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ON REMARKABLE RELATIONS AND THE PASSAGE TO THE LIMIT IN THE THEORY OF INFINITE SYSTEMS II*

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Abstract. We have found solutions of homogeneous finite Gaussian system of linear algebraic equations with a degenerate matrix. These solutions are given by functional relations that allow us to pass to the limit from finite truncated systems to an infinite system of linear algebraic equations. We give a concept of a fundamental solution of a homogeneous infinite system. This fundamental solution is given by Fedorov's formula. Fedorov's formula expresses the solution using Fedorov's determinant, which is similar to Cramer's determinant for inhomogeneous systems.

Keywords: infinite system, homogeneous, inhomogeneous, method of reduction, strictly particular solution, Cramer's rule, nontrivial solution, principal fundamental solution.

AMS Subject Classification: 15A06, 15A15.

1. Introduction

This article is a continuation to the earlier article [3]. It focuses on the main and the most difficult issue, namely, the problem of the passage to the limit from a finite systems solution to an infinite systems solution. In [3] we mainly investigated an inhomogeneous system, in this article we investigate a homogeneous system. The existence of nontrivial solutions of homogeneous systems plays a special role for infinite systems because the uniqueness of its solution depends on its existence. Nontrivial solutions of the infinite system may exist even when an infinite determinant of this system does not equal to zero. This fact is a basic difference between infinite and finite systems. To see how far we advanced the theory of infinite systems, one may read the article [4]. Basic information, concepts and definitions of infinite systems, matrices and determinants can be studied in the articles [1, 5, 6, 8, 9]. The new concepts will be presented in this article as needed.

Apparently, I.P. Natanson [10] first built a sample of a nontrivial solution for homogeneous infinite systems with a nonzero determinant when he replied to a question from professor R.O. Kuzmin. For example, V.S. Rogozhin [11] built a sample of a finite-dimensional space of nontrivial solutions for a homogeneous system with difference indices (index of coefficients of equations is the difference between i and j). We have discovered and described in detail a new class of infinite systems, called periodic infinite systems [5]. It allows us to build an infinite space of nontrivial solutions for a homogeneous periodic infinite system [5, 6]. There is an example of

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such space. Let us consider following infinite system (examples of this kind can be found in the articles [5, 6]):

$$\sum_{p=0}^{\infty} \frac{(2j+2p)!}{(2p)!} x_{j+p} = b^j,\ \ j = \overline{0,\infty}, \quad b = const > 0. \tag{1}$$

The infinite system (1) is a so-called Gaussian system, i.e. all elements below the main diagonal in the coefficient matrix of system (1) are equal to zero meanwhile all elements of the main diagonal are not equal to zero ($a_{j,j} = (2j)!$). It is clear that each series of the elements at the corresponding row of the coefficient matrix diverges and the constant terms b^j are generally unlimited if $b > 1$.

Therefore, it is impossible to solve the system (1) in some normed space.

In fact, there is no method so far that could solve this system: neither approximately nor exactly. Moreover, these systems a priory are excluded from study. However, there is a huge number of these systems having practical applications. For example, the system (1) emerged as a result of solving a non-stationary thermal problem with time-variable boundary conditions [5, 6]. Nevertheless, this system has a solution, moreover, we found a particular solution of the inhomogeneous system (1) and a fundamental solution of the corresponding homogeneous system (1) [5, 6] (hence practically we found its general solution):

$$x_i^{(k)} = \frac{b^i}{(2i)!\mathrm{ch}(\sqrt{\mathrm{b}})} + \frac{(-1)^i \pi^{2i} (2k+1)^{2i} x_0^{(k)}}{(2i)! 2^{2i}}, \quad i, k = 0, 1, 2, ..., \forall x_0^{(k)}. \tag{2}$$

The first part of the solution (2) is a so-called **strictly particular solution** and it is an exact solution. The second part is a **nontrivial solution** (when $x_0^{(k)} \neq 0$) of the corresponding homogeneous system (1) and it is also an exact solution. We obtained the first part of the solution (2) with the reduction method in the narrow sense (simple reduction), i.e. a number of unknowns is equal to a number of equations in the truncated system. The second part of the solution (2) is obtained with the reduction method in the broad sense, i.e. the number of unknowns is greater than the number of equations (i.e. each truncated system of n-th order has a degenerate matrix, although the matrix of the infinite system (1) is nondegenerate). It is clear that a subspace of solutions for the corresponding homogeneous system is infinite because the nontrivial solution of a homogeneous system does not depend on k [6], i.e. it is a fundamental. We may get an analytically exact solution only if we calculated an exact solution of the truncated system of n-th order for each n. This is possible only when the system (1) is a Gaussian system. It should be noted that one can obtain the strictly particular solution only using the reduction method in the narrow sense (simple reduction). The nontrivial solution of homogeneous system could be got only using the reduction method in the broad sense. This is a fundamental difference between two senses of the reduction method.

In this paper we will focus on some remarkable relations for $S_{n-j}(j)$ that arise in dealing with finite truncated homogeneous Gaussian systems. These relations as well as for $B_{n-j}(j)$ [3] allow us to make transition from the solution of the truncated homogeneous system to the solution of the corresponding infinite system. Some fragments of this article were described in our earlier works, for example, in [2, 3, 4]. But these results were shown in order to solve specific problems of these papers. In the present paper these results are collected for one purpose: to answer a question, how to pass to the limit from the solution of the truncated homogeneous Gaussian system to the solution of the corresponding infinite system? We will repeat and clarify proofs for some theorems to maintain the integrity of this work and present new results.

So, the infinite determinant $|A|$ is nonzero. Therefore, the Gaussian elimination is possible [7], and instead of a general infinite system, we solve an infinite Gaussian system ($a_{j,j} \neq 0$ for any j):

$$\sum_{p=0}^{\infty} a_{j,j+p} x_{j+p} = b_j, \quad j = 1, 2, 3, \dots, \tag{3}$$

with the following matrices: the coefficient matrix A (Gaussian matrix) and the augmented matrix respectively

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & . & a_{1,n} & . \\ 0 & a_{2,2} & . & a_{2,n} & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & a_{n,n} & . \\ . & . & . & . & . \end{pmatrix}, \overline{A} = \begin{pmatrix} b_1 & a_{1,1} & . & a_{1,n} & . \\ b_2 & 0 & . & a_{2,n} & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ b_n & 0 & . & a_{n,n} & . \\ . & . & . & . & . \end{pmatrix}. \tag{4}$$

2. The solution of finite truncated systems

Thus, only after changing the general infinite system on the infinite Gaussian system (3) we can apply the reduction method, namely in two of its senses. First, we solve the system (3) with the method of reduction in the narrow sense, i.e. by simple reduction.

Theorem 2.1. *Let the system (3) be truncated with the reduction method in the narrow sense into the finite Gaussian system of the form*

$$\sum_{p=0}^{n-j} a_{j,j+p} \overset{n}{x}_{j+p} = b_j, \quad a_{j,j} \neq 0, \quad j = \overline{1, n}. \tag{5}$$

Then a solution of the finite system (5) is the expression:

$$\overset{n}{x}_j = B_{n-j}, \quad j = 1, 2, \dots, n, \tag{6}$$

where

$$B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=0}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}} B_p, \quad B_0 = \frac{b_n}{a_{n,n}}, \quad j = \overline{1, n-1}. \tag{7}$$

Let us consider the homogeneous infinite Gaussian system ($b_j \equiv 0$ for all j) (3). If we try to solve this system (3) by the reduction in the narrow sense, i.e. using the (2.1), it is difficult to expect to obtain a nontrivial solution. It follows from the (2.1) that for each n we obtain a trivial solution, and it is likely that if n goes to infinity we will get only a trivial solution of the homogeneous infinite Gaussian system (3), which was verified in [3]. Therefore we will solve the homogeneous infinite Gaussian system (3) with the method of reduction in the broad sense. It means that the finite truncated system for any n has at least one unknown with an arbitrary value. It is convenient to assume this unknown to be, for example, x_1.

