arrangement named as mesh. The RBF method can be taken into account as a category of compromise between the FE (finite elements) and the Pseudo-spectral (PS) methods. On the one hand, the RBF method is based on an expansion into basis functions that have a spatial location just similar to FE method. In this point of view, these basis functions can be grouped in a definite section to locally increase the accuracy of the method.

On the other hand, the basic functions used in the RBF expansion are high-order functions that conventionally cover the whole domain like with the PS technique. It was remarked that RBFs converge to PS methods in their at radial function boundary, making RBFs a generalized formulation and methodology to PS methods, for scattered nodes and non-flat radial functions [9]. RBFs have quite a lot of rewards over PS methods: in spite of subscription of flexibility in terms of the domain shape, they allow a local node refinement, an easy singularity-free generalization to N dimensions and a shape parameter letting user extend the solution space to regions outside of the polynomial space, particularly susceptible to the Runge phenomenon [12]. The existence, uniqueness, and convergence of the RBFs approximation was discussed in detail by [13, 31, 34]. Many detailed discussions have been carried out regarding meshless methods and their related applications for solving complex PDEs [8, 18, 19, 20, 38, 39], for fractional equations [3, 7, 15, 16, 22, 29, 30, 42] and for integral equations [14, 17].

1.2. **The outline of current research.** In the current research, our objective is to focus on the numerical simulation of time fractional advection-dispersion equation via RBFs meshless method. The paper taxonomy is structured as follows. In Section 2, the discretization process of the problem in time is described variable via finite difference scheme of order $O(\tau^{2-\alpha})$ for $0 < \alpha < 1$ and also approximated by using the RBFs in the space direction. In Section 3, stability and convergence the the time discrete scheme and the error computes of this method are presented. In Section 4, we report the numerical experiments of solving the aforementioned equation with the proposed method for some test problems. Finally in Section 5, the paper ends with a conclusion and some remarks for future work. Note that the numerical results have been calculated by Matlab programming.

2. Implementation of the method

In this section we explain the numerical scheme for the solution of Eq. (2). Hence to construct numerical scheme, we consider N points $\{x_j = jh | j = 1, 2, 3, ..., N\}$ in the bounded domain $[a, b]$, where x_1, x_N are the boundary points, and the grid points in the time interval $[0, T]$ are tagged as $t_n = n\tau, n = 0, 1, 2, 3, ..., M$, where $h = (b-a)/N, \tau = T/M$ and $u^n(x_i) = u(x_i, t_n)$.

2.1. **Discretization of time.** In Eq. (2) is the Caputo fractional derivative of $u(x, t)$, which can be written as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t-\xi)^\alpha} d\xi, & 0 < \alpha < 1, \\ \\ \frac{\partial u(x,t}{\partial t}, & \alpha = 1, \end{cases} \tag{5}$$

we use the finite difference scheme to analogize the time fractional derivative term

$$\begin{aligned} \frac{\partial^\alpha u(x,t_{n+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t_{n+1}-\xi)^\alpha} d\xi \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k\tau}^{(k+1)\tau} \frac{\partial u(x,\xi)}{\partial \xi} \frac{1}{(t_{n+1}-\xi)^\alpha} d\xi \\ &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k\tau}^{(k+1)\tau} \frac{\partial u(x,\xi_k)}{\partial \xi} \frac{1}{(t_{n+1}-\xi)^\alpha} d\xi. \end{aligned} \tag{6}$$

