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INTRODUCING A NEW APPROACH TO SOLVE NONLINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this paper, we present a new class of orthogonal basis functions (NOBFs) for solving the nonlinear Volterra-Fredholm integral equations of the second kind (NVFIE2). To do this, first, we construct the operational matrix of integration. Then, by using the proposed method, we reduce the original problem to a nonlinear system of algebraic equations. Finally, to show the applicability of this method, we give some numerical examples.

Keywords: block-pulse functions, Volterra-Fredholm integral equations, integral operational matrix, introducing new basis.

AMS Subject Classification: 83-02, 99A00.

1. Introduction

Volterra-Fredholm integral equations (VFIE) occur in many scientific applications such as the population dynamics, spread of epidemics, and semi-conductor devices [14]. In the literature, many methods have been used for solving these equations with sufficient accuracy and efficiency [7, 8, 9, 12, 17]. Molbahrami in [10] suggested direct computation method for solving a general nonlinear Fredholm integro-differential equation under the mixed conditions. Babolian et.al. [1] have applied Haar wavelets and collocation method to solve NVFIE2. Maleknejad et.al. [5] have suggested a computational method for system of Volterra-Fredholm integral equations. Applying new basis functions for solving VFIE have introduced by Paripour et.al. in [11]. In [6], Maleknejad et.al. proposed a new computational method to obtain the numerical solution of NVFIE. Kauthen in [4] used continuous time collocation method for solving VFIE. Ezzati et.al. applied Chebyshev polynomials for solving NVFIE in [2]. In[15], Yalsinbas developed numerical method for solving NVFIE by using Taylor polynomials. In [16], the authors have been used Legendre wavelets to solve VFIE. In [13], by using Chebyshev wavelets operational matrix, the authors proposed a new method for solving integral equations.

In this paper, first, we introduce a new class of orthogonal basis functions. Then, we propose a new approach based on the operational matrix of integration of these basis to solve NVFIE2

$$f(x) = g(x) + \lambda_1 \int_0^x K_1(x,t)[f(x)]^{n_1} dt + \lambda_2 \int_0^1 K_2(x,t)[f(x)]^{n_2} dt, \quad (1)$$

where the function $g \in L^2[0,1]$, the kernels $K_1, K_2 \in L^2([0,1]\times[0,1])$ are known functions, and f is the unknown function must be determined and n_1, n_2 are positive integers. Clearly, the

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operational matrix of integration, P, is as follows:

$$\int_0^t \Psi(s)ds \simeq P\Psi(t),$$

where $\Psi(t) = [\psi_1(t), \psi_2(t), ..., \psi_n(t)]$ and the matrix $P_{n\times n}$ can be determined on the basis of the new class of orthogonal functions uniquely. The set $\psi_1(t), \psi_2(t), ..., \psi_n(t)$ are the orthogonal basis functions, on the interval [0,1]. Clearly, by expanding f and known function in Eq.(1) according to new functions, Eq.(1) reduces to a system of algebraic equations.

The remainder of this article has been organized as follows: In Section 2, we will review the fundamental concepts from Block pulse functions. In Section 3, we will introduce a new class of orthogonal basis functions. In Section 4, we drive the operational matrix of integration of the new basis functions. In Section 5, we present function approximation by using these orthogonal basis functions. The efficiency of the proposed method is shown in Section 6 by some numerical examples. Finally, Section 7 gives our concluding remarks.

2. Preliminaries

Definition 2.1. *Block-pulse functions (BPFs), $\phi_i(t), i = 1, 2, ..., n$, on the interval $[0, 1)$ are defined as* [3]*:*

$$\psi_i(t) = \begin{cases} 1, & ih \le t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $i = 1, 2, 3, ..., n$ is an arbitrary positive integer number and $h = \frac{1}{n}$.

The basic properties of BPFs are as follows:
(1) The BPFs are disjointed with each other in $[0, 1)$, i.e

$$\psi_i(t)\psi_j(t) = \delta_{ij}\psi_i(t),$$

where i = 1, 2, . . . ,n and j = 1, 2, . . . ,n and δ_{ij} is Kronicker delta.
(2) The BPFs are orthogonal function in the interval $t \in [0, 1)$, i.e.:

$$\int_0^1 \psi_i(t)\psi_j(t)dt = 0 \quad fori \neq j \quad and \quad i, j = 1, 2, 3, ..., n.$$

