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HILFER FRACTIONAL SPECTRAL PROBLEM VIA BESSEL OPERATOR

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Abstract. In this paper, we deal with a modified fractional Hilfer Sturm–Liouville operator for Bessel potential and we show the self-adjointness of the operator, orthogonality of distinct eigenfunctions and reality of eigenvalues. Also, we obtain an integral representation of solution and we give a numerical method for obtaining numerical results by changing α, β values illustrated by graphics.

Keywords: Hilfer, Bessel, spectral theory, fractional, Sturm-Liouville.

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1. Introduction

Fractional differential equations involve fractional order derivatives of the dependent variables, like $\frac{d^{\alpha}y(x)}{dx^{\alpha}}$, which are defined for $\alpha > 0$. Here α is not necessarily an integer, and can be rational, and complex-valued. Since fractional calculations have been applied to many areas of science, the importance of fractional differential equations has increased for the last years. Nowadays, the applications of fractional differential equation models are available in many areas such as physics, mathematics, engineering, biology, and earth sciences [12, 13, 14, 15]. Fractional differential equations continue to develop its new and updated definitions [1, 4, 9, 10, 13, 14, 17, 22, 21, 23, 25]. One of those is the generalized Riemann–Liouville fractional derivative operator in [12], it is also called fractional Hilfer derivative, and defined as follows

$$\left(D_{a\pm}^{\alpha,\beta}y\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a\pm}^{(1-\beta)(1-\alpha)}y\right)\right)(x),\ x>0,$$

where $\alpha \in \mathbb{R}$, $\alpha \in (0,1]$, $\beta \in \mathbb{R}$ and $\beta \in [0,1]$, I is the classical fractional Riemann-Liouville derivative. According to this definition, the derivative is defined by two parameters α, β, where α is the order and β is the type of the derivative. β plays an important role by changing the type of the derivative, namely fractional Hilfer derivative is Riemann-Liouville fractional derivative when $\beta = 0$, and Caputo fractional derivative when $\beta = 1$. The studies on fractional Hilfer derivative have been done by [9, 10, 12, 26]. A modified fractional Hilfer derivative [24] is defined as follows differently from the above definition

$$\begin{aligned}\left(D_{a+}^{\alpha,\beta}f\right)(x) &= \left(I_{a+}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a+}^{(1-\beta)(1-\alpha)}f\right)\right)(x),\\ \left(D_{b-}^{\alpha,\beta}f\right)(x) &= \left(I_{b-}^{(1-\beta)(1-\alpha)}\frac{d}{dx}\left(I_{b-}^{\beta(1-\alpha)}f\right)\right)(x).\end{aligned}$$

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A new fractional operator called Ψ fractional integral defined in [24] and this new definition is a generalization of the modified fractional Hilfer derivative.

Sturm-Liouville (S–L) problems become even more important in a lot of areas of science, and engineering. S–L problems are divided into regular and singular types. Differential equations such as Bessel, hydrogen atom, Hermitte, Jakobi, Legendre, and Chebyshev equations can be obtained from Sturm-Liouville equations. There are many studies on these issues [2, 3, 5, 6, 7, 8, 11, 16, 18, 19, 20, 27, 28, 29]. Regular fractional S–L problem is studied in [15]. [5, 6] consider the spectral theory of singular fractional S–L problems.

Let $\nu \in \mathbb{C}$. The following differential equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-\nu^2\right)w=0$$

is known as Bessel's equation of order ν. The solutions of the Bessel equation can be found with the Frobenius method. The general solution of the Bessel equation of order ν is

$$w\left(z\right)=AJ_\nu\left(z\right)+BY_\nu\left(z\right), \quad A,B\in\mathbb{C}$$

where $J_\nu\left(z\right)$ and $Y_\nu\left(z\right)$ are called the Bessel functions of the first and second kind of order ν and argument z, respectively.

In this paper, our aim is to consider modified Hilfer fractional singular Sturm-Liouville equation with Bessel potential, show the spectral properties of this problem and find the representation of the solution of this equation via Laplace transforms. Furthermore, we give some applications and their graphs of solutions of the equation.