Now, the first order time derivative taking into account the forward difference formula can be approximated:

$$\frac{\partial u(x,\xi_k)}{\partial \xi} = \frac{u(x,\xi_{k+1}) - u(x,\xi_k)}{\tau} + R_1^{k+1}(x),$$

where $\xi_k \in [t_k, t_{k+1}]$. In view of Taylor's Theorem, the truncation error can be calculated as:

$$|R_1^{k+1}(x)| \leq C_1\tau \quad or \quad R_1^{k+1} = O(\tau).$$

Now, we obtain the following implicit discrete scheme for Eq. (6)

$$\begin{aligned}
\frac{\partial^\alpha u(x,t_{n+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n}\Big(\frac{u(x,t_{k+1}) - u(x,t_k)}{\tau} + O(\tau)\Big)\int_{k\tau}^{(k+1)\tau}\frac{1}{(t-\xi)^\alpha}d\xi \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n}\Big(\frac{u(x,t_{k+1}) - u(x,t_k)}{\tau} + O(\tau)\Big)\int_{k\tau}^{(k+1)\tau}\frac{dr}{r^\alpha} \\
&= \begin{cases} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(u^{n+1} - u^n) + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum\limits_{k=1}^{n}\Big[(k+1)^{1-\alpha} - k^{1-\alpha}\Big](u^{n+1-k} - u^{n-k}) & n \geq 1 \\ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(u^1 - u^0) & n = 0 \end{cases} \\
&= \begin{cases} a_0\Big[(u^{n+1} - u^n) + \sum\limits_{k=1}^{n} b_k(u^{n+1-k} - u^{n-k})\Big], & n \geq 1, \\ a_0(u^1 - u^0), & n = 0, \end{cases} + O(\tau^{2-\alpha}), \qquad (7)
\end{aligned}$$

where $a_0 = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}$, $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}, (k = 0, 1, ..., n)$, $u^0 = u(x, t = 0) = g(x)$.

Substituting Eq. (7) into Eq. (2), we discretize time derivative of time FADE using a classic finite difference formula with accuracy of order $2 - \alpha$ and space derivatives between successive two time levels n and $n + 1$ as:

$$\begin{aligned}
&a_0 u^{n+1} - \beta_1 \nabla^2 u^{n+1} + \beta_2 \nabla u^{n+1} = \\
&\begin{cases} a_0\Big[u^n - \sum\limits_{k=1}^{n} b_k(u^{n+1-k} - u^{n-k})\Big] + f^{n+1}, & n \geq 1, \\ a_0 u^0 + f^1, & n = 0, \end{cases} + R^{k+1}. \qquad (8)
\end{aligned}$$

where ∇ is the gradient differential operator and $f^{n+1} = f(x, t_{n+1})$; $n = 0, 1, ..., M$. We will prove that the bound for the error R^{k+1} of semi-discrete scheme is $O(\tau^{2-\alpha})$. Defining U^k as the approximation of u^k and eliminating the small term R^{k+1}, then a semi-discrete scheme is achieved as follows:

$$\begin{aligned}
&a_0 U^{n+1} - \beta_1 \nabla^2 U^{n+1} + \beta_2 \nabla U^{n+1} = \\
&\begin{cases} a_0\Big[U^n - \sum\limits_{k=1}^{n} b_k(U^{n+1-k} - U^{n-k})\Big] + f^{n+1}, & n \geq 1, \\ a_0 U^0 + f^1, & n = 0. \end{cases} \qquad (9)
\end{aligned}$$

2.2. **Spatial derivative discretization by RBFs approximation scheme.** In order to apply RBFs approximation scheme, we collocate N different points $\{x_j | j = 1, ..., N\}$, where x_1 and x_N are boundary points and the other $(N-2)$ points are inner points $\{x_j | j = 2, ..., N-1\}$. The approximate expansion of $u(x_i, t_n)$ at a point of interest $\mathbf{x}_i$ is as follows:

$$U_i^{n+1} = U(x_i, t_{n+1}) = \sum_{j=1}^{N} \lambda_j^{n+1} \phi(r_{ij}) + \lambda_N^{n+1} x_j + \lambda_{N+1}^{n+1}, \tag{10}$$

where $\{\lambda_j^n\}$ are unknown coefficients of the n^{th} time layer $\phi(r_{ij})$ radial basis function, $r_{ij} = |x_i - x_j|$. The some types of radial basis functions are generally listed in the following:

