REPUBLIC OF AZERBAIJAN

On the rights of the manuscript

ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

STUDY OF BULLING OUT AND RELEASE OF ENERGY OF A THREE-LAYER PIEZOELECTRIC+METAL+PIEZOELECTRIC CIRCULAR PLATE WITH INTERLAYER CRACKS

Specialty:	2002.01 - Mechanics of deformable solid
Field of science:	Mathematics
Applicant:	**Fazila Ilyas kizi Cafarova**

Baku – 2021

The work was performed at the department of "General technical subjects" of the Faculty of Physics and Technical Subjects of Ganja State University.

Scientific supervisor: Doc. of Phy.-Math. sc., professor
Orujali Huseyngulu oglu Rzayev

Official opponents: Doc. of Sc. in Math., professor
Ramiz Aziz oglu Iskenderov

PhD in Phy.-Math. Sc., ass. professor
Khalid Binnet oglu Mamedov

PhD in math., ass. professor
Famil Alizade oglu Seifullayev

Dissertation Council FD 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University

Chairman of the Dissertation Council:
Full member of ANAS, Doc. of Ph.-Math. sc., professor

___________ Magomed Farman oglu Mekhdiyev

Scientific secretary of the dissertation council:
Doc. of Sc. in Mech., ass. professor

___________ Laura Faik kizi Fatullayeva

Chairman of the scientific seminar:
Doc. of Sc. in Math., professor

___________ Natik Garakishi oglu Akhmedov

GENERAL DESCRIPTION OF WORK

Relevance and degree of study of the topic. The development of technology in various leading branches of modern industry, such as aircraft construction, mechanical engineering, shipbuilding, as well as in the military industry, urgently requires the use of composite materials, including layered materials containing layers (or components) of piezoelectric materials. In the world scientific literature, piezoelectric materials are denoted by PZT.

Successful applications of these laminates, as well as structural elements made of these materials, require a thorough theoretical study of their delamination and destruction. The presented dissertation work is devoted to the theoretical study of the problems of delamination-buckling, as well as the destruction of a three-layer PZT + metal + PZT circular plate with interphase circular cracks.

Typically, thin films or PZT plates are used as sensors or actuators to monitor the performance of structural elements such as plates and shells made from traditional materials. When these sensors and actuators are glued, non-glued zones often appear on the surface of the carrier material. Namely, these non-glued zones are the centers of local buckling (or loss of stability) of the PZT layers under the action of the corresponding electromechanical compressive forces. Consequently, appropriate theoretical research is needed to control and prevent the noted local buckling. However, until recently, such studies were completely absent, despite the fact that the problems of stability of plates and rods made of PZT materials attracted the attention of researchers. These researchers found that the piezoelectricity of the material of the plates and rods increases the values of the corresponding compressive critical forces. There are also a number of other studies related to the dynamics, statics and stability of systems consisting of PZT and elastic components.

All studies performed for circular cracks in piezoelectric materials were performed under two assumptions:

- the layers containing the crack are infinite in the radial direction;

- types and levels of external electro-mechanical loading allow the use of the classical linear theory of electro-elasticity for piezoelectric materials in the study of relevant problems.

It is known that the classical linear theory of electro-elasticity for piezoelectric materials cannot take into account the influence of external forces acting in a direction parallel to the plane on which the crack is located on the values of the stress intensity factor (SIF) and energy release rate (ERR) at the crack front. We note that the corresponding approach, which allows one to take into account the influence of the above forces on the values of the SIF and ERR for purely elastic materials, was investigated within the framework of certain distinctions.

It should be noted that this approach is based on the linearization of geometrically nonlinear equations. In linearization, the stress state caused by external forces and acting in a direction parallel to the plane on which the cracks are located is taken as the initial stresses, and the stress state caused by the forces acting on the crack surface is taken as an additional stress state. The above linearization is carried out with respect to the values of this additional state. For the validity of the noted linearization, it is assumed that the values of the forces causing the initial state are significantly greater than the values of the forces causing the additional stress state. As a result of the described linearization, a so-called three-dimensional linearized equation is obtained, which has coefficients that contain a force acting in a direction parallel to the plane on which the crack is located. With the help of these coefficients, it becomes possible to take into account the influence of external forces acting in a direction parallel to the plane of the location of the cracks on the magnitude of the recovery factor and ERR at the fronts of the cracks.

Despite many studies in this area, there are still no studies on the effect of initial stresses in the above sense on the SIF and ERR at the front of a circular crack in a piezoelectric material or on the interface plane between piezoelectric and elastic materials.

In the dissertation work, questions related to the problems of destruction are investigated, i.e. to the problems of interfacial circular cracks in a three-layer "PZT + elastic + PZT" circular plate. Namely, a problem is considered for determining the electro-mechanical energies of the specified plate with interfacial circular cracks, and also, a problem is considered for determining the rate of energy release (i.e. ERR-Energy Rebeate Rate: Energy Release Rate (ERR)) at the crack front.

Object and subject of research. In the presented dissertation work, the ERR is investigated at the front of circular interfacial cracks located in a three-layer "PZT + elastic + PZT" circular plate. Here, in addition, the influence of the radial compressive or tensile initial stress on the value of the ERR is investigated.

The purpose and objectives of the study. The purpose of the thesis is to determine the electro-mechanical energies of a three-layer "PZT + elastic + PZT" circular plate with interfacial circular cracks, as well as to determine the rate of energy release at the crack front.

In the dissertation work, two types of problems are considered for a three-layer "PZT + elastic + PZT" circular plate with interphase circular cracks using geometrically nonlinear equations of the theory of electro-elasticity for piezoelectric materials in the framework of a piecewise-homogeneous body model. These tasks are as follows:

- research on delamination-buckling of face PZT layers in the vicinity of interfacial cracks and study of the effect of mutual influence of electric and mechanical fields on the value of the corresponding critical parameters;
- investigation of ERR at the front of an interphase crack and determination of the influence of the initial stress, as well as determination of the effect of mutual influence of electric and mechanical fields on the value of the SIF and ERR.

Research method. The finite element method (FEM), which is widely used in solving fundamental problems of the theory of elasticity, is used to solve the problems posed.