2. Preliminaries

Definition 2.1. [21] *Let $0<\alpha\leq 1$. The left and right fractional integrals in the Riemann-Liouville sense of order α are given as follows:*

$$\left(I_{a,+}^{\alpha}f\right)(x)=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^x\left(x-s\right)^{\alpha-1}f\left(s\right)ds, \quad x>a,$$

$$\left(I_{b,-}^{\alpha}f\right)(x)=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_x^b\left(s-x\right)^{\alpha-1}f\left(s\right)ds, \quad x<b,$$

respectively, where Γ denotes the gamma function.

Definition 2.2. [21] *Let $0<\alpha\leq 1$. The left and right fractional derivatives in the Riemann-Liouville sense of order α are defined as*

$$\left(D_{a,+}^{\alpha}f\right)(x)=D\left(I_{a,+}^{1-\alpha}f\right)(x) \qquad x>a,$$

$$\left(D_{b,-}^{\alpha}f\right)(x)=-D\left(I_{b,-}^{1-\alpha}f\right)(x) \qquad x<b,$$

similar formulas for left- and right-sided Caputo derivatives of order α:

$$\left({}^{C}D_{a,+}^{\alpha}f\right)(x)=\left(I_{a,+}^{1-\alpha}Df\right)(x) \qquad x>a,\, 0<\alpha\leq 1,$$

$$\left({}^{C}D_{b,-}^{\alpha}f\right)(x)=\left(I_{b,-}^{1-\alpha}\left(-D\right)f\right)(x) \qquad x<b,.\, 0<\alpha\leq 1.$$

Definition 2.3. [10] *The right and left sided fractional derivatives* $D_{a\pm}^{\alpha,\beta}$ *of order* $\alpha\,(0<\alpha<1)$ *and type* β $(0\leq\beta\leq1)$ *with respect to* x *are defined by*

$$\left(D_{a\pm}^{\alpha,\beta}f\right)(x)=\left(\pm I_{a\pm}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a\pm}^{(1-\beta)(1-\alpha)}f\right)\right)(x) \tag{1}$$

whenever the second member of (1) *exists.*

Theorem 2.1. [24] *For sufficiently good functions* $f(x)$ *and* $g(x)$, *the new operator defined above satisfy the following properties*

$$\int_a^b g(x)\left(D_{a+}^{\alpha,\beta}f\right)(x)\,dx=-\int_a^b f(x)\left(D_{b-}^{\alpha,\beta}g\right)(x)\,dx+I_{a+}^{(1-\alpha)(1-\beta)}f(x)\,I_{b-}^{\beta(1-\alpha)}g(x)\Big|_a^b. \tag{2}$$

Definition 2.4. [24] *The right-sided modified fractional Hilfer derivative* $D_{a+}^{\alpha,\beta}$ *and left-sided modified fractional Hilfer derivative* $D_{b-}^{\alpha,\beta}$ *are defined by*

$$\left(D_{a+}^{\alpha,\beta}f\right)(x)=\left(I_{a+}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a+}^{(1-\beta)(1-\alpha)}f\right)\right)(x), \tag{3}$$

$$\left(D_{b-}^{\alpha,\beta}f\right)(x)=\left(I_{b-}^{(1-\beta)(1-\alpha)}\frac{d}{dx}\left(I_{b-}^{\beta(1-\alpha)}f\right)\right)(x) \tag{4}$$

where $\alpha\in(0,1)$ *and* $\beta\in[0,1]$.

Definition 2.5. [21] *The Laplace transform of a function* $f(t)$ *is defined by*

$$F(s)=\mathfrak{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)\,dt,$$

where $s\in\mathbb{R}^+$.

Property 1. [10] According to the definition of the left Hilfer derivative we can write the Laplace transform of Hilfer derivative as follow

$$\mathfrak{L}\left\{D_{0+}^{\alpha,\beta}f(x)\right\}(s)=s^{\alpha}\mathfrak{L}\{f(x)\}(s)-s^{\beta(\alpha-1)}\left(I_{a+}^{(1-\alpha)(1-\beta)}\frac{d}{dx}f(0+)\right). \tag{5}$$

Property 2. The convolution of a function $f(t)$ and $g(t)$ supported on only $[0,\infty)$ is defined by

$$(f*g)(t)=\int_0^t f(s)\,g(t-s)\,ds,$$

where $f,g:[0,\infty)\to\mathbb{R}$.