Theorem 2.2. *Let the system (3) be truncated with the reduction method in the broad sense into a finite Gaussian system of the form:*

$$\sum_{p=0}^{n-j} a_{j,j+p} \overset{n}{x}_{j+p} = 0, \quad a_{jj} \neq 0, \quad j = \overline{1, n-1}. \tag{8}$$

Then the following expression is a solution of(8:

$$\overset{n}{x}_j = \frac{(-1)^{j-1}x_1}{\prod_{k=1}^{j-1} S_{n-j+k}}, \quad j = \overline{2,n}, \tag{9}$$

where

$$S_{n-j} = \frac{a_{j,j+1}}{a_{j,j}} + \sum_{p=2}^{n-j} \frac{(-1)^{p+1}a_{j,j+p}}{a_{j,j}\prod_{k=1}^{p-1} S_{n-j-k}}, \; S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \quad j = \overline{1,n-1}, \tag{10}$$

and x_1 *is an arbitrary real number.*

Corollary 2.1. *Neighboring unknowns of the homogeneous finite Gaussian system (8) coupled by the following relation:*

$$\overset{n}{x}_j = -S_{n-j}\overset{n}{x}_{j+1}, \quad j = \overline{1,n-1}, \tag{11}$$

where S_{n-j} *are recursively defined by the formula (10).*

Theorem 2.3. *Let the inhomogeneous infinite Gaussian system (3) be truncated with the reduction method in the broad sense into the following inhomogeneous finite Gaussian system:*

$$\sum_{p=0}^{n-j} a_{j,j+p}\overset{n}{x}_{j+p} = b_j, \quad a_{jj} \neq 0, \quad j = \overline{1,n-1}. \tag{12}$$

Then a solution of (12) is the expression:

$$\overset{n}{x}_j = B_{n-j} + \frac{(-1)^{j-1}B_{n-1}}{\prod_{k=1}^{j-1} S_{n-j+k}} + \frac{(-1)^{j-1}x_1}{\prod_{k=1}^{j-1} S_{n-j+k}}, \quad j = \overline{2,n}, \tag{13}$$

where

$$B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=1}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}}B_p \quad B_1 = \frac{b_{n-1}}{a_{n-1,n-1}}, \quad j = \overline{1,n-1}, \tag{14}$$

$$S_{n-j} = \frac{a_{j,j+1}}{a_{j,j}} + \sum_{p=2}^{n-j} \frac{(-1)^{p+1}a_{j,j+p}}{a_{j,j}\prod_{k=1}^{p-1} S_{n-j-k}}, \; S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \quad j = \overline{1,n-1}, \tag{15}$$

x_1 *is an arbitrary real number, and for unification of notations we consider that* $B_0 = 0$.

Corollary 2.2. *Neighboring unknowns of the homogeneous finite Gaussian system(12) coupled by the following relation:*

$$\overset{n}{x}_j = B_{n-j} + S_{n-j}B_{n-j-1} - S_{n-j}\overset{n}{x}_{j+1}, \quad j = \overline{1,n-1}, \tag{16}$$

where B_{n-j} *and* S_{n-j} *are recursively defined by formulas (14) and (15), and* $B_0 = 0$ *as above.*

It is obvious that for the homogeneous finite system, the relation (11) follows from (16).

Remark 2.1. Clearly, that the expressions (10) and (15) are the same, hence the numbers S_{n-j} does not depend on the fact that the system is homogeneous or inhomogeneous.

Remark 2.2. From the (2.3) it follows that there is a sum of particular solutions of homogeneous and inhomogeneous systems in the solution (13) which was obtained with the reduction method in the broad sense.

Remark 2.3. Truly, if we truncate an infinite system with the reduction method in the broad sense, then we will get a finite system where the last row of its matrix is entirely consists of zeros. Hence, when we use the reduction method in the broad sense, we approximate the original infinite system with the finite system with degenerate matrix even if it has a nondegenerate matrix. It will be shown below, that this approach is the most effective when solving homogeneous infinite systems.

In [3] we have got remarkable relations for B_{n-j} of the form:

Theorem 2.4. *For numbers B_{n-j} we have the following relations:*

$$I. \quad B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=1}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}} B_p \quad B_1 = \frac{b_{n-1}}{a_{n-1,n-1}};$$

$$II. \quad B_{n-j} = \frac{|A_{n-1}^{(j)}|}{|A_{n-1}|},$$

where $|A_{n-1}|$ is a determinant for a finite system of type (5) of order $n-1$, $|A_{n-1}^{(j)}|$ – is a Cramer's determinant for the same system (it obtained by replacing the j column of $|A_{n-1}|$ with the right-hand side from the system of type (5);

$$III. \quad B_{n-j} = \begin{vmatrix} \frac{b_j}{a_{j,j}} & \frac{b_{j+1}}{a_{j+1,j+1}} & \frac{b_{j+2}}{a_{j+2,j+2}} & \dots & \frac{b_{n-1}}{a_{n-1,n-1}} \\ \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & \dots & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & \dots & 0 \\ . & . & . & \dots & . \\ \frac{a_{j,j+j}}{a_{j,j}} & \frac{a_{j+1,j+j}}{a_{j+1,j+1}} & \frac{a_{j+2,j+j}}{a_{j+2,j+2}} & \dots & 0 \\ . & . & . & \dots & \\ \frac{a_{j,n-2}}{a_{j,j}} & \frac{a_{j+1,n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,n-2}}{a_{j+2,j+2}} & \dots & 0 \\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,n-1}}{a_{j+2,j+2}} & \dots & 1 \end{vmatrix} = B_{n-j}(j); \tag{17}$$

$$IV. \quad B_{n-j} = \sum_{p=0}^{n-j-1} (-1)^p A_p(j) \frac{b_{j+p}}{a_{j+p,j+p}}, \quad where$$

$$A_p(j) = \sum_{k=0}^{p-1} \frac{(-1)^{p-1-k} a_{j+k,j+p}}{a_{j+k,j+k}} A_k(j), \; A_0(j) = 1 \;\; \forall j, \tag{18}$$

and $j = \overline{1, n-1}$ for all relations.

The determinant $B_{n-j}(j)$ can be called *generalized Cramer's determinant*, that is justified by the relations II and III. In addition, without loss of generality, we can assume that the coefficients $a_{j,j} = 1$ in (17), in the relation IV and in (18). Otherwise, the ratios $\frac{a_{j,j+p}}{a_{j+p,j+p}}$, $\frac{b_j}{a_{j,j}}$ should be taken for $a_{j,j}$ and b_j respectively.

It is shown in [3] that if the strictly particular solution x_j of the inhomogeneous infinite system (3) exists, then it is of the form

$$x_j = B(j) = \lim_{n\to\infty} B_{n-j} = \sum_{p=0}^{\infty} (-1)^p A_p(j) \frac{b_{j+p}}{a_{j+p,j+p}}, \tag{18'}$$

where $A_p(j)$ is calculated by the recurrence formula (18).

Actually in (18) the numbers $A_p(j)$ are values of the variable determinant of p order, where j is an integer argument.

In case $A_p(j) = A_p = const$, we called these determinants the *characteristic determinants* of an infinite Gaussian system with difference indices [5, 6]. Hence, in the general case, we called them with the same term, see [3].

We assume that we can increase order of the determinant (17) without limit in accordance with the infinite system (3). Then, when the first row is deleted from the determinant (17) and then appropriate last row is added, we will get the determinant $|A(j)|$ of $n-j$ order, i.e.

$$|A_{n-j}(j)| = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & \dots & 0 & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & \dots & 0 & 0 \\ \frac{a_{j,j+3}}{a_{j,j}} & \frac{a_{j+1,j+3}}{a_{j+1,j+1}} & \frac{a_{j+2,j+3}}{a_{j+2,j+2}} & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ \frac{a_{j,n-2}}{a_{j,j}} & \frac{a_{j+1,n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,n-2}}{a_{j+2,j+2}} & \dots & 1 & 0 \\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,n-1}}{a_{j+2,j+2}} & \dots & \frac{a_{n-2,n-1}}{a_{n-2,n-2}} & 1 \\ \frac{a_{j,n}}{a_{j,j}} & \frac{a_{j+1,n}}{a_{j+1,j+1}} & \frac{a_{j+2,n}}{a_{j+2,j+2}} & \dots & \frac{a_{n-2,n}}{a_{n-2,n-2}} & \frac{a_{n-1,n}}{a_{n-1,n-1}} \end{vmatrix}. \tag{19}$$

Below the symbol of the determinant $|A(j)|$ is omitted.