Generalized Multiquadric (GMQ) $(c^2 + r^2)^{\beta}$,

Inverse Multiquadric (IMQ) $\frac{1}{\sqrt{c^2+r^2}}$,

Inverse Quadratic (IQ) $(c^2 + r^2)^{-1}$,

Multiquadric (MQ) $\sqrt{c^2 + r^2}$,

Hyperbolic secant (sech) $sech(cr)$,

Thin Plate Spline (TPS) $r^{2m} \log(r)$,
where constant c is known as the shape parameter of the RBFs for controlling the shape of functions which is found experimentally for each RBF. The appropriate choice of shape parameter is a significant duty in approximating functions by RBFs and scientists continuously have concerned about choice a good shape parameter.

Besides N equations resulting from collocating Eq. (10) at N points, additional two equations are required by the following regularization conditions

$$\sum_{j=1}^{N} \lambda_j^{n+1} = \sum_{j=1}^{N} \lambda_j^{n+1} x_j = 0, \tag{11}$$

Combining Eq. (10) with Eq. (11), in a matrix form, it is to illustrate that:

$$\{U\}^{n+1} = A\{\lambda\}^{n+1}, \tag{12}$$

where $\{U\}^{n+1} = [U_1^{n+1}, ..., U_N^{n+1}, 0, 0]^T$ and $\{\lambda\}^{n+1} = [\lambda_1^{n+1}, ..., \lambda_N^{n+1}]^T$ and the matrix $A = (a_{ij})_{(N+2)\times(N+2)}$ is defined by

$$A = \begin{bmatrix} \Phi & P_{N\times 2} \\ P^T & \mathbf{0}_{2\times 2} \end{bmatrix},$$

where $\Phi = [\phi(r_{ij})]_{N\times N}$ and

$$P = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix}_{N\times 2}.$$

Rewriting of Eq. (8) in the matrix form can be represented as follows:

$$B\{\lambda\}^1 = \{b\}^1, \tag{13}$$

in which

$$B = \left[\begin{array}{cc} L(\Phi) & L(P) \\ P^T & \mathbf{0} \end{array} \right]_{(N+2)\times(N+2)}, \tag{14}$$

where L represents an operator given by

$$L(*) = \begin{cases} [a_0 + \beta_2\nabla - \beta_1\nabla^2](*), & 1 < i < N, \\ (*), & i = 1 \ or \ N. \end{cases}$$

and $\{b\}^1 = [b_1^1 ... b_N^1, 0, 0]^T$ where $b_1^1 = g_1^1$, $b_N^1 = g_2^1$ and $b_i^1 = a_0 U_i^0 + f_i^0, \qquad i = 2, 3, ..., N-1.$

Also, for $n \geq 1$

$$B\{\lambda\}^{n+1} = \{b\}^{n+1}, \tag{15}$$

$\{b\}^{n+1} = [b_1^{n+1} ... b_N^{n+1}, 0, 0]^T$ are achieved by Eq. (8) as

$$b_i^{n+1} = \begin{cases} g_1^{n+1} & i = 1, \\ a_0 \left[U_i^n - \sum\limits_{k=1}^{n} b_k (U_i^{n+1-k} - U_i^{n-k}) \right] + f_i^{n+1} & 1 < i < N, \\ g_2^{n+1} & i = N. \end{cases} \tag{16}$$

After obtaining the accurate answers for the algebraic system of equations $B\{\lambda\}^{k+1} = \{b\}^{k+1}$ at each time step, the solution can be constructed using Eq. (12).

3. Error estimation

First of all, we mention the following preliminary of functional analysis that shall be utilized for discretization of time variable.