3. Main results

3.1. **Spectral Properties Hilfer Fractional Bessel Operator.**

In this section, firstly we shall present the modified Hilfer fractional derivative. We show spectral properties of modified Hilfer fractional derivative and give the representation of the solution via Laplace transforms.

Let $\alpha\in(0,1)$ and $\beta\in[0,1]$. Modified fractional Hilfer S–L operator for Bessel potential is given as

$$L_\alpha=D_{0+}^{\alpha,\beta}p(x)\,D_{1-}^{\alpha,\beta}+\left(q(x)-\frac{v^2-1/4}{x^2}\right),$$

where $D^{\alpha,\beta}$ is given by equations $(3-4)$.

Consider fractional S–L equation

$$L_\alpha y_\lambda(x) = \lambda y_\lambda(x) \tag{6}$$

where $p(x) \neq 0$, p is a real valued continuous function in interval $(0,1]$, v is a nonnegative integer and $q \in L_2(0,1)$.

The boundary conditions for the equation (6) is as follows:

$$y(0) = 0, \tag{7}$$

$$c_1 I_{0+}^{(1-\alpha)(1-\beta)} p(1)\, D_{1-}^{\alpha,\beta} y(1) + c_2 I_{1-}^{\beta(1-\alpha)} y(1) = 0, \tag{8}$$

where $c_1^2 + c_2^2 \neq 0$.

Now, we consider the properties of the operator L_α by means of the following theorems.

Theorem 3.1. *Modified fractional Hilfer Sturm-Liouville operator L_α for Bessel potential is self-adjoint on $(0,1]$.*

Proof. Considering the following equation, and also $L_\alpha \in L^2[0,1]$, we have

$$\langle L_\alpha \varphi, \phi \rangle = \int_0^1 L_\alpha \varphi(x)\, \phi(x)\, dx$$

$$= \int_0^1 \phi(x) \left[D_{0+}^{\alpha,\beta} p(x)\ D_{1-}^{\alpha,\beta} \varphi(x) + \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) \varphi(x) \right] dx$$

$$= \int_0^1 \phi(x)\, D_{0+}^{\alpha,\beta} p(x)\ D_{1-}^{\alpha,\beta} \varphi(x)\, dx + \int_0^\pi \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) \varphi(x)\, \phi(x)\, dx.$$

From the equation (2), boundary conditions (7),(8) and we get

$$\langle L_\alpha \varphi, \phi \rangle = -\int_0^1 p(x)\, D_{1-}^{\alpha,\beta} \varphi(x)\, D_{1-}^{\alpha,\beta} \phi(x)\, dx$$

$$+ I_{0+}^{(1-\alpha)(1-\beta)} p(x)\, D_{1-}^{\alpha,\beta} \varphi(x)\, I_{1-}^{\beta(1-\alpha)} \phi(x) \Big|_0^1 + \int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) \varphi(x)\, \phi(x)\, dx. \tag{9}$$

Similarly, we have

$$\langle \varphi, L_\alpha \phi \rangle = -\int_0^1 p(x)\ D_{1-}^{\alpha,\beta} \phi(x)\, D_{1-}^{\alpha,\beta} \varphi(x)\, dx$$

$$+ I_{0+}^{(1-\alpha)(1-\beta)} p(x)\, D_{1-}^{\alpha,\beta} \phi(x)\, I_{1-}^{\beta(1-\alpha)} \varphi(x) \Big|_0^1 + \int_0^1 \left(q(x) - \frac{v^2 - 1/4}{x^2} \right) \phi(x)\, \varphi(x)\, dx. \tag{10}$$

It can be easily seen that the right hand sides of the equations (9) and (10) are equal, therefore

$$\langle L_\alpha \varphi, \phi \rangle = \langle \varphi, L_\alpha \phi \rangle.$$

The proof is completed.

□

Theorem 3.2. *The eigenvalues of modified fractional Hilfer Sturm-Liouville problem with Bessel potential (6)-(8) are real.*

Proof. Firstly, assuming that λ eigenvalues are complex. If we use the self-adjointness of the operator L_α, then we have

$$\begin{aligned} \langle L_\alpha u, u\rangle &= \langle u, L_\alpha u\rangle \\ \langle \lambda u, u\rangle &= \langle u, \lambda u\rangle \\ \left(\lambda - \overline{\lambda}\right)\langle u, u\rangle &= 0. \end{aligned}$$

Since $\langle u, u\rangle \neq 0$,

$$\lambda = \overline{\lambda}$$

and hence λ eigenvalues are real.