We construct a sequence of determinants $A_p(j)$ $0 \le p \le n-j$, assuming that $A_0(j) = 1$ for all j, and for other p values we take the leading principal minors of the determinant (19), i.e. $A_0(j) = 1, \quad A_1(j) = \frac{a_{j,j+1}}{a_{j,j}}, \quad A_2(j) = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} \end{vmatrix},$

$$A_p(j) = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 & \dots & 0 & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & \dots & 0 & 0 \\ . & . & . & . & . \\ \frac{a_{j,j+p-2}}{a_{j,j}} & \frac{a_{j+1,j+p-2}}{a_{j+1,j+1}} & \dots & 1 & 0 \\ \frac{a_{j,j+p-1}}{a_{j,j}} & \frac{a_{j+1,j+p-1}}{a_{j+1,j+1}} & \dots & \frac{a_{j+p-2,j+p-1}}{a_{j+p-2,j+p-2}} & 1 \\ \frac{a_{j,j+p}}{a_{j,j}} & \frac{a_{j+1,j+p}}{a_{j+1,j+1}} & \dots & \frac{a_{j+p-2,j+p}}{a_{j+p-2,j+p-2}} & \frac{a_{j+p-1,j+p}}{a_{j+p-1,j+p-1}} \end{vmatrix}. \tag{20}$$

Here we want to emphasize that the determinants $A_p(j)$ are also the principal determinants (the leading principal minors) of (19) if it is infinite, and besides j may start from zero.

The recurrence relations (18) can be easily proved by induction using the sequence (20).

Theorem 2.5. *Let A be a n-th order Gaussian matrix*

$$A = \begin{pmatrix} 1 & a_{0,1} & a_{0,2} & \dots & a_{0,n-1} \\ 0 & 1 & a_{1,2} & \dots & a_{1,n-1} \\ 0 & 0 & 1 & \dots & a_{2,n-1} \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \end{pmatrix}.$$

Then the inverse matrix A^{-1} is of the form:

$$A^{-1}=\begin{pmatrix}1 & -A_1(0) & A_2(0) & ... & (-1)^{n-1}A_{n-1}(0)\\ 0 & 1 & -A_1(1) & ... & (-1)^{n-2}A_{n-2}(1)\\ 0 & 0 & 1 & ... & (-1)^{n-3}A_{n-3}(2)\\ . & . & . & ... & .\\ 0 & 0 & 0 & ... & 1\end{pmatrix} \tag{21}$$

where the $A_p(j)$ are the characteristic determinants (20) and $A_0(j)=1$ for all j, $0\le j\le n-1$.

Proof. Let A' be a matrix in (21). We will prove that $A'=A^{-1}$. To do this, we will show that a product of matrices A', A is equal to the identity matrix, i.e. $A'A=\overline{A}(\overline{a}_{j,i})=E$, where E is the identity matrix.

It is obvious that $\overline{a}_{j,j}=1$, i.e. the diagonal elements of $\overline{A}(\overline{a}_{j,i})$ are equal to one and $\overline{a}_{j,i}=0$ for $j>i$. It means that all elements below the main diagonal are equal to zero.

Let's calculate the rest of elements $\overline{a}_{j,j+p}$, where $p=1,2,...,n-1-j$. To do this we take the j-th row of matrix A': $(\underbrace{0,0,...,0}_{j},1,-A_1(j),A_2(j),...,(-1)^{p-1}A_{p-1}(j),(-1)^pA_p(j))$ and the j-th column of matrix A: $(a_{0,j+p},a_{1,j+p},...,a_{j,j+p},a_{j+1,j+p},...,a_{j+p-2,j+p},a_{j+p-1,j+p},1)^T$, $j=0,1,...,n-p-1$. The scalar product of them gives the element $a_{j,j+p}=a_{j,j+p}-a_{j+1,j+p}A_1(j)+a_{j+2,j+p}A_2(j)+...+(-1)^{p-1}a_{j+p-1,j+p}\ A_{p-1}(j)+(-1)^pA_p(j)$. Using the recurrence relations (18), we get $a_{j,j+p}=0$. Hence we can obtain $A'=A^{-1}$. □

Corollary 2.3. *The characteristic determinant $A_p(j)$ is equal to the complementary minor $M_{p+j,j}$ of the element $a_{j+p,j}$ in the matrix A, i.e. $A_p(j)=M_{p+j,j}$, $j=0,1,...,n-p-1$.*

Proof. From the definition of the inverse matrix for a Gaussian matrix having unit elements on main diagonal we have

$$A^{-1}=\begin{pmatrix}1 & -M_{1,0} & M_{2,0} & ... & (-1)^{p+j}M_{p+j,0}\\ 0 & 1 & -M_{2,1} & ... & (-1)^{p+j-1}M_{p+j-1,1}\\ 0 & 0 & 1 & ... & (-1)^{p+j-2}M_{p+j-2,2}\\ . & . & . & ... & .\\ 0 & 0 & 0 & ... & 1\end{pmatrix},$$

where $M_{p+j,j}$ is the minor of the element $a_{p+j,j}$ in the matrix A, $p=1,2,...,n-1-j$.

Using the (21), we have proved the corollary. □

To generalize the (2.5) for infinite Gaussian matrix, it is necessary to use the (2.3). The (2.3) and the relations (18) imply that for the fixed j and p an infinite complementary minors are finite and exist.

So, using the induction we generalize the (2.5) to the infinite case:

Theorem 2.6. *Let A be an infinite Gaussian matrix*

$$A=\begin{pmatrix}1 & a_{0,1} & a_{0,2} & ... & a_{0,n-1} & .\\ 0 & 1 & a_{1,2} & ... & a_{1,n-1} & .\\ 0 & 0 & 1 & ... & a_{2,n-1} & .\\ . & . & . & ... & . & .\\ 0 & 0 & 0 & ... & 1 & .\\ . & . & . & ... & . & .\end{pmatrix}$$

Then the inverse matrix A^{-1} is of the form

$$A^{-1} = \begin{pmatrix} 1 & -A_1(0) & A_2(0) & . & (-1)^i A_i(0) & . & (-1)^{n-1}A_{n-1}(0) & . \\ 0 & 1 & -A_1(1) & . & (-1)^{i-1}A_{i-1}(1) & . & (-1)^{n-2}A_{n-2}(1) & . \\ 0 & 0 & 1 & . & (-1)^{i-2}A_{i-2}(2) & . & (-1)^{n-3}A_{n-3}(2) & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & . & (-1)^{n-j-1}A_{n-j-1}(j) & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & . & 1 & . \\ . & . & . & . & . & . & . & . \end{pmatrix} \qquad (21')$$

where the $A_p(j)$ are the characteristic determinants (20) and $A_0(j) = 1$ for all fixed j.

3. The solution of finite truncated systems

Let's find a relation for S_{n-j} similar to the relation III of (2.4). Firstly, we will prove two lemmas, they will connect the S_{n-j} with characteristic determinants (19) or (20). Let's note that the S_{n-j} are a functions of integer argument as well as numbers $B_{n-j}(j)$.

Lemma 3.1. *Let's denote the ratio of characteristic determinants $\frac{A_{n-j}(j)}{A_{n-j-1}(j+1)}$ by $C_{n-j}(j)$. Then, the next relation holds:*

$$\frac{A_{n-j}(j)}{A_{n-j-k}(j+k)} = \prod_{t=0}^{k-1} C_{n-j-t}(j+t), 1 \le k \le n-j, \qquad (22)$$

where $A_{n-j}(j)$ are the characteristic determinants (19) or (20), and we adopt that $A_{-k}(j+k) = 1$.

Proof. We have

$$\frac{A_{n-j}(j)}{A_{n-j-k}(j+k)} = \frac{A_{n-j}(j)}{A_{n-j-1}(j+1)} \frac{A_{n-j-1}(j+1)}{A_{n-j-2}(j+2)} \cdots \frac{A_{n-j-k+1}(j+k-1)}{A_{n-j-k}(j+k)} =$$

$$= C_{n-j}(j)C_{n-j-1}(j+1)...C_{n-j-k+1}(j+k-1) = \prod_{t=0}^{k-1} C_{n-j-t}(j+t).$$

The lemma is proved. □

Lemma 3.2. *With the notation of the (Lemma 3.1.), we have*

$$A_{n-j}(j) = \prod_{k=0}^{n-j} C_{n-j-k}(j+k), 0 \le j \le n-1. \qquad (23)$$

Proof. From (22) it follows that:

$$A_{n-j}(j) = A_{n-j-k}(j+k) \prod_{t=0}^{k-1} C_{n-j-t}(j+t).$$

Suppose that $n-j-t = 0$ and taking into account $A_0(n) = 1$, we have proved (23). □

Theorem 3.1. *For numbers S_{n-j}, we have*

$$S_{n-j} = \frac{A_{n-j}(j)}{A_{n-j-1}(j+1)} = S_{n-j}(j), 0 \le j \le n-1, \qquad (24)$$

where $A_{n-j}(j)$ are the characteristic determinants (19).