3.1. Background on functional analysis. Let Ω demonstrate a bounded and open domain in $\mathbb{R}^2$ and let dx be the Lebesgue measure on $\mathbb{R}^2$. For $p < \infty$, we denote by $L^p(\Omega)$ the space of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $\int\limits_{\Omega} |u(x)|^p dx \leq \infty$. It is a Banach space for the norm

$$||u||_{L^p(\Omega)} = \left(\int\limits_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

The space $L^p(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int\limits_{\Omega} u(x)v(x)dx,$$

and norm in L^2,

$$||v|| = [(v, v)]^{\frac{1}{2}} = \left[\int\limits_{\Omega} v(x)v(x)dx \right]^{\frac{1}{2}}.$$

Also we assume that Ω is an open domain in $\mathbb{R}^d$, $\gamma = (\gamma_1, ..., \gamma_d)$ is a d-tuple of nonnegative integers and $|\gamma| = \sum\limits_{i=1}^{p} \gamma_i$. Accordingly, we put

$$D^\gamma v = \frac{\partial^{|\gamma|} v}{\partial x_1^\gamma \partial x_2^\gamma ... \partial x_d^\gamma}.$$

In this regard, one can obtain:

$$H^1(\Omega) = \{v \in L^2(\Omega), \ \tfrac{dv}{dx} \in L^2(\Omega)\},$$
$$H_0^1(\Omega) = \{v \in H^1(\Omega), \ v|_{\partial(\Omega)} = 0\},$$

$$H^m(\Omega) = \{v \in L^2(\Omega),\ D^\gamma v \in L^2(\Omega) \texttt{ for all positive integer} |\gamma| \leq m\}.$$

The definition of inner product in Hilbert space is below as:

$$(u, v)_m = \sum_{|\gamma| \leq m} \int_\Omega D^\gamma u(x) D^\gamma v(x) dx,$$

which induces the norm

$$||u||_{H^m(\Omega)} = \left(\sum_{|\gamma| \leq m} ||D^\gamma u||^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

The Sobolev space $W^{1,p}(I)$ is defined to be

$$W^{1,p}(I) = \{u \in L^p(I);\ \exists g \in L^P(I) : \int_I u\varphi' = \int_I g\varphi', \forall \varphi \in C^1(I)\}.$$

Also, in this paper, we introduce the following inner product and the associated energy norms in L^2 and H^1

$$||v|| = (v, v)^{1/2}, \quad ||v||_1 = (v, v)_1^{1/2}, \quad |v|_1 = (\tfrac{\partial v}{\partial x}, \tfrac{\partial v}{\partial x})^{1/2}.$$

by inner products of $L^2(\Omega)$ and $H^1(\Omega)$

$$(u, v) = \int u(x)v(x)dx, \qquad (u, v)_1 = (u, v) + (\tfrac{\partial u}{\partial x}, \tfrac{\partial v}{\partial x}),$$

respectively.

3.2. **Stability and convergence.** In this section, the stability and convergence of the proposed numerical solution will be described. The equation (9) can be restated according to below expression:

$$U^{k+1} - \mu_1 \nabla^2 U^{k+1} - \mu_2 \nabla U^{k+1} = (1 - b_1)U^k + \sum_{j=1}^{k} (b_j - b_{j+1}) U^{k-j} + b_k U^0 + F^{k+1}, \tag{17}$$

where $\mu_1 = \frac{\Gamma(2-\alpha)}{\tau^\alpha} \beta_1$, $\mu_2 = -\frac{\Gamma(2-\alpha)}{\tau^\alpha} \beta_2$, $F = \frac{\Gamma(2-\alpha)}{\tau^\alpha} f$. Firstly, we state the following three lemmas for discretization of the time fractional derivative [41, 26].

Lemma 3.1. *Let $g(t) \in C^2[0, t_k]$ and $0 < \alpha < 1$ then*

$$\left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{g(t)}{(x-t)^\alpha} dt - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[(1 - b_0) g(t_k) + \sum_{j=1}^{k-1} (b_{k-j-1} - b_{k-j}) g(t_j) + b_{k-1}) g(t_0) \right] \right|$$
$$\leq \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha}) \right] \max_{0 \leq t \leq t_k} |g''(t)| \tau^{2-\alpha},$$

where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$.