The main provisions for the defense.

a) statement of the problem, development and application of the numerical finite element method (FEM) for solving the corresponding boundary value problems;

b) development of criteria for determining critical parameters related to local buckling of the face layers in the vicinity of circular cracks;

c) development of algorithms for determining the influence of initial stresses on the value of ERR at the front of interphase circular cracks in the case of mode I;

d) carrying out specific numerical studies on the influence of the parameters of the problem on the value of critical efforts, as well as on the value of the ERR;

e) analysis of the obtained numerical results and identification of new effects emanating from the mutual influence of electric and mechanical fields, as well as from the anisotropy of the piezoelectric face layers.

Scientific novelty of the research. Scientific novelty and significance of the results of the work are:

- in the formulation of problems on the study of local stratification-buckling of the face layers in the vicinity of interphase circular cracks, as well as the ERR at the front of these cracks for mode I;
- in the development of FEM for the numerical solution of the corresponding boundary value problems of the theory of electro-elasticity for piezoelectric materials;
- in the implementation of FEM in a software package for obtaining numerical results;
- in the establishment of some electromechanical effects associated with the electromechanical properties of piezoelectric materials, with boundary conditions related to the magnitude of the electric field, with the geometric parameters of the problems.

All the considered problems of a three-layer "PZT+elastic+PZT" circular plate with two interphase circular cracks were solved for the first time within the framework of a piecewise-homogeneous body model using exact three-dimensional geometrically nonlinear equations of the theory of electro-elasticity for piezoelectric materials in the case of axisymmetric deformation.

The reliability of the results and conclusions obtained is confirmed by:

- the use of precise three-dimensional geometrically nonlinear equations of the theory of electro-elasticity for piezoelectric materials with the use of a piecewise-homogeneous model in the formulation of the problem and the formulation of problems;
- the correctness of the formulation of problems, the accuracy of the variational formulations of these problems when using the FEM;
- the consistency of the numerical results obtained with each other, physical considerations, a fairly good agreement with the known results of other authors in particular cases.

The theoretical and practical significance of the study. Investigations based on a model of a piecewise homogeneous body using exact three-dimensional geometrically nonlinear equations of the theory of electro-elasticity for piezoelectric materials, solving a number of problems for delamination-buckling, as well as for the destruction of a three-layer "PZT + elastic + PZT" circular plate with two interphase circular cracks are the theoretical relevance of the thesis. The results obtained in this work can be applied in structures in which piezoelectric materials with interfacial circular cracks are used.

Approbation and application of the work. The main provisions of the dissertation work were regularly reported at the seminars of the Department of General Technical Subjects of the Faculty of Physics and Technical Subjects of Ganja State University, as well as at the following scientific conferences: Turkish Physical Society, 32nd International Physics Congress (06-09 Sep. 2016, Bodrum, Turkey), Actual problems of modern natural and economic sciences. International Scientific Conference, (04-05 May 2018, Ganja).

Personal contribution of the author. In the works (articles), written in co-authorship with S.D. Akbarov and N. Yakhnioglu, the author solved the set problems using the finite element method, and discussed the results. In works written in collaboration with O. G.

Rzayev, the author is responsible for setting the problem, obtaining numerical results and analyzing them.

Author's publications. The main results of the dissertation were published in 9 papers, including 6 articles, 3 abstracts, and they are listed at the end of the abstract.

The name of the institution where the work was done. The dissertation work was carried out in the department of "General technical subjects" of the Faculty of Physics and Technical Subjects of Ganja State University.

The total volume of the thesis with an indication of the volume of the structural units of the thesis separately. The total volume of the thesis is 200087 characters (title page - 432 characters, content - 2666 characters, introduction - 26332 characters, first chapter - 48000 characters, second chapter - 72000 characters, third chapter - 46000 characters, conclusion - 4657 characters). The dissertation work contains a list of used literature 71 titles, 21 figures and 16 tables.

The work consists of an introduction, three chapters, a conclusion and a list of used literature.

CONTENT OF THE DISSERTATION

The introduction defines the purpose and relevance of the dissertation work, gives a brief overview of works on the topic of the problems under consideration.

The first chapter is devoted to some basic relations of the geometrically nonlinear theory of electro-elasticity in arbitrary curvilinear coordinates at small deformations. By small deformations we mean a version of the theory, when the elongations and shears are small values in comparison with unity and they can be neglected. In addition, according to this provision, we will not take into account the change in elongations, areas and volumes when writing the equilibrium equation and boundary conditions in forces and in electrical displacements. However, when writing the equilibrium equations and boundary conditions in forces and in electrical displacements, we will take into account changes in the orientations (directions) of elementary material fibers due to deformations. At the same time, we will not impose any restrictions on the angles of rotation of material fibers. All considered relationships will be rewritten in cylindrical coordinates. Moreover, *the first five paragraphs* do not mention the presence of an electric field, although the relations and equations given in these paragraphs remain valid for the theory of electro-elasticity for piezoelectric materials.

Section 1.6 provides some necessary information on the equations of electric field theory and on the relationships between mechanical and electric fields related to piezoelectric materials. If we connect the ends of a rod made of piezoelectric material into a voltmeter and begin to squeeze this rod, then the voltmeter needle will move from the zero point and will show a certain voltage, which corresponds to the electric field bridged due to mechanical compression of the piezoelectric rod. In addition, if we load the ends of a piezoelectric rod with electric charges, then a certain mechanical movement will appear (reverse piezoelectric effect).

Note that piezoelectric materials are mainly divided into two groups: polymer piezoelectrics and ceramic piezoelectrics. Ceramic piezoelectrics are brittle materials and deformations in them are elastic, almost until the moment of destruction. However, polymer piezoelectrics have viscoelastic properties.

One of the main properties of piezoelectric materials is their anisotropy. In other words, the mechanical and electrical properties of these materials depend on direction. Instead, piezoelectric materials have continuous dipoles and can be taken as ionized crystals that do not have central symmetry.