(3) If $n \to \infty$, then the BPFs functions are complete; i.e. for every $g \in L^2([0, 1))$,the identity of Parseval holds,

$$\int_0^T g^2(t)dt = \sum_{i=1}^{\infty} g_i^2 \parallel \psi_i(t) \parallel^2,$$

where

$$g_i = \frac{1}{h}\int_0^T g(t)\psi_i(t)dt. \tag{3}$$

We can write the first n terms of BPFs as n-vector form:

$$\Psi(t) = (\psi_1(t), \psi_2(t), ..., \psi_n(t))^T, \quad t \in [0, T). \tag{4}$$

Eq. (4) and disjointness property shows that

$$\Psi(t)\Psi^T(t) = \begin{bmatrix} \psi_1(t) & 0 & \dots & 0 \\ 0 & \psi_2(t) & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & \psi_n(t) \end{bmatrix}_{n\times n} .$$

Furthermore, we have

$$\Psi(t)^T\Psi(t) = 1, \tag{5}$$

$$\Psi(t)\Psi^T(t)G = D_G\Psi(t), \tag{6}$$

where D_G is diagonal matrix with a constant vector in diagonal entries $G = (g_1, g_2, ..., g_n)^T$, and

$$\Psi(t)^T B\Psi(^t) = \tilde{B}\Phi(t), \tag{7}$$

where elements of the diagonal entries of matrix B in an n-vector form is called $\tilde{B}$.

3. Introducing new class of basis functions (NBFs)

Here, we introduce a new class of basis functions (NBFs), and also we show some of their properties.

Definition 2.2. *The m-collection of new basis functions over interval [0,1), the ith left and right functions are introduced as:*

$$\psi 1_i(x) = \begin{cases} \frac{(i+1)^3-(\frac{x}{h})^3}{3i^2+3i+1}, & ih \le x < (i+1)h, \\ 0, & otherwise, \end{cases} \tag{8}$$

$$\psi 2_i(x) = \begin{cases} \frac{(\frac{x}{h})^3-i^3}{3i^2+3i+1}, & ih \le x < (i+1)h, \\ 0, & otherwise, \end{cases} \tag{9}$$

where $i = 0, 2, 3, ..., n-1, h = \frac{1}{n}$, and $\psi 1_i(x)$ and $\psi 2_i(x)$ are the terms ith of $\psi 1(x)$ and $\psi 2(x)$, respectively, and NBFs are defined over [0,1). We have:

$$\psi 1(x) = [\psi 1_0(x), \psi 1_1(x), ..., \psi 1_{n-1}(x)]^T, \psi 2(x) = [\psi 2_0(x), \psi 2_1(x), ..., \psi 2_{n-1}(x)]^T, \tag{10}$$

and

$$\psi(x) = [\psi 1(x), \psi 2(x)]^T. \tag{11}$$

By using Eqs.(8) and (9), we conclude that

$$\psi 1_i(x) + \psi 2_i(x) = \psi_i(x),$$

where $\psi_j(x)$ is the jth BPFs and it is defined in Eq. (2). So, we conclude that

$$\sum_{i=0}^{n-1} \psi 1_i(t) + \sum_{i=0}^{n-1} \psi 2_i(t) = 1,\ 0 \le t < 1.$$

It is obvious that $\psi 1_i(t)$ and $\psi 2_j(t)$, $i, j = 0, 1, 2, ..., n$ are disjoint. The orthogonality of NBFs can be derived immediately from

$$\int_0^1 \psi 1i(x)\psi 1_j(x)dx \simeq \frac{5}{13}h\delta_{i,j}, \int_0^1 \psi 1_i(x)\psi 2_j(x)dx = 0, \tag{12}$$

$$\int_0^1 \psi 2_i(x)\psi 2_j(x)dx \simeq \frac{4}{13}\delta_{i,j}, \int_0^1 \psi 2_i(x)\psi 1_j(x)dx = 0, \tag{13}$$

where $i = 0, 1, 2, ..., n-1, j = 0, 1, 2, , n-1$ and $\delta_{i,j}$ denotes the Kronecker delta function. From definition (3.1), we have :

$$\psi 1_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\psi 2_i(jh) = \begin{cases} 1, & i = j-1, \\ 0, & i \neq j-1, \end{cases}$$

where $i = 0, 1, 2, ..., n-1$ and $j = 0, 1, 2, ..., n-1$.
Consider the first n terms of NBFs in Eq.(10). According to disjointness property of NBFs, we have:

$$\psi 1(x)\psi 1^T(x) \simeq \begin{bmatrix} \psi 1_0(x) & 0 & 0 & ... & 0 \\ 0 & \psi 1_1(x) & 0 & ... & 0 \\ 0 & 0 & \psi 1_2(x) & ... & 0 \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ 0 & 0 & 0 & ... & \psi 1_{n-1}(x) \end{bmatrix}_{n\times n} = diag(\psi 1(x)),$$

$$\psi 1(x)\psi 2^T(x) = 0_{n\times n}.$$

Similarly, we have:

$$\psi 2(x)\psi 2^T(x) \simeq diag(\psi 2(x))$$

$$\psi 2(x)\psi 1^T(x) = 0_{n\times n}.$$

Therefore

$$\psi(x)\psi^T(x) \simeq \begin{bmatrix} diag(\psi 1(x)) & 0 \\ 0 & diag(\psi 2(x)) \end{bmatrix}_{2n\times 2n}.$$

$$\psi(x)\psi^T(x)W \simeq \tilde{W}\psi(x), \tag{14}$$

where W is a $2n$-vector and $\tilde{W} = diag(W)$. Besides, it can be concluded that for an $2n \times 2n$ matrix A, we have:

$$\psi^T(x).A.\psi(x) \simeq \hat{A}^T.\psi(x), \tag{15}$$

where $\hat{A}$ is an $2n$-vector such that its elements are diagonal entries of matrix A.

4. Operational matrix of integration

From orthogonality of $\psi 1(x)$ and $\psi 2(x)$ defined in Eqs. (12) and (13), we obtain

$$\int_0^1 \psi 1_i(x)\psi 2_j(x)dx \simeq \int_0^1 \psi 2_i(x)\psi 1_j(x)dx \simeq \frac{2}{13}h\delta_{i,j}, \tag{16}$$

$$D = \int_0^1 \psi(x)\psi^T(x)dx \simeq \tfrac{h}{13}\begin{bmatrix} 5I_{n\times n} & 2I_{n\times n} \\ 2I_{n\times n} & 4I_{n\times n}\end{bmatrix}_{2n\times 2n}.$$

Also, we have:

$$\int_{ih}^{(i+1)h} \psi 1_i(t)dt \simeq \frac{4}{7}h,$$

$$\int_{ih}^{(i+1)h} \psi 2_i(t)dt \simeq \frac{3}{7}h,$$

$$\int_0^t \psi 1_i(r)dr \simeq \begin{cases} 0, & 0 \le t < ih, \\ \frac{1}{2}(\frac{6}{7})h, & ih \le t < (i+1)h, \\ \frac{4}{7}h, & (i+1)h \le t < 1. \end{cases}$$

Therefore

$$\int_0^t \psi 1_i(r)dr \simeq \frac{h}{14}[0, ..., 0, 6, 8, ..., 8, 0, ..., 0, 8, ..., 8].\psi(t),$$

and similarly

$$\int_0^t \psi 2_i(r)dr \simeq \frac{h}{14}[0, ..., 0, 3, 4, ..., 4, 0, ..., 0, 4, ..., 4].\psi(t).$$

So, we have:

$$\int_0^t \psi(r)dr \simeq P.\psi(t), \tag{17}$$

where $P_{2n\times 2n}$ is matrix of operational integration and given by

$$P = \tfrac{h}{14}\begin{bmatrix} 6 & 8 & ... & 8 & 0 & 8 & ... & 8 \\ 0 & 6 & ... & . & 0 & . & ... & \\ 0 & 0 & ... & 8 & 0 & . & ... & 8 \\ . & 0 & ... & 6 & 0 & . & ... & 0 \\ 3 & 4 & ... & 4 & 0 & 4 & ... & 4 \\ 0 & 3 & ... & 4 & 0 & & ... & 4 \\ 0 & 0 & ... & 4 & 0 & 0 & ... & 4 \\ 0 & 0 & ... & 3 & 0 & 0 & ... & 0 \end{bmatrix}_{2n\times 2n}.$$