Lemma 3.2. *The coefficients b_j $(j = 0, 1, 2, ..)$ defined by (7) fulfills the following:*

- $b_0 = 1, b_j > 0,\ j = 0, 1, 2, ..., b_n \to 0$ *as* $n \to \infty$;
- *we have*

$$b_j > b_{j+1},\ j = 0, 1, 2, ...\ ;$$
$$\sum_{j=0}^{k-1} (b_{j+1} - b_j) + b_k = (1 - b_1) + \sum_{j=1}^{k-1} (b_{j+1} - b_j) + b_k = 1;$$

- *there exists a positive constant $C > 0$ such that:*

$$\tau < C b_j \tau^\alpha,\ j = 0, 1, 2, ...\ .$$

Proof. It is straightforward from the definition $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, where $0 < \alpha < 1$. □

Lemma 3.3. *If* $U^k(x) \in H^1(\Omega)$ $k = 0, 1, ..., M$ *is the solution of Eq.* (17)*, then*

$$\|U^k\| \leq \|U^0\| + b_{k-1}^{-1} \max_{0\leq l\leq M} ||F^l||.$$

Proof. We proceed the result by the principle of mathematical induction on k as the counter of induction. If $k = 0$, we have

$$U^1 = \mu_1 \nabla^2 U^1 + \mu_2 \nabla U^1 + U^0 + F^1 \ . \tag{18}$$

We multiply above equation by U^1 and integrate on Ω,

$$||U^1||^2 - \mu_1(\nabla^2 U^1, U^1) - \mu_2(\nabla U^1, U^1) = (U^0, U^1) + (F^1, U^1).$$

Considering the Cauchy-Schwarz inequality and $U^k(x) \in H^1(\Omega)$, we easily get

$$||U^1|| \leq ||U^0|| + ||F^1|| \leq ||U^0|| + \max_{0\leq l\leq M} ||F^l||,$$

the result is trivially true. Suppose the theorem is true for all j

$$||U^j|| \leq ||U^0|| + b_{j-1}^{-1} \max_{0\leq l\leq M} ||F^l||, \quad j = 1, 2, ..., k. \tag{19}$$

Multiplying Eq. (17) by U^{k+1} and integrating on Ω, one obtains

$$\begin{aligned} &||U^{k+1}||^2 - \mu_1(\nabla^2 U^{k+1}, U^{k+1}) - \mu_2(\nabla U^{k+1}, U^{k+1}) \\ &\qquad = (1 - b_1)(U^k, U^{k+1}) + \sum_{j=1}^{k} (b_j - b_{j+1}) U^{k-j} + b_k(U^0, U^{k+1}) + (F^{k+1}, U^{k+1}). \end{aligned}$$

By means of the Cauchy-Schwarz inequality, $U^k(x) \in H^1(\Omega)$ and $b_{j+1} < b_j < 1$, it leads that

$$||U^{k+1}|| \leq (1 - b_1)||U^k|| + \sum_{j=1}^{k} (b_j - b_{j+1})||U^{k-j}|| + b_k||U^0|| + ||F^{k+1}||. \tag{20}$$

Using Eq. (19), the last expression can be written as follows

$$||U^j|| \leq ||U^0|| + b_{j-1}^{-1} \max_{0\leq l\leq M} ||F^l|| \leq ||U^j|| + b_j^{-1} \max_{0\leq l\leq M} ||F^l||. \tag{21}$$