□

Theorem 3.3. *The eigenfunctions corresponding to distinct eigenvalues of modified fractional Hilfer Sturm-Liouville problem with Bessel potential (6)-(8) are orthogonal with the weight function w_α on $(0,1]$,*

$$\int_0^1 w_\alpha(x)\, y_{\lambda_1}(x)\, y_{\lambda_2}(x)\, dx = 0, \qquad \lambda_1 \neq \lambda_2.$$

Proof. By assumptions modified fractional Hilfer Sturm-Liouville problem with Bessel potential having two different eigenvalues (λ_1, λ_2), we have

$$L_\alpha y_{\lambda_1}(x) = \lambda_1 w_\alpha(x)\, y_{\lambda_1}(x), \tag{11}$$

$$y_{\lambda_1}(0) = 0,$$

$$c_1 I_{0+}^{(1-\alpha)(1-\beta)} p(1)\, D_{1-}^{\alpha,\beta} y_{\lambda_1}(1) + c_2 I_{1-}^{\beta(1-\alpha)} y_{\lambda_1}(1) = 0,$$

$$L_\alpha y_{\lambda_2}(x) = \lambda_2 w_\alpha(x)\, y_{\lambda_2}(x), \tag{12}$$

$$y_{\lambda_2}(0) = 0,$$

$$c_1 I_{0+}^{(1-\alpha)(1-\beta)} p(1)\, D_{1-}^{\alpha,\beta} y_{\lambda_2}(1) + c_2 I_{1-}^{\beta(1-\alpha)} y_{\lambda_2}(1) = 0,$$

multiplying equation (11) and (12) by y_{λ_1}, y_{λ_2} respectively and subtracting from each other, we have

$$(\lambda_1 - \lambda_2)\, w_\alpha(x)\, y_{\lambda_1} y_{\lambda_2} = y_{\lambda_1} L_\alpha y_{\lambda_2} - y_{\lambda_2} L_\alpha y_{\lambda_1}.$$

Integrating from 0 to 1, and applying relation (2), we find that

$$(\lambda_1 - \lambda_2)\int_0^1 w_\alpha(x)\, y_{\lambda_1}(x)\, y_{\lambda_2}(x)\, dx = 0,$$

where $\lambda_1 \neq \lambda_2$.

□

Lemma 3.1. *Consider the singular fractional Hilfer Sturm-Liouville equation*

$$D_{0+}^{\alpha,\beta}\left(D_{0+}^{\alpha,\beta}y(x)\right)+\left(q(x)-\frac{v^2-1/4}{x^2}\right)y(x)=\lambda y. \tag{13}$$

where $q(x)\in L^2(0,1)$*. Then the representation of solution for* (13) *can be written in the form*

$$y(x)=a_1t^{(\alpha-1)(1-\beta)}E_{2\alpha,\alpha-\beta(\alpha-1)}\left(\lambda t^{2\alpha}\right)+$$

$$+a_2t^{2\alpha-1-\beta(\alpha-1)}E_{2\alpha,2\alpha-\beta(\alpha-1)}\left(\lambda t^{2\alpha}\right)$$

$$-\int_0^t(t-s)^{2\alpha-1}E_{2\alpha,2\alpha}\left(\lambda(t-s)^{2\alpha}\right)\left(q(s)-\frac{v^2-1/4}{s^2}\right)y(s)\,ds. \tag{14}$$

Proof. By taking Laplace transform on both sides of equation (13) and using equation (5), we have

$$\mathfrak{L}\left\{D_{0+}^{\alpha,\beta}\left(D_{0+}^{\alpha,\beta}y(x)\right)\right\}=$$

$$=\mathfrak{L}\left\{\left(\lambda-q(x)+\frac{v^2-1/4}{x^2}\right)y(x)\right\}\Rightarrow$$

$$s^{\alpha}\mathfrak{L}\left\{D_{0+}^{\alpha,\beta}y(x)\right\}-s^{\beta(\alpha-1)}\left[I_{0+}^{(1-\alpha)(1-\beta)}\frac{d}{dx}D_{0+}^{\alpha,\beta}y(0+)\right]$$