Currently, piezoelectric materials used in technology and industry are made from a mixture of "Lead + Zirconium (zirkonium) + titanium (titanium)". Since "lead" is spelled "Plumbo" in Greek, such piezoelectric materials are called PZT, i.e. with the first letters of the name of the materials included in the mixture. PZT materials differ from each other in the amount of zirconium and titanium in the composition of the mixture and are renamed as PZT-2, PZT-4, PZT-5, etc.

In addition to these, crystals that do not have central symmetry in the crystal lattice are used to reveal piezoelectric effects, since Barium Titanat (BaTiO3), Kuartz (SiO2), Cinko Oksid (ZnO) and others.

The second chapter is devoted to the study of delamination – buckling near interlayer circular cracks of a three-layer PZT + METAL + PZT circular plate. Here the axisymmetric case is investigated and it is assumed that the plate is compressed in the radial direction by uniformly distributed normal forces acting on the lateral surface of the plate-disk. The statement of the problem is written in the framework of the model of a piecewise homogeneous body with the use of geometrically nonlinear equations of the theory of electroelasticity for piezoelectric materials. The formulated problems are solved using the boundary shape perturbation method, with the help of which the nonlinear problem is reduced to the corresponding linearized problems. Moreover, these linearized problems are solved numerically using the Finite Element Method (FEM). Numerical results are analyzed on the critical values of the compressive forces for various types of PZT and metals from elastic materials and for various values of geometric parameters characterizing the ratio of the crack radius to the radius of a circular disk, etc.

Section 2.1 gives the problem statement. Consider a three-layer circular disc with the geometry shown in Fig. 1.a and for generality, we assume that the material of all layers of the disk (or plate) are piezoelectric. In addition, we assume that the materials of the face layers of the plates are the same and there are circular cracks between the middle and face layers, the location of which is shown in Fig. 1.b.

We connect the cylindrical coordinate system $Or\theta z$ with the lower face planes of the plate (Fig. 1) and determine the position of the points of this plate through the Lagrangian coordinates in this coordinate system. Thus, according to Fig. 1, the plate occupies the $\{0 \le r \le l/2,\ 0 \le \theta \le 2\pi;\ 0 \le z \le h\}$ area and the circular cracks are located in the $\{z = h_F \pm 0,\ 0 \le r \le l_0/2\}$ and $\{z = h_c + h_F \pm 0,\ 0 \le r \le l_0/2\}$ intervals, where $l-$ is the disk diameter, $h-$ is the disk thickness, $l_0 -$ is the diameter of the circular crack, $h_F -$ is the thickness of the face layer, and the $h_c -$ is the thickness of the middle layer.

In addition, let us assume that in the natural state of the crack surface there is a very slight and axisymmetric initial imperfection (or buckling). In fig. 1b, the upper and lower surfaces of the upper fracture are indicated through S_u^+ and S_u^-, and the same surfaces for the lower fracture are indicated through S_L^+ and S_L^- respectively.

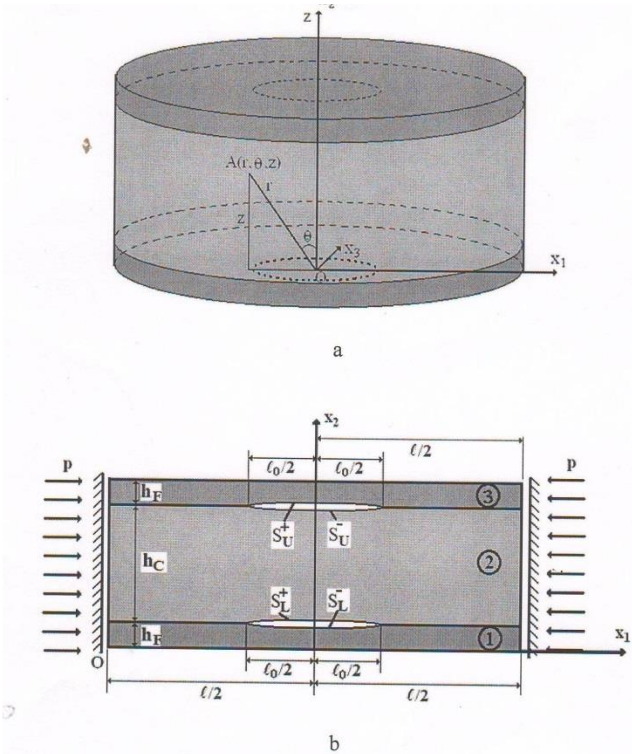
The equation of these surfaces can be written as follows:

$$z = h_F \pm \varepsilon f(r) \quad \text{для } S_L^{\pm},$$

$$z = h_c + h_F \pm \varepsilon f(r) \quad \text{для } S_u^{\pm}, \tag{1}$$

where ε $(0 < \varepsilon << 1)$ is a dimensionless small parameter that characterizes the degree of initial imperfection of fracture surfaces, $f(r)$ is a function that characterizes the shape of the initial imperfection.

Let us assume that the considered three-layer plate is compressed along the lateral surface of this plate with uniformly distributed radial normal forces with the intensity P. Let us try to study the development of the above initial imperfection of the crack surfaces with an increase in the P value, and this study is carried out using the geometrically nonlinear equations of the theory of electro-elasticity for piezoelectricc materials.



Pic. 1. Geometry of plates and interfacial cracks.

So, let us write this equation and, at the same time, with superscripts (3) and (1) we denote the quantities related to the upper and lower layers, respectively, and the superscript (2) denotes the quantities related to the middle layer. Note that since it is assumed that the geometry of the plate and the initial imperfection of the crack surfaces, as well as the distribution of external forces have symmetry about the Oz axis, therefore, all subsequent research will be carried out for this axisymmetric case.

Thus, we write down the complete system of equations of the theory of electro-elasticity for piezoelectric materials in a geometrically nonlinear formulation in the axisymmetric case.