Regarding Lemma 3.2, we get $b_j < b_i < 1;\ 1 \leq i \leq j$ and easily it results

$$\begin{aligned} (1 - b_1)||U^k|| + \sum_{j=1}^{k} (b_j - b_{j+1})||U^{k-j}|| &= \sum_{j=0}^{k-1} (b_j - b_{j+1})||U^{k-j}|| \\ &\leq \sum_{j=0}^{k-1} (b_j - b_{j+1}) \left[||U^0|| + b_{k-j-1}^{-1} \max_{0\leq l\leq M} ||F^l|| \right] \\ &\leq (1 - b_k)||U^0|| + (1 - b_k) b_k^{-1} \max_{0\leq l\leq M} ||F^l|| \\ &= (1 - b_k)||U^0|| + \ (b_k^{-1} - 1) \max_{0\leq l\leq M} ||F^l|| \ . \end{aligned} \tag{22}$$

Therefore, in view of Eqs. (20)-(22), the following inequality is achieved

$$||U^{k+1}|| \leq ||U^0|| + b_{k-1}^{-1} \max_{0\leq l\leq M} ||F^l||.$$

Induction completes the proof of the lemma. □

Theorem 3.1. *The fractional implicit numerical method defined by Eq.* (17) *is un-conditionally stable.*

Proof. Let $\widehat{U}^k(x);\ k = 1, ..., M$, be the solution of the method (17) with the initial condition $\widehat{U}^0 = U(x, 0)$, then the error $\varepsilon^k = U^k(x) - \widehat{U}^k(x)$ fulfills

$$\varepsilon^{k+1} - \mu_1 \nabla^2 \varepsilon^{k+1} + \mu_2 \nabla \varepsilon^{k+1} = (1 - b_1)\varepsilon^k + \sum_{j=1}^{k} (b_j - b_{j+1})\varepsilon^{k-j} + b_k \varepsilon^0 + F^{k+1},$$

and $\varepsilon^{k+1}|_{\partial\Omega} = 0$. Now recalling Lemma 3.3., it can be easily shown that

$$\|\varepsilon^k\| \leq \|\varepsilon^0\|,\ k = 1, ..., M.$$

The proof is completed. □

Theorem 3.2. *Suppose that* $\{u(x, t_k)\}_{k=1}^{M}$ *is the exact solution of Eqs.*(2)-(4) *and* $\{U^k(x)\}_{k=1}^{M}$ *be the time-discrete solution of Eq.*(17) *with initial condition* $U^0(x) = u(x, 0)$. *Then we have the following error estimate,*

$$||u(x, t_k) - U^k(x)|| \leq C\tau^{2-\alpha},$$

where C is a positive constant.

Proof. First of all, we define $\xi^k = u(x, t_k) - U^k(x)$ at $t = t_k;\ k = 1, 2, , M$. Now, by subtracting Eq.(8) from Eq.(9) lead to

$$\xi^{k+1} - \mu_1 \nabla^2 \xi^{k+1} + \mu_2 \nabla \xi^{k+1} = \xi^k - \sum_{j=1}^{k} b_j (\xi^{k+1-j} - \xi^{k-j}) + R^{k+1},$$

$\xi^0(x) = 0$ and $\xi^0(x)|_{\partial\Omega} = 0$. By using Lemma 3.3., it is quite evident that

$$||\xi^k|| \leq b_{k-1}^{-1} \max_{0 \leq l \leq M} ||R^l|| \leq b_{k-1}^{-1} \tau^2,$$

Because $b_{k-1}^{-1}\tau^\alpha$ is bounded [28], thus we result

$$||\xi^k|| = ||u(x, t_k) - U^k(x)|| \leq C\tau^{2-\alpha},$$

and the proof is finished. □

4. Numerical experiments

In this section to demonstrate the effectiveness of our approach, we current the numerical results of the suggested approach. We tested the accurateness and stability of the method defined in this paper by doing the aforesaid numerical method for different values of h, τ and c. To check the accuracy of method, we compute the following error norm:

$$L_\infty = \max_{1 \leq j \leq M-1} |U(x_j, T) - u(x_j, T)|.$$

The computational orders (denoted by C_1-Order and C_2-Order) in time variable and in space variable respectively can be evaluated as below

$$C_1 - order = \frac{\log(\frac{E_1}{E_2})}{\log(\frac{\tau_1}{\tau_2})},$$

$$C_2 - order = \log_2 \left(\frac{||L_\infty(16\tau, 2h)||}{||L_\infty(\tau, h)||} \right),$$

in which E_i is the error value that corresponds to grid with mesh size τ_i [35, 4].