$$=\lambda\mathfrak{L}\{y(x)\}-\mathfrak{L}\left\{\left(q(x)-\frac{v^2-1/4}{x^2}\right)y(x)\right\}$$

$$s^{\alpha}\left[s^{\alpha}\mathfrak{L}\{f(x)\}(s)-s^{\beta(\alpha-1)}\left(I_{0+}^{(1-\alpha)(1-\beta)}\frac{d}{dx}y(0+)\right)\right]$$

$$+s^{\beta(\alpha-1)}\left[I_{0+}^{(1-\alpha)(1-\beta)}\frac{d}{dx}D_{0+}^{\alpha,\beta}y(0+)\right]$$

$$=\lambda\mathfrak{L}\{y(x)\}-\mathfrak{L}\left\{\left(q(x)-\frac{v^2-1/4}{x^2}\right)y(x)\right\}.$$

It can be readily obtained

$$\mathfrak{L}\{y(x)\}=\frac{s^{\alpha+\beta(\alpha-1)}a_1}{s^{2\alpha}-\lambda}+\frac{s^{\beta(\alpha-1)}a_2}{s^{2\alpha}-\lambda}-\frac{1}{s^{2\alpha}-\lambda}*\left(q(x)-\frac{v^2-1/4}{x^2}\right)y(x). \tag{15}$$

In here by taking inverse Laplace transform for the fractional equation on both sides of (15), we can obtain (14), where

$$a_1=I_{0+}^{(1-\alpha)(1-\beta)}\frac{d}{dx}y(0+)\ \text{ and }\ a_2=I_{0+}^{(1-\alpha)(1-\beta)}\frac{d}{dx}D_{0+}^{\alpha,\beta}y(0+),$$

where $*$ denotes the convolution symbol. □

4. Applications

4.1. Analysis of the Adomian Decomposition Method.

Let's consider the equation (6) for the applicability of the method

$$D_{0+}^{\alpha,\beta}\left(p(x)\frac{d}{dx}y(x)\right)+\left(q(x)-\frac{v^2-1/4}{x^2}-\lambda\right)y(x)=0,\ x\in[0,\pi]. \tag{16}$$

We may rewrite equation (16) for convenience in the form

$$Ty(x)=N$$

where $T = D_{0+}^{\alpha,\beta} p(x) D^1$ is specified as an inverse operator and $N = N(\lambda, x, y, y')$ is a linear operator involves all other terms. Here $D^1 = \frac{d}{dx}$. The inverse of the operator T has the following form

$$T^{-1} = \int_0^x \frac{1}{p(t)} I_{0+}^{\alpha}(.)\,dt.$$

Applying T^{-1} on the left side of equation (16), then we get

$$\begin{aligned}
\left(T^{-1}T\right)(y(x)) &= \int_0^x \frac{1}{p(t)} I_{0+}^{\alpha}\left(D_{0+}^{\alpha,\beta}\left(p(t)\frac{d}{dt}y(t)\right)\right)dt, \\
&= \int_0^x \frac{1}{p(t)}\left(p(t)\frac{d}{dt}y(t) - y_{\alpha,\beta}(t)\right)dt, \\
&= y(x) - y(0) - \int_0^x \frac{y_{\alpha,\beta}(t)}{p(t)}dt,
\end{aligned}$$

where the initial conditions $y(0)$ and $y_{\alpha,\beta}(x) = \frac{x^{-(1-\alpha)(1-\beta)}}{\Gamma(\alpha+\beta-\alpha\beta)} I_{0+}^{(1-\alpha)(1-\beta)} p(0)\frac{d}{dx}y(0)$ should be known.

Hence, applying T^{-1} to both side of the equation (16), then we get

$$y(x) = y(0) + \int_0^x \frac{y_{\alpha,\beta}(t)}{p(t)}dt + T^{-1}N\left(\lambda, x, y, y'\right)$$

$$= y(0) + \int_0^x \frac{y_{\alpha,\beta}(t)}{p(t)}dt + \int_0^x \frac{1}{p(t)} I_{0+}^{\alpha}\left[N\left(\lambda, t, y, y'\right)\right]dt. \tag{17}$$

The Adomian's decomposition method presumes the solution function $y(x)$ can be represented by the following series,