Equilibrium equation:

$$\frac{\partial t_{rr}^{(k)}}{\partial r}+\frac{\partial t_{zr}^{(k)}}{\partial z}+\frac{1}{r}\left(t_{rr}^{(k)}-t_{\theta\theta}^{(k)}\right)=0,\ \frac{\partial t_{rz}^{(k)}}{\partial r}+\frac{\partial t_{zz}^{(k)}}{\partial z}+\frac{1}{r}\left(t_{rz}^{(k)}\right)=0,$$

$$\frac{\partial D_R^{(k)}}{\partial r}+\frac{1}{r}D_R^{(k)}+\frac{\partial D_Z^{(k)}}{\partial z}=0. \tag{2}$$

where

$$t_{rr}^{(k)}=\sigma_{rr}^{(k)}\left(1+\frac{\partial u_r^{(k)}}{\partial r}\right)+\sigma_{rz}^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+M_{rr}^{(k)},$$

$$t_{\theta\theta}^{(k)}=\sigma_{\theta\theta}^{(k)}\left(1+\frac{u_r^{(k)}}{r}\right)+M_{\theta\theta}^{(k)},$$

$$t_{zr}^{(k)}=\sigma_{zr}^{(k)}\left(1+\frac{\partial u_r^{(k)}}{\partial r}\right)+\sigma_{zz}^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+M_{zr}^{(k)},$$

$$t_{rz}^{(k)}=\sigma_{rr}^{(k)}\frac{\partial u_z^{(k)}}{\partial r}+\sigma_{rz}^{(k)}\left(1+\frac{\partial u_z^{(k)}}{\partial r}\right)+M_{rz}^{(k)},$$

$$t_{zz}^{(k)}=\sigma_{zz}^{(k)}\left(1+\frac{\partial u_z^{(k)}}{\partial z}\right)+\sigma_{rz}^{(k)}\frac{\partial u_z^{(k)}}{\partial r}+M_{zz}^{(k)},$$

$$D_R^{(k)}=\left(1+\frac{\partial u_r^{(k)}}{\partial r}\right)D_r^{(k)}+\frac{\partial u_r^{(k)}}{\partial z}D_z^{(k)},$$

$$D_Z^{(k)}=\frac{\partial u_z^{(k)}}{\partial r}D_r^{(k)}+\left(1+\frac{\partial u_z^{(k)}}{\partial z}\right)D_z^{(k)}. \tag{3}$$

Electromechanical relationships for piezoelectric materials:

$$\sigma_{rr}^{(k)}=c_{1111}^{(k)}s_{rr}^{(k)}+c_{1122}^{(k)}s_{\theta\theta}^{(k)}+c_{1133}^{(k)}s_{zz}^{(k)}-e_{111}^{(k)}E_r^{(k)}-e_{311}^{(k)}E_z^{(k)},$$

$$\sigma_{\theta\theta}^{(k)}=c_{2211}^{(k)}s_{rr}^{(k)}+c_{2222}^{(k)}s_{\theta\theta}^{(k)}+c_{2233}^{(k)}s_{zz}^{(k)}-e_{122}^{(k)}E_r^{(k)}-e_{322}^{(k)}E_z^{(k)},$$

$$\sigma_{zz}^{(k)} = c_{3311}^{(k)} s_{rr}^{(k)} + c_{3322}^{(k)} s_{\theta\theta}^{(k)} + c_{3333}^{(k)} s_{zz}^{(k)} - e_{133}^{(k)} E_r^{(k)} - e_{333}^{(k)} E_z^{(k)},$$

$$\sigma_{rz}^{(k)} = c_{13211}^{(k)} s_{rz}^{(k)} - e_{113}^{(k)} E_r^{(k)} - e_{313}^{(k)} E_z^{(k)},$$

$$D_r^{(k)} = e_{111}^{(k)} s_{rr}^{(k)} + c_{122}^{(k)} s_{\theta\theta}^{(k)} + e_{133}^{(k)} s_{zz}^{(k)} + \varepsilon_{11}^{(k)} E_r^{(k)} + \varepsilon_{13}^{(k)} E_z^{(k)},$$

$$D_z^{(k)} = e_{311}^{(k)} s_{rr}^{(k)} + e_{322}^{(k)} s_{\theta\theta}^{(k)} + e_{333}^{(k)} s_{zz}^{(k)} + \varepsilon_{31}^{(k)} E_r^{(k)} + \varepsilon_{33}^{(k)} E_z^{(k)},$$

$$M_{rr}^{(k)} = \varepsilon_0^{(k)} \left(E_r^{(k)} E_r^{(k)} - \frac{1}{2} E^{(k)^2} \right),$$

$$M_{\theta\theta}^{(k)} = \varepsilon_0^{(k)} \left(\frac{1}{2} E^{(k)^2} \right), \quad M_{zz}^{(k)} = \varepsilon_0^{(k)} \left(E_z^{(k)} E_z^{(k)} - \frac{1}{2} E^{(k)^2} \right),$$

$$M_{rz}^{(k)} = M_{zr}^{(k)} = \varepsilon_0^{(k)} \left(E_r^{(k)} E_z^{(k)} \right), \quad E^{(k)^2} = \left(E_r^{(k)} \right)^2 + \left(E_z^{(k)} \right)^2,$$

$$E_r^{(k)} = -\frac{\partial \phi^{(k)}}{\partial \tau}, \quad E_z^{(k)} = -\frac{\partial \phi^{(k)}}{\partial z}. \tag{4}$$

Relationship between deformations and displacements:

$$s_{rr}^{(k)} = \frac{\partial u_r^{(k)}}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r^{(k)}}{\partial r} \right)^2 + \frac{1}{2}\left(\frac{\partial u_z^{(k)}}{\partial r} \right)^2, \quad s_{\theta\theta}^{(k)} = \frac{u_r^{(k)}}{r} + \frac{1}{2}\left(\frac{u_r^{(k)}}{r} \right)^2,$$

$$s_{zz}^{(k)} = \frac{\partial u_z^{(k)}}{\partial z} + \frac{1}{2}\left(\frac{\partial u_z^{(k)}}{\partial z} \right)^2 + \frac{1}{2}\left(\frac{\partial u_r^{(k)}}{\partial z} \right)^2,$$

$$s_{rz}^{(k)} = \frac{1}{2}\left(\frac{\partial u_r^{(k)}}{\partial z} + \frac{\partial u_z^{(k)}}{\partial r} + \frac{\partial u_r^{(k)}}{\partial z}\frac{\partial u_r^{(k)}}{\partial r} + \frac{\partial u_z^{(k)}}{\partial z}\frac{\partial u_z^{(k)}}{\partial r} \right). \tag{5}$$