5. Function approximation

We can expand an arbitrary real and bounded function $g(x) \in L^2[0,1)$ as:

$$g(x) \simeq \sum_{i=0}^{n-1} G1_i \psi1_i(x) + \sum_{i=0}^{n-1} G2_i \psi2_i(x) = G1^T \psi1(x) + G2^T \psi2(x) = G^T \psi(x), \tag{18}$$

where G is a $2n$-vector given by $G = [G1^T, G2^T]$, and $\psi1(x)$ and $\psi2(x)$ are defined in Eq.(10). The coefficients in $G1$ and $G2$ can be computed by the function $g(x)$ at points ih and $(i+1)h$ for arbitrary h and i. We can substitute

$$G1_i = g(ih),$$
$$G2_i = g((i+1)h).$$

Lemma 5.1. *Assume that $g(x) \in L^2[0,1)$ can be approximated by using NBFs as:*

$$g(x) \simeq \sum_{i=0}^{n-1} G1_i \psi1_i(x),$$

then $[g(x)]^p$, $p \in N$ can be approximated as:

$$[g(x)]^p \simeq \sum_{i=0}^{n-1} {G1^p}_i \psi1_i(x),$$

Proof. The proof is similar to the proof of [11] □

Now, let $k(x;t)$ be an arbitrary function of two variables defined on $L^2([0;1) \times [0;1))$. Clearly it can be expanded by NBFs as the following form:

$$k(x;t) \simeq \psi^T(x) k \psi(t),$$

where $\psi(x)$ and $\psi(t)$ are NBFs vectors with $h = \frac{1}{n}$. Also k is the $2n \times 2n$ coefficients matrix as follows:

$$K = \begin{bmatrix} k11 & k12 \\ k21 & k22 \end{bmatrix},$$

where

$$[k11]_{nm} = K(nh, mh),$$
$$[k12]_{nm} = K(nh, (m+1)h),$$
$$[k21]_{nm} = K((n+1)h, mh),$$
$$[k22]_{nm} = K((n+1)h, (m+1)h).$$

Now, consider the nonlinear Volterra-Fredholm integral equations:

$$f(x) = g(x) + \lambda_1 \int_0^x K_1(x,t)[f(x)]^{n_1} dt + \lambda_2 \int_0^1 K_2(x,t)[f(x)]^{n_2} dt, \ 0 \le x < 1, \tag{19}$$

where $g(x) \in L^2[0,1)$, $K(x,t) \in L^2([0,1) \times [0,1))$ are known functions, $f(x)$ is the unknown function and n_1, n_2 are positive integers. By approximating functions $g(x)$, $K(x,t)$, $[f(x)]^{n_1}$, $[f(x)]^{n_2}$ in the matrix form, we have:

$$g(x) \simeq G^T \psi(x), \tag{20}$$
$$K_1(x,t) \simeq \psi^T(x) K_1 \psi(t), \tag{21}$$
$$K_2(x,t) \simeq \psi^T(x) K_2 \psi(t), \tag{22}$$
$$f(x) \simeq F^T \psi(x), \tag{23}$$

$$[f(x)]^{n_1} \simeq F_{n_1}^T \psi(x), \tag{24}$$

$$[f(x)]^{n_2} \simeq F_{n_2}^T \psi(x), \tag{25}$$

where F_{n_1} and F_{n_2} are column vector such that elements are n_1, n_2th powers of the elements of the vector F respectively and T denotes transpose of the vector. By substituting Eqs. (20)-(25) into Eq. (19), we obtain:

$$F^T\psi(x) = G^T\psi(x) + \lambda_1 \int_0^x \psi^T(x)K_1\psi(t)\psi^T(t)F_{n_1}dt + \lambda_2 \int_0^1 \psi^T(x)K_2\psi(t)\psi^T(t)F_{n_2}dt$$

$$F^T\psi(x) = F^T\psi(x) + \lambda_1\psi^T(x)K_1 \int_0^x \psi(t)\psi^T(t)F_{n_1}dt + \lambda_2\psi^T(x)K_2 \int_0^1 \psi(t)\psi^T(t)F_{n_2}dt,$$

by using Eqs. (14) and (12) we have:

$$F^T\psi(x) = G^T\psi(x) + \lambda_1\psi^T(x)K \int_0^x \tilde{F}_{n_1}\psi(t)dt + \lambda_2\psi^T(x)K_2DF_{n_2}.$$

Also, by using Eq (17), we have:

$$F^T\psi(x) = G^T\psi(x) + \lambda_1\psi^T(x)K\tilde{F}_{n_1}P\psi(x) + \lambda_2\psi^T(x)K_2DF_{n_2}.$$

By assuming $B = K\tilde{U}_{n_1}P$, and using Eq. (15), we will get:

$$F^T\psi(x) = G^T\psi(x) + \lambda_1\tilde{B}^T\psi(x) + \lambda_2(K_2DF_{n_2})^T\psi(x).$$

so

$$F = G + \lambda_1\tilde{B} + \lambda_2K_2DF_{n_2}.$$

This equation is a system of algebraic nonlinear equations. Clearly, we can solve this equation by using known methods as Newtons iteration method.