Proof. We expand the determinant (20) along the first row. We have

$$A_p(j)=\frac{a_{j,j+1}}{a_{j,j}}A_{p-1}(j+1)-1\cdot\begin{vmatrix}\frac{a_{j,j+2}}{a_{j,j}} & 1 & \dots & 0 & 0\\ \frac{a_{j,j+3}}{a_{j,j}} & \frac{a_{j+2,j+3}}{a_{j+2,j+2}} & \dots & 1 & 0\\ . & . & \dots & . & .\\ \frac{a_{j,j+p-1}}{a_{j,j}} & \frac{a_{j+2,j+p-1}}{a_{j+2,j+2}} & \dots & \frac{a_{j+p-2,j+p-1}}{a_{j+p-2,j+p-2}} & 1\\ \frac{a_{j,j+p}}{a_{j,j}} & \frac{a_{j+2,j+p}}{a_{j+2,j+2}} & \dots & \frac{a_{j+p-2,j+p}}{a_{j+p-2,j+p-2}} & \frac{a_{j+p-1,j+p}}{a_{j+p-1,j+p-1}}\end{vmatrix}.$$

Next, we expand the last determinant along its first row, and then we continue to do so several times. We have

$$A_p(j)=\frac{a_{j,j+1}}{a_{j,j}}A_{p-1}(j+1)-\frac{a_{j,j+2}}{a_{j,j}}A_{p-2}(j+2)+\dots(-1)^{p-1}\cdot\begin{vmatrix}\frac{a_{j,j+p-1}}{a_{j,j}} & 1\\ \frac{a_{j,j+p}}{a_{j,j}} & \frac{a_{j+p-1,j+p}}{a_{j+p-1,j+p-1}}\end{vmatrix}.$$

On the other hand, the characteristic determinants $A_1(j)$ and $A_0(j)$ respectively equal to $A_1(j+p-1)=\dfrac{a_{j+p-1,j+p}}{a_{j+p-1,j+p-1}}$ and $A_0(j+p)=1$ by definition. Therefore, we have:

$$A_p(j)=\frac{a_{j,j+1}}{a_{j,j}}A_{p-1}(j+1)-\frac{a_{j,j+2}}{a_{j,j}}A_{p-2}(j+2)+\dots$$
$$\dots+(-1)^{p-2}\frac{a_{j,j+p-1}}{a_{j,j}}A_1(j+p-1)+(-1)^{p-1}\frac{a_{j,j+p}}{a_{j,j}}A_0(j+p).$$

Dividing the last expression by $A_{p-1}(j+1)$, we will get

$$\begin{gathered}\frac{A_p(j)}{A_{p-1}(j+1)}=\frac{a_{j,j+1}}{a_{j,j}}-\frac{a_{j,j+2}}{a_{j,j}}\frac{A_{p-2}(j+2)}{A_{p-1}(j+1)}+\dots\\ \dots+(-1)^{p-2}\frac{a_{j,j+p-1}}{a_{j,j}}\frac{A_1(j+p-1)}{A_{p-1}(j+1)}+(-1)^{p-1}\frac{a_{j,j+p}}{a_{j,j}}\frac{A_0(j+p)}{A_{p-1}(j+1)}.\end{gathered}\tag{25}$$

Let $n-j=1$ ($j=n-1$), then the next equalities hold true:

$$\frac{A_1(n-1)}{A_0(n)}=A_1(n-1)=\frac{a_{n-1,n}}{a_{n-1,n-1}}=S_1(n-1),$$

i.e. for the initial value $n-j=1$ the relation (24) holds.

Let's prove the theorem by induction. Assume (24) holds for $n-j\le p-1$, then we show that (24) holds for $n-j=p$. Let $p=n-j$, then we can rewrite (25) as

$$\frac{A_{n-j}(j)}{A_{n-j-1}(j+1)}=\frac{a_{j,j+1}}{a_{j,j}}+\sum_{p=2}^{n-j}(-1)^{p-1}\frac{a_{j,j+p}}{a_{j,j}}\frac{1}{\frac{A_{n-j-1}(j+1)}{A_{n-j-p}(j+p)}}.\tag{26}$$

It is clear that the statement of theorem holds for the ratio $\dfrac{A_{n-j-1}(j+1)}{A_{n-j-p}(j+p)}$ by induction hypothesis, i.e. $C_{n-j-k}(j+k)=S_{n-j-k}(j+k)$, $k>0$. Let's apply (1) to (25) and take into account the (16), then we have

$$\frac{A_{n-j}(j)}{A_{n-j-1}(j+1)}=\frac{a_{j,j+1}}{a_{j,j}}+\sum_{p=2}^{n-j}(-1)^{p-1}\frac{a_{j,j+p}}{a_{j,j}\prod_{k=1}^{p-1}S_{n-j-k}}=S_{n-j}(j).$$

Theorem is proved. □

We use (3) and get the next corollary:

Corollary 3.1. *For determinants $A_{n-j}(j)$ the next relation holds true*

$$A_{n-j}(j) = \prod_{k=0}^{n-j} S_{n-j-k}(j+k). \tag{27}$$

Let's express a solution of the homogeneous system (12) ($b_j = 0, j = \overline{0, n-1}$) in terms of determinants, like a Cramer's rule.

Theorem 3.2. *A solution of the homogeneous finite system (12) is given by*

$$x_j = \frac{(-1)^j x_0 A_{n-j}(j)}{A_n(0)}, \quad j = 0, 1, ..., n, \tag{28}$$

where $A_{n-j}(j)$ are the characteristic determinants (19) and x_0 is an arbitrary real number.

Proof. From (24), we have

$$A_{n-j}(j) = S_{n-j}(j) A_{n-j-1}(j+1).$$

Solving this recurrence equation in the same way as the equation (11), we have

$$A_{n-j}(j) = \frac{A_n(0)}{\prod_{k=0}^{j-1} S_{n-k}(k)}. \tag{29}$$

We find $\prod\limits_{k=0}^{j-1} S_{n-k}(k)$ from (29) and then substitute it to (9) ($j = 0, 1, ..., n$), we will get what is required to prove. □

In summary, we formulate a theorem for numbers $S_{n-j}(j)$ in the same way as the (2.4).

Theorem 3.3. *For numbers $S_{n-j}(j)$ we have the following relations:*

$$I. \quad S_{n-j} = \frac{a_{j,j+1}}{a_{j,j}} + \sum_{p=2}^{n-j} \frac{(-1)^{p+1} a_{j,j+p}}{a_{j,j} \prod\limits_{k=1}^{p-1} S_{n-j-k}}, \ S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \quad j = \overline{0, n-2};$$

$$II. \quad S_{n-j}(j) = \frac{A_{n-j}(j)}{A_{n-j-1}(j+1)}, \quad j = \overline{0, n-1},$$

where $A_{n-j}(j)$ are the characteristic determinants (19) of order $n-j$ and

$$A_{n-j}(j) = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & ... & 0 & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & ... & 0 & 0 \\ \frac{a_{j,j+3}}{a_{j,j}} & \frac{a_{j+1,j+3}}{a_{j+1,j+1}} & \frac{a_{j+2,j+3}}{a_{j+2,j+2}} & ... & 0 & 0 \\ . & . & . & ... & . & . \\ \frac{a_{j,n-2}}{a_{j,j}} & \frac{a_{j+1,n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,n-2}}{a_{j+2,j+2}} & ... & 1 & 0 \\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,n-1}}{a_{j+2,j+2}} & ... & \frac{a_{n-2,n-1}}{a_{n-2,n-2}} & 1 \\ \frac{a_{j,n}}{a_{j,j}} & \frac{a_{j+1,n}}{a_{j+1,j+1}} & \frac{a_{j+2,n}}{a_{j+2,j+2}} & ... & \frac{a_{n-2,n}}{a_{n-2,n-2}} & \frac{a_{n-1,n}}{a_{n-1,n-1}} \end{vmatrix}, j = \overline{0, n-1}; \tag{30}$$

$$III. \quad \prod_{k=0}^{n-j} S_{n-j-k}(j+k) = A_{n-j}(j), \quad j = \overline{0, n-1};$$

$$IV. \quad A_p(j) = \sum_{k=0}^{p-1} \frac{(-1)^{p-1-k} a_{j+k,j+p}}{a_{j+k,j+k}} A_k(j), \; A_0(j) = 1 \text{ for all } \forall j, \; j = \overline{0, n-1};$$

$$V. \quad \overset{n+1}{x_j} = \frac{(-1)^j x_0 A_{n-j}(j)}{A_n(0)}, \; j = \overline{0, n}.$$

Here we only note that we obtain the relation V when the homogeneous system of n equations with $n+1$ unknowns x_0, x_1,...,x_n is considered and x_0 is an arbitrary real number. Moreover, the relation V looks like a Cramer's rule (in case of homogeneous system). This fact will be discussed in more detail below. Hence, (3.3) shows an implementation of the method of reduction in the broad sense to solve a homogeneous system.