It should be noted that when obtaining the corresponding equations of the classically linear theory of electro-elasticity for piezoelectric materials, the following two assumptions are taken into account: (I) the difference between the areas of elementary surfaces before and after deformations, as well as between the volume of elementary elements before and after deformations ; (II) the rotation of "materialized" basis vectors due to deformation is not taken into account. However, when deriving equations (2) - (5), it is assumed that the deformations are so small that (I) the assumption remains valid, however, the (II) assumption is abandoned. In other words, when writing equations (2) - (5), the difference between the directions of the "materialized" basis vectors before and after deformations is taken into account and the rejection of (II) assumptions means taking into account the rotations of the basis vectors caused by deformation when determining mechanical stresses, electrical displacements and recording the equations of fields, as well as when recording the boundary conditions with respect to the efforts.

Then the boundary and contact conditions are formulated. In this case, with respect to the surfaces of the cracks, we can write the following boundary conditions:

$$t_{rr}^{(2)}\Big|_{S_u^+} n_r^+ + t_{zz}^{(2)}\Big|_{S_u^-} n_z^- = 0,\; t_{rr}^{(2)}\Big|_{S_L^+} n_r^+ + t_{zr}^{(2)}\Big|_{S_L^+} n_z^+ = 0,$$

$$t_{rz}^{(2)}\Big|_{S_L^+} n_r^+ + t_{zz}^{(2)}\Big|_{S_L^+} n_z^+ = 0, \tag{6}$$

$$t_{rr}^{(1)}\Big|_{S_L^-} n_r^- + t_{zr}^{(1)}\Big|_{S_L^-} n_z^- = 0,\; t_{rz}^{(1)}\Big|_{S_L^-} n_r^- + t_{zz}^{(1)}\Big|_{S_L^-} n_z^- = 0.$$

Note that conditions (6) are satisfied for $0 \le r \le l_0 / 2$ however, for $l_0 / 2 \le r \le l / 2$ the following contact conditions hold:

$$t_{zz}^{(3)}\Big|_{z=h_F+h_C} = t_{zz}^{(2)}\Big|_{z=h_F+h_C}, \qquad t_{zr}^{(3)}\Big|_{z=h_F+h_C} = t_{zr}^{(2)}\Big|_{z=h_F+h_C},$$

$$u_z^{(3)}\Big|_{z=h_F+h_C} = u_z^{(2)}\Big|_{z=h_F+h_C}, \qquad u_r^{(3)}\Big|_{z=h_F+h_C} = u_r^{(2)}\Big|_{z=h_F+h_C},$$

$$t_{zz}^{(2)}\Big|_{z=h_F} = t_{zz}^{(1)}\Big|_{z=h_F}, \qquad t_{zr}^{(2)}\Big|_{z=h_F} = t_{zr}^{(2)}\Big|_{z=h_F},$$

$$u_z^{(2)}\Big|_{z=h_F} = u_z^{(1)}\Big|_{z=h_F}, \qquad u_r^{(2)}\Big|_{z=h_F} u_r^{(1)}\Big|_{z=h_F}. \tag{7}$$

Section 2.2. a method for solving the problem is presented, while the solution includes several stages of mathematical procedures. First, the required values are presented in the form of a series in a small parameter and the equations for each approximation are obtained, the values of the zero approximation and the values related to the first approximation are determined.

To solve the problem formulated in the previous section, we apply the method developed in the monograph[1] for purely elastic and viscoelastic materials. According to this method, all the sought values are represented as a series in the small parameter $\Re$, which is included in equation (1) and characterizes the degree of imperfection of the fracture surfaces:

$$\left\{\sigma_{rr}^{(k)},\ldots,u_r^{(k)},\ldots,D_r^{(k)},\ldots,\phi^{(k)}\right\}= \\ =\sum_{n=0}^{\infty}\varepsilon^n\left\{\sigma_{rr}^{(k),n},\ldots,u_r^{(k),n},\ldots,D_r^{(k),n},\ldots,\phi^{(k),n}\right\} \tag{8}$$

Obtaining an expression for the $n_r^{\pm}$ and $n_z^{\pm}$ components of the normal vector to the fracture surface, represented by the equation in (1) and presenting the same expressions as a series in the small ε parameter, then substituting (8) into the nonlinear equation and the ratio given in the previous paragraph, producing cumbersome mathematical transformations and developments, we obtain the corresponding equation and ratio for each approximation in (8) separately. Here, we write these equations and relations only for the zero and first approximations when writing the equations related to the zero approximation, we neglect the nonlinearity in them. So we write these equations and relations for the zero approximation:

Equilibrium equations:

$$\frac{\partial\sigma_{rr}^{(k),0}}{\partial r}+\frac{\partial\sigma_{zr}^{(k),0}}{\partial z}+\frac{1}{r}\left(\sigma_{rr}^{(k),0}-\sigma_{\theta\theta}^{(k),0}\right)=0,$$

$$\frac{\partial\sigma_{rz}^{(k),0}}{\partial r}+\frac{\partial\sigma_{zz}^{(k),0}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(k),0}=0,$$

$$\frac{\partial D_r^{(k),0}}{\partial r}+\frac{1}{r}D_r^{(k),0}+\frac{\partial D_z^{(k),0}}{\partial z}=0 \tag{9}$$

Relationship between deformations and mechanical movements:

---

[1] Akbarov S.D. Stability loss and buckling delamination: Three-dimensional linearized approach for elastic and viscoelastic composites. New York: Springer, 2013, 448p.

$$s_{rr}^{(k),0}=\frac{\partial u_r^{(k),0}}{\partial r},\quad s_{\theta\theta}^{(k),0}=\frac{u_r^{(k),0}}{r},\quad s_{zz}^{(k),0}=\frac{\partial u_z^{(k),0}}{\partial z},$$

$$s_{rz}^{(k),0}=\frac{1}{2}\left(\frac{\partial u_r^{(k),0}}{\partial z}+\frac{\partial u_z^{(k),0}}{\partial r}\right), \tag{10}$$