6. Numerical examples

Here, we apply the proposed method in Section 5 to solve some nonlinear VFIE of the second kind. To show the efficiency of the presented method, we compare the numerical results with the exact solution.

Example 6.1. [11] *Consider the following NFIE of the second kind*:

$$u(x) = -\frac{1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x^4 - \frac{5}{4} + \int_0^x (x-t)u^2(t)dt + \int_0^1 (x+t)u(t)dt, \tag{26}$$

with the exact solution $u(x) = x^2 - 2, 0 \leq x < 1.$

To compare the numerical results with the exact solution, see Table 1 and Figure 1 .

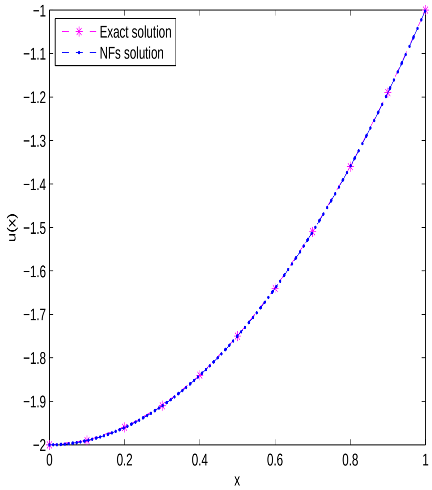

Figure 1. Exact solution and approximate solution for Example 6.1, m=32.

Table 1. The numerical results of Example 6.1 for m=32.

x_i	*Exactsolution*	*Aproximate solution*	*Absolute error*
0.1	-1.99000000	-1.99014527	0.00014527
0.2	-1.96000000	-1.96023105	0.00023105
0.3	-1.91000000	-1.91023558	0.00023558
0.4	-1.84000000	-1.84015858	0.0001585
0.5	-1.75000000	-1.75000000	0
0.6	-1.64000000	-1.64015457	0.00015457
0.7	-1.51000000	-1.51023360	0.00023360
0.8	-1.36000000	-1.36023492	0.00023492
0.9	-1.19000000	-1.19015731	0.00015731

Example 6.2. [11] *Consider NVIE of the second kind*

$$f(t) = sin(\pi t) + \frac{1}{5}\int\limits_0^t cos(\pi t)sin(\pi u)f^3(u)du, \tag{27}$$

with the exact solution $f(t) = sin(\pi t) + \frac{20-\sqrt{391}}{3}cos(\pi t),\ 0 \le t < 1.$

In Figure 2, we show the numerical results. Also, in Table 2, we compared the numerical solution obtained by proposed method in Section 5 and the method of [11].

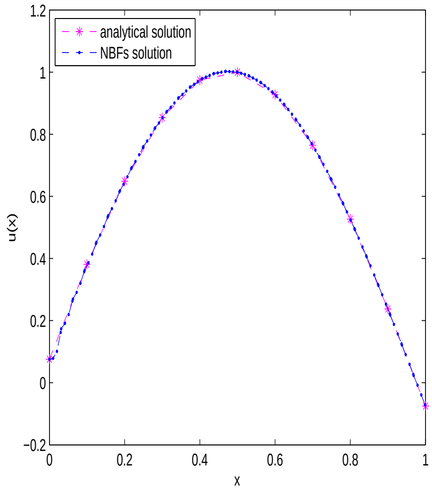

Figure 2. A comparison between exact and approximate solutions for Example 6.2, m=32.

Table 2. The approximate results of Example 6.2 with m=32.

x_i	Exact solution	Aproximate solution	Absolute error	The method of [11]
0	0.07542668	0.07531365	0.00011302	0.090008
0.1	0.38075203	0.37637141	0.00438062	0.30670
0.2	0.64880672	0.64523101	0.00357571	0.58640
0.3	0.85335168	0.85097223	0.00237944	0.81029
0.4	0.97436464	0.97324575	0.00111888	0.95475
0.5	1.00000000	1.00000000	0	0.10000
0.6	0.92774838	0.92740060	0.00034778	0.93312
0.7	0.76468229	0.76455509	0.00012720	0.76912
0.8	0.52676377	0.52702376	0.00025999	0.52969
0.9	0.23728195	0.23772625	0.00044430	0.24007

Example 6.3. [11] *In this example, we consider NFIE of the second kind*

$$f(t) = e^{t+1} - \int_0^1 e^{t-2u} f^3(u)du, \tag{28}$$

with the exact solution $f(t) = e^t, \ 0 \leq t < 1$.