Example 4.1. *Consider the following non-homogeneous FADE*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad x \in [0,1], \quad 0 \le t \le T, \quad 0 < \alpha < 1,$$

in which $f(x,t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t[\sin(x) + \cos(x)]$. *The analytic solution of the above problem is given by* $u(x,t) = t\sin(x)$ [35].

The initial and boundary conditions can be achieved from the analytic solution. We solve this example with the method presented in this paper with several values of h, τ, for $a = 0$, $b = 1$, c at final time $T = 1$. The L_∞ error, C_1-order and C_2-order of applied method are shown in Tables 2 and 3. Based on detailed comparisons in Table 2, we conclude that the convergence order in time is $2 - 0.2 = 1.8$ or $2 - 0.7 = 1.3$. Also, in view of Table 3, we conclude that the convergence order of our proposed numerical approach in space is good agreemente with [35]. Figure 1 displays the numerical solution of Example 4.1. using MQ with $c = 0.5$.

Table 2. Errors and computational orders obtained for Example 4.1. with $h = 0.1$ and MQ-RBF.

τ	$\alpha = 0.2$			$\alpha = 0.7$		
	L_∞	c	C_1-order	L_∞	c	C_1-order
1/10	2.3610×10^{-3}	0.50	–	1.6240×10^{-2}	0.65	–
1/20	7.2306×10^{-4}	0.55	1.7073	6.5437×10^{-3}	0.52	1.3141
1/40	2.1852×10^{-4}	0.65	1.7264	2.6312×10^{-3}	0.68	1.3144
1/80	6.7150×10^{-5}	0.90	1.7023	1.0467×10^{-3}	0.75	1.3299
1/160	2.0148×10^{-5}	0.95	1.7368	4.1838×10^{-4}	0.95	1.3230
1/320	6.1053×10^{-6}	1.00	1.7225	1.6572×10^{-4}	1.00	1.3361
TCO			1.8			1.3

Table 3. Errors and computational orders obtained for Example 4.1. with $c = 0.5$ and MQ-RBF .

h	τ	$\alpha = 0.2$		$\alpha = 0.7$	
		L_∞	C_2-order	L_∞	C_2-order
1/4	1/4	8.246×10^{-2}	–	6.473×10^{-2}	–
1/8	1/64	3.632×10^{-3}	4.5049	3.689×10^{-3}	4.4636
1/16	1/1024	1.536×10^{-4}	4.5635	8.019×10^{-5}	4.3820
1/8	1/8	4.161×10^{-2}	–	1.296×10^{-2}	–
1/16	1/128	2.689×10^{-3}	3.9518	3.541×10^{-4}	5.1938

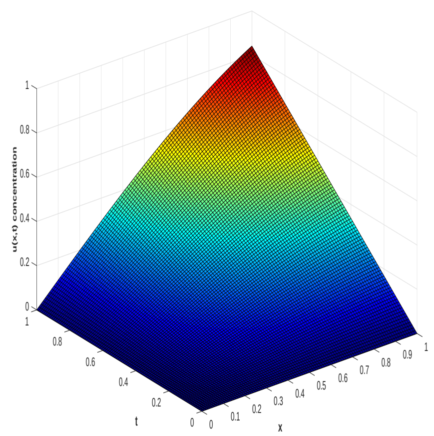

Figure 1. Numerical solution of Example 4.1. with $\alpha = 0.7$, $\tau = 0.001$ and $h = 0.01$.