4. The transition from the homogeneous finite system solutions to the solution of the homogeneous infinite system

If a nontrivial solutions of homogeneous infinite system exists, then we solve this system in the same way as the inhomogeneous system in [3]. Here it is necessary to use the method of reduction in the broad sense, i.e. the truncated system for any order is considered as a system with degenerate matrix. What is the reason for that, if we assumed that infinite determinant is nonzero? This is due to the next two facts. Firstly, let nontrivial solution $\{x_j\}_0^\infty$ exists, then it is obvious that the set $\{cx_j\}_0^\infty$ is also a solution of homogeneous system for any constant $c \neq 0$. It means that an existence of one solution implies an existence of infinitely many solutions.

Secondly, if we use the method of reduction in the narrow sense to solve the finite truncated system of nth order, then each finite system has only a trivial solution. Therefore, it is hard to expect a nontrivial solution at the limit.

Hence, we consider the $S_{n-j}(j)$ instead of the $B_{n-j}(j)$. We will use (3.3) to pass to the limit from the finite system (8) ($j = 0, 1, ..., n-1$) to infinite Gaussian system (3) ($j = \overline{0, \infty}$). We assume that the following two conditions hold as well as we assumed in [3]:

Condition 4.1. *a) Suppose that the limit $\lim\limits_{n\to\infty} S_{n-j}(j) = S(j) \neq 0$ exists for any fixed j. This condition guarantees that the method of reduction in the broad sense converges, as it will be shown bellow;*

Condition 4.2. *b) Suppose that in the relation I it is possible to pass term-by-term to the limit in the sense of formula*

$$\lim_{n\to\infty} \sum_{p=2}^{n-j} \frac{(-1)^{p+1} a_{j,j+p}}{a_{j,j} \prod\limits_{k=1}^{p-1} S_{n-j-k}} = \sum_{p=2}^{\infty} \frac{(-1)^{p+1} a_{j,j+p}}{a_{j,j} \prod\limits_{k=1}^{p-1} \lim\limits_{n\to\infty} S_{n-j-k}}, \qquad j = \overline{0, \infty}. \tag{31}$$

for any fixed j.

As it will be seen below, if the condition a) holds then the condition b) is a sufficient for a nontrivial solution of the homogeneous original system (3) to be construct by numbers $S(j)$. Thus, the performance of only one condition a) is not sufficient for us to get a nontrivial solution of homogeneous infinite system by the method of reduction in a broad sense.

Theorem 4.1. *Let the conditions a) and b) hold, then there are infinitely many nontrivial solutions $\{cx_j\}_0^\infty$ of homogeneous ($b_j = 0$) Gaussian system (3), where $c \neq 0$ is an arbitrary real number.*

Proof. We pass to the limit as $n \to \infty$ in the relation I of (3.3). Under the conditions of (4), the next equality holds true for each j:

$$\sum_{p=0}^{\infty} \frac{(-1)^p a_{j,j+p}}{a_{j,j} \prod\limits_{k=0}^{p-1} S(j+k)} = 0, \quad j = 0, 1, 2, \dots, \tag{32}$$

for the unification of notations we consider that $\prod\limits_{k=0}^{-1} S(j+k) = 1$ for each j.

Here, in (32) we note that, without loss of generality, we may assume that $a_{j,j} = 1$.

Now for any real $c = x_0 \neq 0$, we compose a numbers $x_j = \frac{(-1)^j x_0}{\prod\limits_{k=0}^{j-1} S(k)}$. Then let us to show that these numbers satisfy the homogeneous system (3). Indeed, if we put them into system (3), then we get

$$\sum_{p=0}^{\infty} a_{j,j+p} x_{j+p} = \frac{(-1)^j x_0}{\prod\limits_{k=0}^{j-1} S(k)} \sum_{p=0}^{\infty} \frac{(-1)^p a_{j,j+p}}{\prod\limits_{k=0}^{p-1} S(j+k)} = 0,$$

here we used that $\prod\limits_{k=0}^{j+p-1} S(k) = \prod\limits_{k=0}^{j-1} S(k) \prod\limits_{k=j}^{p-1} S(j+k)$, $x_0 \neq 0$, $S(j) \neq 0$ and equality (32). Theorem is proved. □

Corollary 4.1. *Let the conditions a) and b) hold. Then the neighboring components of nontrivial solution $\{x_j\}_0^{\infty}$ of homogeneous system (3) coupled by the following relation:*

$$x_j = -S(j) x_{j+1}, \quad j = \overline{0, \infty}, \tag{33}$$

where $S(j) = \lim\limits_{n\to\infty} S_{n-j}(j)$.

Remark 4.1. *If the conditions a) and b) hold, then the method of reduction in the broad sense converges to nontrivial solution $\{x_j\}_0^{\infty}$. It means that the passage to the limit from solution (9) of the finite system (8 to solution of the homogeneous infinite system (3) holds true:*

$$\lim_{n\to\infty} \overset{n}{x}_j = x_j = \lim_{n\to\infty} (-S_{n-j}(j) \overset{n}{x}_{j+1}) = -S(j) x_{j+1}, \quad j = \overline{0, \infty},$$

where S_{n-j} are recursively defined by the relation I of (3.3), and $S(j) = \lim\limits_{n\to\infty} S_{n-j}(j)$.

Indeed, taking into account (4), if we pass to the limit in (11) then we get the last equality.

Remark 4.2. *In fact for any nontrivial solution the relation such as (33) holds true. Indeed, if $\overline{S}(j) = -\frac{x_j}{x_{j+1}}$ is a new unknown, then we get relation (33). Hence numbers $\overline{S}(j)$ characterize a nontrivial solution. For example, if $\overline{S}(j) = 0$ for some j or these numbers do not exist, then homogeneous system has only a trivial solution. If $\overline{S}(j) = \lim\limits_{n\to\infty} S_{n-j}(j) = S(j)$, then we get some special nontrivial solution, as it will be shown below.*

Definition 4.1. *Let numbers $\overline{S}(j)$ exist for each j and satisfy relation (33. Here the set $\{x_j\}_0^{\infty}$ composes a nontrivial solution of the homogeneous Gaussian system (3). Then numbers $\overline{S}(j)$ are called* a characteristic numbers *of a corresponding solution. If $\overline{S}(j) = \lim\limits_{n\to\infty} S_{n-j}(j) = S(j)$, then we call these numbers a principal characteristic numbers of a principal nontrivial solution $\{x_j\}_0^{\infty}$.*

Hence, if the limit $\lim\limits_{n\to\infty} S_{n-j}(j) = S(j) \neq 0$ is a principal characteristic number, then the method of reduction in the broad sense converges to a principal nontrivial solution $\{x_j\}_0^{\infty}$ of the homogeneous system (3), and solutions $\{cx_j\}_0^{\infty}$ we call *a principal fundamental solutions.*

It is worth noting that a characteristic numbers $S(j)$ of a solutions $\{cx_j\}_0^\infty$ and $\{x_j\}_0^\infty$ are the same for any constant $c \neq 0$. Therefore, a principal fundamental solution is determined up to a constant multiplier c. In general, if we know only one nontrivial solution, then there are an infinitely many nontrivial solutions.

The importance of the next theorem lies in the fact that the passage to the limit in expression (15) and the existence of a principal characteristic numbers of corresponding nontrivial solution of the homogeneous system are equivalent.