Boundary conditions on crack surfaces:

$$\sigma_{zr}^{(3),0}\Big|_{z=h_F+h_C}=0,\ \sigma_{zz}^{(3),0}\Big|_{z=h_F+h_C}=0,\ \sigma_{zr}^{(2),0}\Big|_{z=h_F+h_C}=0,$$

$$\sigma_{zz}^{(2),0}\Big|_{z=h_F+h_C}=0,\ \sigma_{zr}^{(2),0}\Big|_{z=h_F}=0,\ \sigma_{zr}^{(1),0}\Big|_{z=h_F}=0,\ \sigma_{zz}^{(1),0}\Big|_{z=h_F}=0,$$

$$\text{with}\quad 0\le r\le l_0/2. \tag{11}$$

Contact conditions between layers of plates:

$$\sigma_{zz}^{(3),0}\Big|_{z=h_F+h_C}=\sigma_{zz}^{(2),0}\Big|_{z=h_F+h_C},\ \sigma_{zr}^{(3),0}\Big|_{z=h_F+h_C}=\sigma_{zr}^{(2),0}\Big|_{z=h_F+h_C},$$

$$u_z^{(3),0}\Big|_{z=h_F+h_C}=u_z^{(2),0}\Big|_{z=h_F+h_C},\ u_r^{(3),0}\Big|_{z=h_F+h_C}=u_r^{(2),0}\Big|_{z=h_F+h_C},$$

$$\sigma_{zz}^{(2),0}\Big|_{z=h_F}=\sigma_{zz}^{(1),0}\Big|_{z=h_F},\ \sigma_{zr}^{(2),0}\Big|_{z=h_F}=\sigma_{zr}^{(1),0}\Big|_{z=h_F},$$

$$u_z^{(2),0}\Big|_{z=h_F}=u_z^{(1),0}\Big|_{z=h_F},\ u_r^{(2),0}\Big|_{z=h_F}=u_r^{(1),0}\Big|_{z=h_F},$$

$$\text{with}\quad l_0/2\le r\le l/2. \tag{12}$$

Then the boundary conditions are given on the front planes of the face layers, the boundary conditions on the cylindrical lateral surface, the boundary conditions on the surface of the cracks with respect to the electric displacement or with respect to the electric potential, the contact conditions for the electric displacement and the electric potential, the boundary conditions for the face planes of the face layers for electric displacement or for electric potential, boundary conditions on the cylindrical lateral surface for electric displacement and for electric potential.

Now let's consider the equation and the relation related to the first approximation. Note that in obtaining these equations and relations it is assumed that $\sigma_{rz}^{(k),0}=\sigma_{zz}^{(k),0}=0$ and

$$\left\{\partial u_r^{(k),0}\Big|_{\partial r};\partial u_z^{(k),0}\Big|_{\partial r};\partial u_r^{(k),0}\Big|_{\partial z};\partial u_z^{(k),0}\Big|_{\partial z}\right\}<<1.$$

Thus, we obtain the following equations and relations for the first approximation:

equilibrium equation:

$$\frac{\partial t_{rr}^{(k),1}}{\partial r}+\frac{\partial t_{zr}^{(k),1}}{\partial z}+\frac{1}{r}\left(t_{rr}^{(k),1}-t_{\theta\theta}^{(k),1}\right)=0,$$

$$\frac{\partial t_{rz}^{(k),1}}{\partial r}+\frac{\partial t_{zz}^{(k),1}}{\partial z}+\frac{1}{r}t_{rz}^{(k),1}=0,$$

$$\frac{\partial D_{R}^{(k),1}}{\partial r}+\frac{1}{r}D_{R}^{(k),1}+\frac{\partial D_{Z}^{(k),1}}{\partial z}=0. \qquad (13)$$

where

$$t_{rr}^{(k),1}=\sigma_{rr}^{(k),1}+\sigma_{rr}^{(k),0}\frac{\partial u_{r}^{(k),1}}{\partial r}+M_{rr}^{(k),1},$$

$$t_{\theta\theta}^{(k),1}=\sigma_{\theta\theta}^{(k),1}+\sigma_{\theta\theta}^{(k),0}\frac{u_{r}^{(k),1}}{r}+M_{\theta\theta}^{(k),1},$$

$$t_{zr}^{(k),1}=\sigma_{zr}^{(k),1}+M_{zr}^{(k),1}, t_{rz}^{(k),1}=\sigma_{rr}^{(k),1}+\sigma_{rr}^{(k),0}\frac{\partial u_{z}^{(k),1}}{\partial r}+M_{rz}^{(k),1},$$

$$t_{zz}^{(k),1}=\sigma_{zz}^{(k),1}+M_{zz}^{(k),1}, D_{R}^{(k),1}=D_{r}^{(k),1}+D_{r}^{(k),0}\frac{\partial u_{r}^{(k),1}}{\partial r}+\frac{\partial u_{r}^{(k),1}}{\partial z}D_{z}^{(k),0},$$

$$D_{Z}^{(k),1}=D_{z}^{(k),1}+D_{z}^{(k),0}\frac{\partial u_{z}^{(k),1}}{\partial z}+D_{r}^{(k),0}\frac{\partial u_{z}^{(k),1}}{\partial r},$$

$$M_{rr}^{(k),1}=E_{r}^{(k),0}E_{r}^{(k),1}-E_{z}^{(k),0}E_{z}^{(k),1}, M_{\theta\theta}^{(k),1}=E_{r}^{(k),0}E_{r}^{(k),1}-E_{z}^{(k),0}E_{z}^{(k),1},$$

$$M_{zz}^{(k),1}=E_{z}^{(k),0}E_{z}^{(k),1}-E_{r}^{(k),0}E_{r}^{(k),1}. \qquad (14)$$

Relationship between deformations and mechanical movements:

$$s_{rr}^{(k),1}=\frac{\partial u_{r}^{(k),1}}{\partial r}, s_{\theta\theta}^{(k),1}=\frac{u_{r}^{(k),1}}{r}, s_{zz}^{(k),1}=\frac{\partial u_{z}^{(k),1}}{\partial z},$$

$$s_{rz}^{(k),1}=\frac{1}{2}\left(\frac{\partial u_{r}^{(k),1}}{\partial z}+\frac{\partial u_{z}^{(k),1}}{\partial r}\right). \qquad (15)$$

Again, the corresponding boundary and contact conditions are presented.