Example 4.2. *Now, we consider the following non-homogeneous FADE,*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \qquad x \in [0,1], \quad 0 \le t \le T,$$

subject to the initial condition $u(x,0) = 0$, $f(x,t) = 2x - 2 + \frac{2}{\sqrt{\pi}}\sqrt{t}$ with $\alpha = 0.5$. Then the analytic solution of this example is $u(x,t) = x^2 + t$. The initial and boundary conditions can be achieved from the analytic solution. We solve this example with the method presented in this paper with several values of h, τ, for $a = 0$, $b = 1$, c at final time $T = 1$. The L_∞ error and C_1-order of applied method are shown in Table 4 Based on detailed comparisons in Table 4, we conclude that the convergence order in time is $2 - 0.5 = 1.5$. Table 5 displays the comparison between errors obtained for MQ, IQ and GA RBFs with $h = 0.$ and $\alpha = 0.5$ for Example 4.2. Figure 2 shows the numerical solution of Example 4.2. using MQ with $c = 0.65$.

Table 4. Errors and computational orders obtained for Eample 4.2. with $h = 0.1$ and MQ-RBF.

τ	MQ		
	L_∞	c	C_1-order
1/10	2.3746×10^{-3}	0.65	–
1/20	8.5153×10^{-4}	0.50	1.4778
1/40	3.0195×10^{-4}	0.75	1.4959
1/80	1.0457×10^{-5}	0.90	1.5306
1/160	1.2075×10^{-6}	1.00	1.5617
TCO			1.5

Table 5. Comparison between errors obtained for different RBFs with $h = 0.1$ and $\alpha = 0.5$ for Example 4.2.

τ	MQ		IQ		GA	
	L_∞	c	L_∞	c	L_∞	c
1/10	4.7498×10^{-3}	0.50	5.2461×10^{-3}	0.5	5.3673×10^{-2}	6.5
1/20	2.6432×10^{-3}	0.50	2.8147×10^{-3}	0.5	2.8573×10^{-4}	6.5
1/40	1.7483×10^{-3}	0.50	1.3498×10^{-3}	0.5	1.5268×10^{-4}	6.5
1/60	1.4316×10^{-4}	0.50	2.7498×10^{-4}	0.5	2.7561×10^{-4}	6.0
1/80	1.1387×10^{-4}	0.50	1.2642×10^{-4}	0.5	1.4791×10^{-4}	6.0
1/100	4.8145×10^{-5}	0.50	5.6233×10^{-5}	0.5	5.6233×10^{-5}	5.5

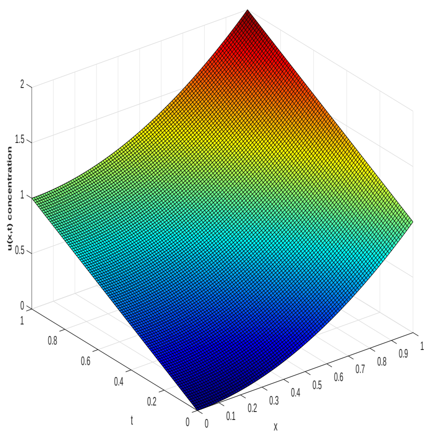

Figure 2. Numerical solution of Example 4.2. with $\alpha = 0.5$, $\tau = 0.001$ and $h = 0.01$.

5. Conclusion

In the research, an attempt was made to develop an implicit meshless approach based on the RBF for numerical simulation of time FADEs, which is a category of fractional partial differential equation (FPDE). The stability and also convergence of the proposed meshless approach are assessed hypothetically and mathematically. Numerous numerical instances with different problem domains are utilized to investigate the developed meshless model accuracy and effectiveness. As can be inferred from mentioned investigations, the convergence order of this current method concerning to time is $O(\tau^{2-\alpha})$. All in all, the current meshless formulation is very operative for modeling and simulation of fractional differential equations, and it has well prospective to advance a robust simulation tool for problems in engineering and science which are governed by the numerous types of fractional differential equations. In the future studies, we will focus on problems with much more complexity.

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