Theorem 4.2. *Let the condition a) holds. The passage to the limit in relation I is possible if and only if a numbers $S(j)$ $j = 0, 1, ...$ are the principal characteristic numbers of a corresponding solution.*

Proof. N e c e s s i t y. Let the passage to the limit in relation I is possible, then in accordance with (4) and (4.1), we conclude that a numbers $S(j)$ $j = 0, 1, ...$ are the principal characteristic numbers of a corresponding solution.

S u f f i c i e n c y. Let the set of numbers $S(j)$ $j = 0, 1, ...$ be a set of the principal characteristic numbers of a corresponding solution. It means that an equality (33) holds and a numbers $x_j = \frac{(-1)^j x_0}{\prod_{k=0}^{j-1} S(k)}$ satisfy to homogeneous system (3). Let's prove that an expression (31) holds true. Taking into account the notation $S_{n-j} = S_{n-j}(j)$ and condition a), we pass to the limit in the relation I of (4), then we have

$$\lim_{n\to\infty} S_{n-j} = S(j) = a_{j,j+1} + \lim_{n\to\infty} \sum_{p=2}^{n-j} \frac{(-1)^{p+1} a_{j,j+p}}{\prod\limits_{k=1}^{p-1} S_{n-j-k}(j+k)}, \quad j = 0, 1, 2, \dots. \tag{34}$$

From the (32), we have

$$S(j) = a_{j,j+1} + \sum_{p=2}^{\infty} \frac{(-1)^{p+1} a_{j,j+p}}{\prod\limits_{k=0}^{p-1} S(j+k)}, \quad j = 0, 1, 2, \dots, \tag{35}$$

We subtract an equality (35) from an equality (34) and taking into account the $S(j) = \lim\limits_{n\to\infty} S_{n-j}(j)$, we have:

$$\lim_{n\to\infty} \sum_{p=2}^{n-j} \frac{(-1)^{p+1} a_{j,j+p}}{\prod\limits_{k=1}^{p-1} S_{n-j-k}} = \sum_{p=2}^{\infty} \frac{(-1)^{p+1} a_{j,j+p}}{\prod\limits_{k=1}^{p-1} S(j+k)} = \sum_{p=2}^{\infty} \frac{(-1)^{p+1} a_{j,j+p}}{\prod\limits_{k=1}^{p-1} \lim\limits_{n\to\infty} S_{n-j-k}}, \qquad j = \overline{0, \infty}.$$

Theorem is proved. □

Therefore, if the conditions a) and b) hold, then the method of reduction in the broad sense converges to a principal nontrivial solution of the homogeneous Gaussian system (3). Hence, a principal nontrivial solution is a special solution by the its construction, and it is a unique. Actually, this solution is the basis of the vector subspace of nontrivial solutions of the homogeneous system. Hence, in analogy with periodic system, we may call this solution $\{cx_j\}_0^\infty$ *a fundamental solution* [5, 6]. It means that we call a principal nontrivial solution *a principal fundamental solution.*

In order to obtain a better idea of the fundamental solutions of homogeneous system, we consider next homogeneous periodic system, more precisely, a homogeneous system with difference indices [5, 6]. Let the next homogeneous infinite system with difference indices (a coefficients depend on the one index) be given:

$$\sum_{p=0}^{\infty} a_p x_{j+p} = 0, \quad j = 0, 1, \ldots . \tag{a}$$

As shown in [5, 6], the expression

$$x_j = \frac{(-1)^j x_0}{S^j}, \quad j = 0, 1, \ldots \tag{b}$$

is a solution of system (a), where $\frac{1}{S}$ is an arbitrary solution of the characteristic equation $f(x) = 0$, i.e.

$$f(x) = \sum_{p=0}^{\infty} \frac{(-1)^j a_p}{S^p} = 0, \tag{c}$$

where x_0 is an arbitrary real number.

The solution of (b) type we call *a fundamental solution* of the system (a) with difference indices. The equality $x_j = -Sx_{j+1}$ holds true for any fundamental solutions (b), and x_j are a components of a fundamental solution of the system (a).

If $\frac{1}{S_k}$ is a k-th root of multiplicity ν_k of a characteristic equation (c), then an expressions of the form

$$\begin{gathered} x_i^{(k)} = \frac{(-1)^i x_0^{(k)}}{S_k^i}, \quad x_i^{(k)} = \frac{(-1)^i i x_0^{(k)}}{S_k^i}, \quad x_i^{(k)} = \frac{(-1)^i i^2 x_0^{(k)}}{S_k^i}, \ldots \\ \ldots x_i^{(k)} = \frac{(-1)^i i^{\nu_k - 1} x_0^{(k)}}{S_k^i}, \quad i > 0, \ k = 1, 2, \ldots, N, \end{gathered} \tag{d}$$

are a linearly independent solutions of system (a), where $x_0^{(k)}$ are an arbitrary real numbers and N is the number of zeros of characteristic equation (c), counted without multiplicity.

We have shown that a solutions of (d) type form a complete system of fundamental solutions of system (a).

Unfortunately, it is impossible to define the limits themselves, i.e. the numbers $S(j)$, from the relation I as well as for $B(j)$, because defining them is equivalent to finding the solution of the original homogeneous system (3). Therefore, we will use the next theorem to calculate a characteristic numbers.

Theorem 4.3. *A necessary and sufficient condition for nontrivial solution $\{x_j\}_0^\infty$ of the homogeneous Gaussian system (3) to exist is that the its characteristic numbers $S(j)$ satisfy the equality (32) for each j.*

Proof. N e c e s s i t y. Let y_j be an arbitrary nontrivial solution of system (3). Then, by (4.1, the relation (33) holds true.

Solving the recursive equation (33) in reverse order, we have:

$$y_j = \frac{(-1)^j y_0}{\prod\limits_{k=0}^{j-1} S(k)}, \tag{36}$$

where y_0 is an arbitrary real number, and $\prod_{k=0}^{-1} S(k) = 1$ as we considered before.

We have

$$\prod_{k=0}^{j+p-1} S(k) = \prod_{k=0}^{j-1} S(k) \prod_{k=j}^{j+p-1} S(k) = \prod_{k=0}^{j-1} S(k) \prod_{k=0}^{p-1} S(j+k).$$

Taking into account the last relations, we put (36) into the origin system (3), then we get

$$\sum_{p=0}^{\infty} a_{j,j+p} y_{j+p} = \sum_{p=0}^{\infty} \frac{(-1)^{j+p} y_0 a_{j,j+p}}{\prod\limits_{k=0}^{j+p-1} S(k)} = \frac{(-1)^j y_0}{\prod\limits_{k=0}^{j-1} S(k)} \sum_{p=0}^{\infty} \frac{(-1)^p a_{j,j+p}}{\prod\limits_{k=0}^{p-1} S(j+k)} = 0,$$

where $j = 0, 1, ...$ In general case we have $y_0 \neq 0$, then the necessary condition (32) for each j follows from the last equation. Necessity is proved.

S u f f i c i e n c y. Let numbers $S(j)$ be a solutions of equations (32) for each j. Then we compose a numbers x_j of the form (36):

$$x_j = \frac{(-1)^j x_0}{\prod\limits_{k=0}^{j-1} S(k)}.$$

We put these values into the homogeneous system (3) and verify that all the equations are satisfied, because the (32) holds. Theorem is proved. □

In order to find the principal characteristic numbers and the principal fundamental solution, we will use the remarkable properties of $S_{n-j}(j)$. We discuss it in the next paragraph.

5. The existence of nontrivial solutions of the homogeneous infinite systems

As we noted before, the method of reduction in both sense gives an essentially different solutions of infinite systems. This method in the narrow sense gives a special solution for inhomogeneous infinite system – *a strictly particular solution*, and the method of reduction in the broad sense gives a special nontrivial solution for homogeneous infinite system – *a principal fundamental solution* of the homogeneous system.

It is worth mentioning the critical importance of strictly particular solution properties [2, 3, 4]. These properties may be useful for a principal fundamental solution, they are as follows.

Property 5.1. We obtain a strictly particular solution with the method of reduction in the narrow sense. Thus, the existence of a strictly particular solution proves the convergence of the reduction method.

Property 5.2. A consistent inhomogeneous Gaussian system always has a unique particular solution. It is the strictly particular solution, and it can be expressed by Cramer's formula. It follows that Cramer's formula for infinite system is obtained from Cramer's formula for finite truncated Gaussian system by use of the passage to the limit.