The numerical results are shown in Figure 3. In Table 3, we compare the numerical results obtained from the proposed method and the method of [11].

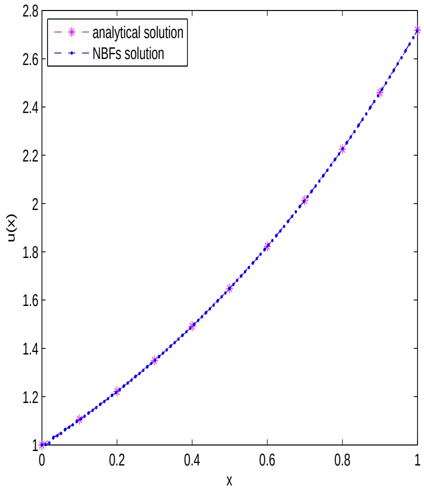

Figure 3. A comparison between exact and approximate solutions for Example 6.3, m=32.

Table 3. The approximate results for Example 6.3 with m=32.

x_i	Exact solution	Aproximate solution	Absolute error	The method of [11]
0	1.00	1.00006105	0.00006105	1.00121629
0.1	1.10517091	1.10378743	0.00138348	1.10580428
0.2	1.2214027	1.22021692	0.00118583	1.22232420
0.3	1.34985880	1.34903388	0.00082492	1.35112684
0.4	1.49182469	1.49143597	0.00038872	1.49345921
0.5	1.64872127	1.64882193	0.00010065	1.65072661
0.6	1.8221188	1.82190416	0.00021463	1.82424236
0.7	2.01375270	2.01343983	0.00031286	2.01610170
0.8	2.22554092	2.22528372	0.00025720	2.22818176
0.9	2.45960311	2.45951527	0.00008784	2.46257191

Example 6.4. [14] *Consider the following integral equation*

$$f(t) = e^t - \frac{t}{192}(e^2+1) + \int_0^1 tuf^2(u)du, \tag{29}$$

with the exact solution $f(t) = e^t,\ 0 \leq t < 1.$

To compare the approximate results with the exact solution, see Table 4 and Figure 4.

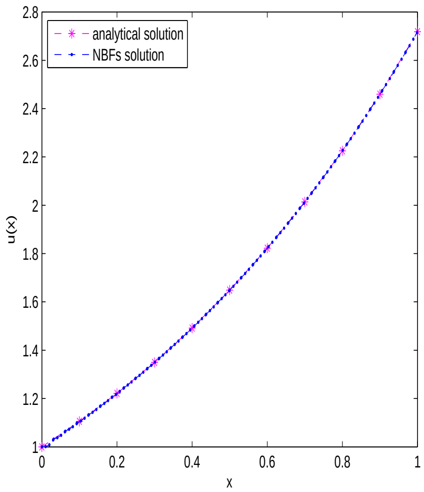

Figure 4. A comparison between exact and approximate solutions for Example 6.4, m=32.

Table 4. The approximate results for Example 6.4 with m=32.

x_i	*Exact*	*Aproximate*	*Absolute error*
0	1.00	1	0
0.1	1.10517091	1.10370285	0.00146806
0.2	1.2214027	1.22010776	0.00129499
0.3	1.34985880	1.34889936	0.00095944
0.4	1.49182469	1.49127526	0.00054943
0.5	1.64872127	1.64863410	0.00008716
0.6	1.8221188	1.82168838	0.00043041
0.7	2.01375270	2.01319494	0.00055776
0.8	2.22554092	2.22500857	0.00053247
0.9	2.45960311	2.45920825	0.00039486

7. Conclusions (mandatory)

In this article, we introduced a new class of orthogonal basis functions to solve nonlinear VFIE of the second kind. This method is reduces a given nonlinear VFIE of the second kind to a system of nonlinear algebraic equations with the sparse coefficient matrix. The efficiency and simplicity of this method is illustrated by solving some numerical examples with known exact solutions.

The advantage of this method are low cost of setting up the equations without applying any projection method such as Galerkin, collocation, etc. Also, the system equation of this method is a lower triangular system. Therefore the count of operations is very low. Finally, this method can be extended and applied to systems of Volterra-Fereholm integral equations of the second kind. Also the property in lemma 5.1 is the other advantage.

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