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{18}$$

and the term N is defined by an infinite series of Adomian polynomials

$$N = \sum_{n=0}^{\infty} A_n, \tag{19}$$

where A_n are the Adomian polynomials as following

$$A_n = \frac{1}{n!}\frac{d^n}{d\mu^n}\left[N\left(\sum_{i=0}^{\infty}\mu^i y_i(x)\right)\right]_{\mu=0}.$$

Considering $(17), (18)$ and (19), we have

$$\sum_{n=0}^{\infty} y_n(x) = y(0) + \int_0^x \frac{y_{\alpha,\beta}(t)}{p(t)}dt + \sum_{n=0}^{\infty} T^{-1}\left(A_n(x)\right).$$

Consequently, we may give the recursive relation as following

$$\begin{aligned} y_0(x) &= y(0) + \int_0^x \frac{y_{\alpha,\beta}(t)}{p(t)} dt, \\ y_{n+1}(x) &= T^{-1}(A_n(x)), \ n \geq 0, \end{aligned}$$

where

$$A_n(x) = \left(-q(x) + \frac{v^2 - 1/4}{x^2} + \lambda\right) y_n(x).$$

Now, we give some numerical assessments by means of this method.

4.2. **Numerical Results.**

Let's consider the fractional eigenvalue problem

$$D_{0+}^{\alpha,\beta}(y'(x)) + \left(q(x) - \frac{v^2 - 1/4}{x^2} - \lambda\right) y(x) = 0, \tag{20}$$

subject to

$$y(0) = 0, \ I_{0+}^{\beta} y'(0) = 0. \tag{21}$$

Equation (20) can be written as the following closed form,

$$Ty(x) + \left(q(x) - \frac{v^2 - 1/4}{x^2} - \lambda\right) y(x) = 0.$$

The inverse operator of T has the form

$$T^{-1} = \int_0^x \frac{1}{p(t)} I_{0+}^{\alpha}(.)\, dt.$$

Applying T^{-1} to equation (20) and using initial condition at $x = 0$, we have

$$\begin{aligned} I_{0+}^{\alpha} D_{0+}^{\alpha,\beta} y'(x) &= I_{0+}^{\alpha}\left[\left(-q(x) + \frac{v^2 - 1/4}{x^2} + \lambda\right) y(x)\right] \\ y'(x) - y_{\alpha,\beta} &= -I_{0+}^{\alpha}\left[\left(-q(x) + \frac{v^2 - 1/4}{x^2} + \lambda\right) y(x)\right] \\ y(x) - y(0) &= \int_0^x y_{\alpha,\beta}(t)\, dt + \int_0^x I_{0+}^{\alpha}\left[\left(-q(t) + \frac{v^2 - 1/4}{t^2} + \lambda\right) y(t)\right] dt \\ y(x) &= y(0) + \int_0^x y_{\alpha,\beta}(t)\, dt + \int_0^x I_{0+}^{\alpha}\left[\left(-q(t) + \frac{v^2 - 1/4}{t^2} + \lambda\right) y(t)\right] dt \end{aligned}$$

where

$$y_{\alpha,\beta}(t) = \frac{t^{-(1-\beta)(1-\alpha)}}{\Gamma(\alpha + \beta - \alpha\beta)} I_{0+}^{(1-\beta)(1-\alpha)} y'(0).$$

Applying decomposition series and performing necessary operations, we have,

$$
\begin{aligned}
y_0(x) &= y(0) + \frac{1}{\Gamma(\alpha+\beta-\alpha\beta)} \int_0^x t^{-(1-\beta)(1-\alpha)} I_{0+}^{(1-\beta)(1-\alpha)} y'(0)\, dt \\
y_{n+1}(x) &= \int_0^x I_{0+}^{\alpha} \left[\left(-q(t) + \frac{v^2 - 1/4}{t^2} + \lambda \right) y_n(t) \right] dt, \quad n \geq 0.
\end{aligned}
$$

Let's give the numerical results for the eigenfunctions of the problem $(20)-(21)$ with graphics under different orders, different types and different potentials by using the ADM method for $n = 15$ iterations.