Property 5.3. The strictly particular solution does not contain the nontrivial solution of the corresponding homogeneous system. That is why this solution was called *a strictly particular solution*

Property 5.4. The strictly particular solution is *the principal solution* [2] of the infinite system, if it exists. This principal solution is obtained when we combine the reduction method with the method of successive approximations whose convergence does not depend on reduction method convergence.

Property 5.5. The trivial solution of the homogeneous Gaussian infinite system is also its strictly particular solution. Hence, we can not obtain the trivial solution of the homogeneous system with the method of reduction in the narrow sense.

Therefore, we will try to solve the homogeneous infinite system (3) with the method of reduction in the broad sense. But before we investigate the homogeneous system, we should say a few words about Properties 1–5. In [3], we proved the important theorems:

Theorem 5.1. *If inhomogeneous Gaussian system (3) has a unique solution, then this solution will certainly be its strictly particular solution, and this solution is given by Cramer's formula;*

Theorem 5.2. *If inhomogeneous Gaussian system (3) is consistent, then its strictly particular solution exists.*

The Property 3 follows directly from the (5.1), because otherwise we have that the determinant of the system is not unique. If we suggest that there is another particular solution and this solution does not contain the nontrivial solution of corresponding homogeneous system as a summand, then we reach a contradiction. It is an actually the proof of (5.2) and Property 2. Thus, (5.2) a is a consequence of (5.1).

We want to say about the proof of (5.2) in [3]. It was found out that the proof is not exactly correct, because in [3] the concept of a principal nontrivial solutions had not been defined yet. Therefore, we gave above the simple proof of it. It may be possible to use the proof of (5.2) in [3] with a few changes as the proof of existence of a principal fundamental solutions, it will be shown below.

5.1. **One-dimensional subspace of the nontrivial solutions of the homogeneous infinite system.**

Theorem 5.3. *If homogeneous Gaussian system (3) has a unique fundamental solution up to an arbitrary multiplier, then this solution is a principal fundamental solution.*

Proof. Let $\{y_i\}_1^\infty$ be a nontrivial solution of the homogeneous system (3), i.e. these numbers satisfy the homogeneous system (3):

$$\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 + ... + a_{1,N}y_N + ... &= 0, \\ a_{2,2}y_2 + a_{2,3}y_3 + a_{2,4}y_4 + ... + a_{2,N}y_N + ... &= 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots&, \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N + ... &= 0, \\ a_{N,N}y_N + ... &= 0, \\ \dots\dots\dots\dots \end{aligned} \tag{37}$$

Next, we truncate the system (37), leaving $N-1$ equations with N unknowns, then we get

$$\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 + ... + a_{1,N-1}y_{N-1} + a_{1,N}y_N &= b_1^N, \\ a_{2,2}y_1 + a_{2,3}y_3 + a_{2,4}y_4 + ... + a_{2,N-1}y_{N-1} + a_{2,N}y_N &= b_2^N, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots&, \\ a_{N-2,N-2}y_{N-2} + a_{N-2,N-1}y_{N-1} + a_{N-2,N}y_N &= b_{N-2}^N, \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N &= b_{N-1}^N, \end{aligned} \tag{38}$$

where $b_j^N = -\sum\limits_{p=N+1}^{\infty} a_{j,p}y_p$, the upper index emphasizes here that all the constant terms in (38) tend to zero as N increases without limit.

We shall explain it. Firstly, because y_i satisfy to the system (37), it follows that $\lim\limits_{N\to\infty} b_j^N = 0$ independently from the fixed j.

Secondly, if N increases, for example by 1, then on the left-hand side of (38) there will be a new component y_{N+1} and a new equation. We will have

$$\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 + ... + a_{1,N}y_N + a_{1,N+1}y_{N+1} &= b_1^{N+1}, \\ a_{2,2}y_1 + a_{2,3}y_3 + a_{2,4}y_4 + ... + a_{2,N}y_N + a_{2,N+1}y_{N+1} &= b_2^{N+1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots &, \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N + a_{N-1,N+1}y_{N+1} &= b_{N-1}^{N+1}, \\ a_{N,N}y_N + a_{N,N+1}y_{N+1} &= b_N^{N+1}, \end{aligned} \tag{39}$$

where $b_j^{N+1} = -\sum\limits_{p=N+2}^{\infty} a_{j,p}y_p$. If N is large enough then it is clear that b_j^N decrease with N.

We use (2.2) to rewrite (16):

$$y_j = B_{N-j} + S_{n-j}B_{N-j-1} - S_{N-j}y_{j+1}, \quad j = \overline{1, N-1}, \tag{40}$$

where B_{N-j} and S_{N-j} are respectively defined by (14) and (15), but in (14) the b_j^N is taken for the b_j, and y_j are the known solutions of homogeneous system (37).

By relation III, B_{N-j} equal to determinant (17), but the b_j^N is taken for the b_j therein. Based on work [3], this is true:

$$\lim_{N\to\infty} B_{N-j} = \begin{vmatrix} \frac{\lim\limits_{N\to\infty} b_j^N}{a_{j,j}} & \frac{\lim\limits_{N\to\infty} b_{j+1}^N}{a_{j+1,j+1}} & \frac{\lim\limits_{N\to\infty} b_{j+2}^N}{a_{j+2,j+2}} & ... & \frac{\lim\limits_{N\to\infty} b_{N-1}^N}{a_{N-1,N-1}} & . \\ \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & ... & 0 & . \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & ... & 0 & . \\ . & . & . & ... & . & . \\ \frac{a_{j,j+j}}{a_{j,j}} & \frac{a_{j+1,j+j}}{a_{j+1,j+1}} & \frac{a_{j+2,j+j}}{a_{j+2,j+2}} & ... & 0 & . \\ . & . & . & ... & . & . \\ \frac{a_{j,N-2}}{a_{j,j}} & \frac{a_{j+1,N-2}}{a_{j+1,j+1}} & \frac{a_{j+2,N-2}}{a_{j+2,j+2}} & ... & 0 & . \\ \frac{a_{j,N-1}}{a_{j,j}} & \frac{a_{j+1,N-1}}{a_{j+1,j+1}} & \frac{a_{j+2,N-1}}{a_{j+2,j+2}} & ... & 1 & . \\ . & . & . & ... & . & . \end{vmatrix}. \tag{41}$$

By the construction, we have $\lim\limits_{N\to\infty} b_j^N = 0$ and this limit does not depend on fixed j. Therefore the infinite determinant in (41) has a zero row on the top. It means that the infinite determinant in (41) exists and is equal to zero, i.e. $\lim\limits_{N\to\infty} B_{N-j} = 0$. We rewrite the (40) in the form

$$y_j - B_{N-j} = S_{n-j}(B_{N-j-1} - y_{j+1}), \quad j = \overline{1, N-1},$$

passing to the limit in it, we have

$$\lim_{N\to\infty}(y_j - B_{N-j}) = \lim_{N\to\infty}[S_{n-j}(B_{N-j-1} - y_{j+1})], \quad j = \overline{1, \infty}. \tag{42}$$

Taking into account $\lim\limits_{N\to\infty} B_{N-j} = 0$, we have $\lim\limits_{N\to\infty}(y_j - B_{N-j}) = y_j$. Then based on (42), we conclude that the numbers $\lim\limits_{N\to\infty} S_{n-j} = S(j)$ exist. It means that the equality $y_j = -S(j)y_{j+1}$ is true, where y_j is a nontrivial solution of the homogeneous system (37). Then based on (4), we get which was to be proved. □

We write the infinite determinant (30) to use the relation V of (3.3)

$$A(j) = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & ... & 0 & 0 & . \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & ... & 0 & 0 & . \\ \frac{a_{j,j+3}}{a_{j,j}} & \frac{a_{j+1,j+3}}{a_{j+1,j+1}} & \frac{a_{j+2,j+3}}{a_{j+2,j+2}} & ... & 0 & 0 & . \\ . & . & . & ... & . & . & . \\ \frac{a_{j,n-2}}{a_{j,j}} & \frac{a_{j+1,n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,n-2}}{a_{j+2,j+2}} & ... & 1 & 0 & . \\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,n-1}}{a_{j+2,j+2}} & ... & \frac{a_{n-2,n-1}}{a_{n-2,n-2}} & 1 & . \\ \frac{a_{j,n}}{a_{j,j}} & \frac{a_{j+1,n}}{a_{j+1,j+1}} & \frac{a_{j+2,n}}{a_{j+2,j+2}} & ... & \frac{a_{n-2,n}}{a_{n-2,n-2}} & \frac{a_{n-1,n}}{a_{n-1,n-1}} & . \\ . & . & . & ... & . & . & . \end{vmatrix}. \tag{43}$$

The infinite determinant (43) plays a significant role in the theory of infinite systems, may be of greater importance than the Cramer's determinant. It hasn't been named before, therefore it is reasonable to name the infinite determinant (43) after its author.