It is also necessary to add the electro-mechanical ratios obtained for each approximation. Let us write these relations for the zero approximation:

$$\sigma_{rr}^{(k),0} = c_{11}^{(k)} s_{rr}^{(k),0} + c_{12}^{(k)} s_{\theta\theta}^{(k),0} + c_{13}^{(k)} s_{zz}^{(k),0} - e_{11}^{(k)} E_r^{(k),0} - e_{31}^{(k)} E_z^{(k),0},$$

$$\sigma_{\theta\theta}^{(k),0} = c_{12}^{(k)} s_{rr}^{(k),0} + c_{22}^{(k)} s_{\theta\theta}^{(k),0} + c_{23}^{(k)} s_{zz}^{(k),0} - e_{12}^{(k)} E_r^{(k),0} - e_{32}^{(k)} E_z^{(k),0},$$

$$\sigma_{zz}^{(k),0} = c_{13}^{(k)} s_{rr}^{(k),0} + c_{23}^{(k)} s_{\theta\theta}^{(k),0} + c_{33}^{(k)} s_{zz}^{(k),0} - e_{13}^{(k)} E_r^{(k),0} - e_{33}^{(k)} E_z^{(k),0},$$

$$\sigma_{rz}^{(k),0} = c_{44}^{(k)} s_{rz}^{(k),0} - e_{15}^{(k)} E_r^{(k),0} - e_{35}^{(k)} E_z^{(k),0},$$

$$D_r^{(k),0} = e_{11}^{(k)} s_{rr}^{(k),0} + e_{12}^{(k)} S_{\theta\theta}^{(k),0} + e_{13}^{(k)} s_{zz}^{(k),0} + e_{11}^{(k)} E_r^{(k),0} + e_{13}^{(k)} E_z^{(k),0},$$

$$D_z^{(k),0} = e_{31}^{(k)} s_{rr}^{(k),0} + e_{32}^{(k)} S_{\theta\theta}^{(k),0} + e_{33}^{(k)} s_{zz}^{(k),0} + e_{31}^{(k)} E_r^{(k),0} + e_{33}^{(k)} E_z^{(k),0},$$

$$M_{rr}^{(k),0} = M_{\theta\theta}^{(k),0} = M_{rz}^{(k),0} = M_{zr}^{(k),0} = 0,$$

$$E_r^{(k),0} = -\frac{\partial \phi^{(k),0}}{\partial r},\ E_z^{(k),0} = -\frac{\partial \phi^{(k),0}}{\partial z}, \qquad (16)$$

Similarly, you can also write down the electro-mechanical relationship for the first approximation.

To determine the critical parameters that determine the delamination-buckling of a circular three-layer disc with two interfacial circular cracks, it is sufficient to use only the zero and first approximations[2].

Section 2.3 gives some essential information about the finite element method.

Section 2.4 provides numerical results and their analyzes. Here the shapes of the initial imperfection of the crack surfaces are selected, the criterion of local buckling, the choice of the layer material and the testing of the calculation algorithm are explained. In the second chapter of the dissertation, many numerical results are presented in the form of tables, from which more specific conclusions can be drawn for more specific cases.

In the third chapter, a three-layer PZT+Elastic+PZT circular plate with interlayer circular cracks is considered and, unlike the

---

[2] Akbarov S.D. Stability loss and buckling delamination: Three-dimensional linearized approach for elastic and viscoelastic composites. New York: Springer, 2013, 448p.

previous chapter, where the crack surfaces do not have any initial imperfections and these surfaces coincide above the interlayer planes. In addition, it is assumed that at first the plate is loaded with uniformly distributed radial normal forces acting on the cylindrical lateral surface and the stress state caused by these loads is called the initial stress. Then, we assume that the crack surfaces are loaded with normal opening forces, resulting in a stress concentration at the crack front, and the Energy Release Rate (ERR) and the Total Electro-Mechanical Energy of the plate under consideration are studied. In this case, the study is carried out using the three-dimensional linearized theory of electroelasticity, which makes it possible to take into account the presence of initial stresses on the value of the ERR.

Section 3.1 first sets out the formulation of the problem for determining the initial state (the first stage), and then the formulation of the problem related to the second stage. It is assumed that in the first stage, the electromechanical state in the considered plate can be determined within the framework of the linear theory of electro-elasticity for piezoelectric materials for the front layers and within the framework of the linear theory of elasticity for the middle layer. The main point of the formulation of the problem related to the second stage are as follows. To increase the influence of the initial electro-mechanical state on the values manifested at the second stage, it is necessary to abandon the linear theory of electro-elasticity for piezoelectric materials. Note that taking this effect into account can be taken into account (within the framework of certain assumptions) within the framework of the linearized theory of electro-elasticity for piezoelectric materials. Based on this position, we will formulate the problem statement related to the second stage within the framework of the indicated linearized theories and thereby provide the possibility of taking into account the influence of the initial state on the value of the sought values manifested by additional loading acting on the surface of the cracks.

Section 3.2 defines the quantities relating to the initial (to the first) and to the perturbed (i.e., to the second stage) state separately.