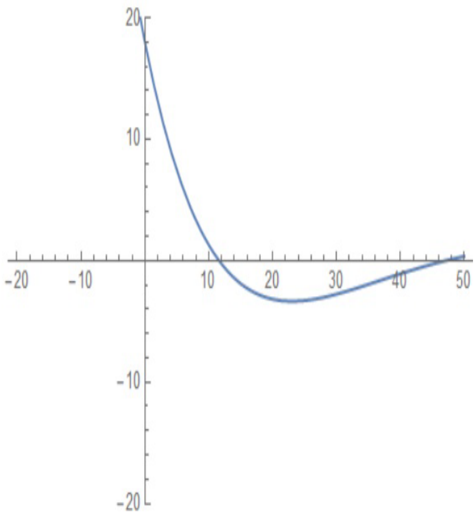

Figure 1. $\alpha = 0.4, \beta = 0.6, v = 0, q = 0, x = \frac{\pi}{2}$.

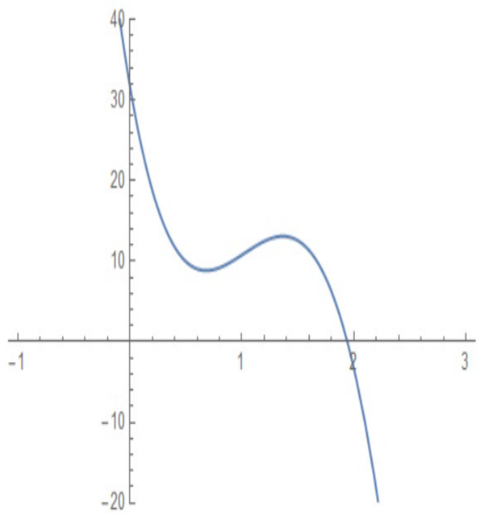

Figure 2. $\alpha = 0.3, \beta = 0.6, v = 0, q = 0, x = \frac{\pi}{2}$.

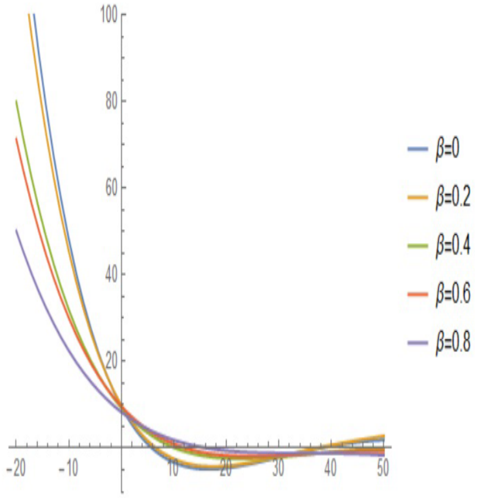

Figure 3. $\alpha = 0.4, \beta = 0.6, v = 0, q = 0, x = \frac{\pi}{2}$.

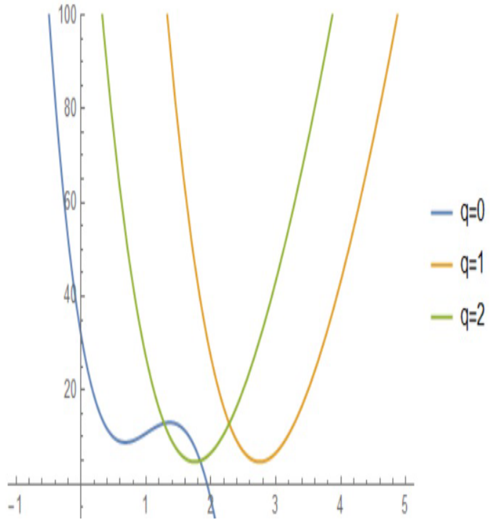

Figure 4. $\alpha = 0.4, \beta = 0.6, v = 0, x = \frac{\pi}{2}$.

5. Conclusion

Consequently, we define modified fractional Hilfer Sturm-Liouville problem with Bessel potential, and then we show the self-adjointness of the operator, orthogonality of distinct eigenfunctions and reality of eigenvalues. Also, we give the Adomian decomposition method for finding approximations of eigenvalues and eigenfunctions. We compare these results under different values.

We observe the approximations to the eigenvalues by changing values of α and β . Also, we observe the behavior of eigenfunctions by changing the type of the derivative β in $Fig.3$, the results are rather interesting and in accordance with the originality of the Hilfer fractional derivative for obtaining more accurate numerical results. We observe the behavior of eigenfunctions under different orders of the derivative α in $Fig.1$, $Fig.2$ and $Fig.4$.

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