This determinant, for some specific infinite systems, had been presented in [5, 6]. These works were about a sequence of characteristic determinants which are a leading principal minors of determinant (43).

Definition 5.1. *The infinite determinant (43) obtained by elimination of the first row of generalized Cramer's determinant (17) when it is an infinite, we call a Fedorov's determinant.*

We will show bellow what role Fedorov's determinant plays in the theory of the homogeneous systems. Here, we shall note, that this determinant is also essential in the theory of nonhomogeneous systems. Indeed, the strictly particular solution x_j of inhomogeneous systems is defined by the leading principal minors $A_p(j)$ (see $(18')$) of Fedorov's determinant. And besides, the elements of the inverse matrix of inhomogeneous systems (3) are the leading principal minors of Fedorov's determinant (see $(21')$).

It is important to note that the (5.1) of Fedorov's determinant is a formal, i.e. the Fedorov's determinant may not exist. The finite determinants $A_{n-j}(j)$ that are defined in (30) are of great importance, because they are a leading principal minors of Fedorov's determinant. Actually, the existence of ratio limit $\frac{A_{n-j}(j)}{A_n(0)}$ plays an important role, as the next theorem indicates.

Theorem 5.4. *Let the homogeneous Gaussian system (3) has the principal fundamental solution $\{x_j\}_0^\infty$. Then this solution is expressed by Fedorov's formula*

$$x_j = (-1)^j x_0 \lim_{n\to\infty} \left(\frac{A_{n-j}(j)}{A_n(0)} \right), \quad j = 0, 1, ..., \infty, \tag{44}$$

where $A_{n-j}(j)$ are the leading principal minors of Fedorov's determinant and x_0 is an arbitrary real number. This solution is the unique fundamental solution which expressed by Fedorov's formula.

Proof. We pass to the limit in the relation V of (3.3). Under the conditions of (5.4) and (4) we have

$$\lim_{n\to\infty} \overset{n+1}{x_j} = x_j = \lim_{n\to\infty} \frac{(-1)^j x_0 A_{n-j}(j)}{A_n(0)}, \quad j = \overline{0, \infty}.$$

The uniqueness of such solution follows from the uniqueness of an infinite determinants $A(j)$ and $A(0)$. $\square$

Corollary 5.1. *Let Fedorov's determinants $A(j)$ exist for any fixed j. Then the Fedorov's formula (44) is of the form*

$$x_j = \lim_{n\to\infty} \frac{(-1)^j x_0 A_{n-j}(j)}{A_n(0)} = \frac{(-1)^j x_0 A(j)}{A(0)}, \quad j=\overline{0,\infty},$$

where $A(j)$ is the Fedorov's determinant and $A(0)$ is the initial Fedorov's determinant.

5.2. **The existence of a principal fundamental solution of the homogeneous infinite system.**

Theorem 5.5. *Let the inhomogeneous Gaussian system (3) has the nonunique solution. Then, the principal fundamental solution $\{x_j\}_0^\infty$ of its corresponding homogeneous system exists.*

Proof. Let $\{y_i\}_0^\infty$ be some particular solution of the homogeneous Gaussian system (3). Under the condition of theorem we can write this solution as a sum of the strictly particular solution of the homogeneous system (3) and some nontrivial solution of the corresponding homogeneous system (3). We can leave out without loss of generality only one nontrivial solution up to a constant multiplier, because the other summands may be equated to zero. Therefore, we use the method of reduction in the broad sense, and then let us seize the approach proposed in the proof of the (5.3). We have the solution $\{y_i\}_0^\infty$, it satisfies the system (3). Let's rewrite this system as follows

$$\begin{aligned} a_{0,0}y_0 + a_{0,1}y_1 + a_{0,2}y_2 + a_{0,3}y_3 + ... + a_{0,N}y_N &= b_0 - b_0^N = \overline{b}_0^N, \\ a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + ... + a_{1,N}y_N &= b_1 - b_1^N = \overline{b}_1^N, \\ \dots\dots\dots\dots\dots\dots\dots\dots &\dots\dots\dots, \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N &= b_{N-1} - b_{N-1}^N = \overline{b}_{N-1}^N, \\ \dots\dots\dots\dots\dots\dots &\dots\dots\dots, \end{aligned} \tag{45}$$

where $b_j^N = \sum\limits_{p=N+1}^{\infty} a_{j,p}y_p$.

We note that because y_i satisfies the system (45), it follows that $\lim\limits_{N\to\infty} b_j^N = 0$ independently from the fixed j. Next, we truncate the system (45), leaving N equations with $N+1$ unknowns, then we get a relations like in (40) and passing to the limit in it, we get

$$y_j = \lim_{N\to\infty} \overset{N}{y}_j = \lim_{N\to\infty} B_{N-j} + \lim_{N\to\infty} S_{N-j}(B_{N-j-1} - \overset{N}{y}_{j+1}),\ j=\overline{0,\infty}. \tag{46}$$

By the construction, the $\lim\limits_{N\to\infty} B_{N-j} = B(j)$ is the strictly particular solution. Next, we introduce the notation $x_j = y_j - B(j)$, then $x_j = -\lim\limits_{N\to\infty} S_{N-j}$ follows from (46). It means that $\lim\limits_{N\to\infty} S_{N-j} = S(j)$, i.e. based on (4), the $S(j)$ are forming the characteristic numbers. And solving the last recurrence equation, we get the principal fundamental solution

$$x_j = \frac{(-1)^j x_0}{\prod\limits_{k=0}^{j-1} S(k)}.$$

Theorem is proved completely. □

5.3. **The numerical example.** Consider the following homogeneous Gaussian infinite system:

$$\sum_{p=0}^{\infty} \frac{(2j+2p+1)!}{(2p+1)!} x_{j+p} = 0, \quad j = 0, 1, 2, \dots \tag{47}$$

In fact, the system (47) is a periodic infinite system, that is, the coefficients of the matrix of this system satisfy the condition

$$a_{j,j+p} = a_p a_{j+p,j+p} = \frac{1}{(2p+1)!}(2j+2p+1)!. \tag{48}$$

Significantly using the periodicity of system (47), following [3,4], we can analytically find an independent solutions of (48), that is, fundamental solutions:

$$x_j = C_k \frac{(-1)^j \pi^{2j} k^{2j}}{(2j+1)!}, \quad k = 1, 2, ..., \; j = 0, 1, 2, ...,$$

where $C_k = const$.

It follows that the principal fundamental solution ($k = 1$) will be

$$x_j = \frac{(-1)^j \pi^{2j}}{(2j+1)!}, \quad j = 0, 1, 2, \dots \tag{49}$$

Put in (44) $x_0 = 1$, and compare the exact solution (49) with the calculations by the formula (44) for $n = 30$. The comparison results are given in the table

	x_0	x_1	x_2	...	x_9
(44)	1.0	-1.644915	0.811696	...	-0.0
(49)	1.0	-1.644934	0.811742	...	-0.0

The results of the approximate solution almost coincided. Thus, the formula (44) gives the principal fundamental solution, and it confirms the Theorem (5.4).

6. Conclusion

An analytical solution is obtained for a homogeneous finite algebraic system of any order n using the reduction method in the broad sense. The passage to the limit from a finite homogeneous systems solution to an infinite homogeneous systems solution has been carried out. At the same time, Fedorov's formula for solving a homogeneous system was proposed, similar to Cramer's formula for solving a nonhomogeneous system.

The concept of the generalized infinite Fedorov's determinant is introduced.

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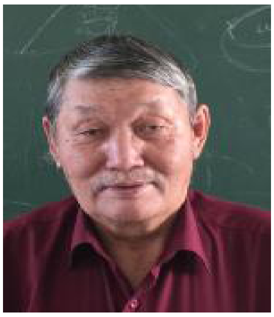

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