According to the Saint-Venant principle, we can assume that at $0 \le r < (l/2 - h)$ in the initial state, the stress in the plate is determined by the following expressions:

$$\sigma_{zz}^{(k)0} = 0, \quad \sigma_{rz}^{(k)0} = 0, \quad s_{zz}^{(k)0} = 0,$$

$$s_{rr}^{(k)0} = s_{\theta\theta}^{(k)0} = const_k, \quad \sigma_{rr}^{(k)0} = \sigma_{\theta\theta}^{(k)0} = const_{1k}. \tag{17}$$

According to the boundary condition with respect to electrical displacement, we can assume that

$$D_z^{(k)0} = D_r^{(k)0} = 0, \qquad k = 1,3. \tag{18}$$

Using (18) and electromechanical relations in the previous section, we obtain:

$$E_r^{(k)0} = a_1^{(k)} s_{rr}^{(k)0} + b_1^{(k)} s_{zz}^{(k)0},$$

$$E_z^{(k)0} = d_1^{(k)} s_{rr}^{(k)0} + c_1^{(k)} s_{zz}^{(k)0}, \tag{19}$$

where

$$a_1^{(k)} = \frac{\varepsilon_{13}^{(k)}\left(e_{31}^{(k)} + e_{32}^{(k)}\right) - \varepsilon_{33}^{(k)}\left(e_{11}^{(k)} + e_{22}^{(k)}\right)}{\varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)} - \varepsilon_{13}^{(k)}\varepsilon_{31}^{(k)}},$$

$$b_1^{(k)} = \frac{\varepsilon_{13}^{(k)} e_{33}^{(k)} - \varepsilon_{33}^{(k)} e_{13}^{(k)}}{\varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)} - \varepsilon_{13}^{(k)}\varepsilon_{31}^{(k)}}, \quad c_1^{(k)} = \frac{\varepsilon_{11}^{(k)} e_{33}^{(k)} - \varepsilon_{31}^{(k)} e_{13}^{(k)}}{\varepsilon_{13}^{(k)}\varepsilon_{33}^{(k)} - \varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)}},$$

$$d_1^{(k)} = \frac{\varepsilon_{11}^{(k)}\left(e_{31}^{(k)} + e_{32}^{(k)}\right) - \varepsilon_{31}^{(k)}\left(e_{11}^{(k)} + e_{12}^{(k)}\right)}{\varepsilon_{13}^{(k)}\varepsilon_{33}^{(k)} - \varepsilon_{11}^{(k)}\varepsilon_{33}^{(k)}}. \tag{20}$$

Taking into account the $\sigma_{zz}^{(j)0} = 0,$ equalities, we can write:

$$s_{zz}^{(j)0} = a_{zr}^{(j)} s_{rr}^{(j)0},$$

$$a_{zr}^{(k)} = \frac{c_{13}^{(k)} + c_{32}^{(k)} - e_{13}^{(k)} a_1^{(k)} - e_{33}^{(k)} d_1^{(k)}}{c_{33}^{(k)} - e_{13}^{(k)} b_1^{(k)} - e_{33}^{(k)} c_1^{(k)}}, \quad k = 1,3,$$

$$a_{zr}^{(2)} = \frac{2\lambda^{(2)}}{\lambda^{(2)} + 2\mu^{(2)}}. \tag{21}$$

Taking into account (21) we get:

$$\sigma_{rr}^{(j)0} = A_r^{(j)} s_{rr}^{(j)0}$$

$$A_r^{(k)} = c_{11}^{(k)} + c_{12}^{(k)} - e_{11}^{(k)} a_1^{(k)} + a_{zr}^{(k)} c_{13}^{(k)} - a_{zr}^{(k)} e_{11}^{(k)} b_1^{(k)} - a_{zr}^{(k)} e_{31}^{(k)} c_1^{(k)}$$

$$A_r^{(2)} = \frac{\lambda^{(2)}}{\lambda^{(2)} + 2\mu^{(2)}}. \tag{22}$$

Taking into account that

$$s_{rr}^{(1)0} = s_{rr}^{(2)0}, \quad 2h_f \sigma_{rr}^{(1)0} + h_C \sigma_{rr}^{(2)0} = hq. \tag{23}$$

Thus, from (22) and (23) we determine the following expression:

$$\sigma_{rr}^{(1)0} = q\left(2\frac{h_F}{h} + \frac{h_C}{h}\frac{A_r^{(2)}}{A_r^{(1)}}\right)^{-1}. \tag{24}$$

So, the quantities related to the initial state are determined through expressions (17)-(24). Recall that these expressions are valid for $0 \le r < (l/2 - h)$ and are obtained when the materials of the face layers are the same.

Boundary contact problems related to the second stage cannot be solved analytically and therefore we use FEM to solve these problems. To use the FEM to solve a boundary value problem, we introduce the following functional[3,4]:

$$\Pi\left(u_r^{(1)}, u_r^{(2)}, u_r^{(3)}, u_z^{(1)}, u_z^{(2)}, u_z^{(3)}, \phi^{(1)}, \phi^{(3)}\right) =$$
$$= \frac{1}{2} 2\pi \sum_{k=1}^{3} \iint_{\Omega^{(k)}} \left[t_{rr}^{(k)} \frac{\partial u_r^{(k)}}{\partial r} + t_{\theta\theta}^{(k)} \frac{u_r^{(k)}}{r} + t_{rz}^{(k)} \frac{\partial u_z^{(k)}}{\partial r} + \right.$$
$$\left. + t_{zr}^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + t_{zz}^{(k)} \frac{\partial u_z^{(k)}}{\partial z} \right] r dr dz +$$
$$+ \frac{1}{2} 2\pi \iint_{\Omega^{(1)}} \left[E_r^{(1)} D_r^{(1)} + E_z^{(1)} D_z^{(1)} \right] r dr dz +$$

[3] Yang Y.S. An introduction to the theory of piezoelectricity. New York. Springer 2005, 282p.

[4] Akbarov S.D. Stability loss and buckling delamination: Three-dimensional linearized approach for elastic and viscoelastic composites. New York: Springer, 2013, 448p.

$$-\frac{1}{2}2\pi\iint_{\Omega^{(3)}}\left[E_r^{(3)}D_r^{(3)}+E_z^{(3)}D_z^{(3)}\right]rdrdz-$$

$$-2\pi\int_0^{l_0/2}pu_z^{(1)}\Big|_{z=h_F}rdr-2\pi\int_0^{l_0/2}pu_r^{(2)}\Big|_{z=h_F}rdr-$$

$$-2\pi\int_0^{l_0/2}pu_z^{(2)}\Big|_{z=h_F+h_C}rdr-2\pi\int_0^{l_0/2}pu_z^{(3)}\Big|_{z=h_F+h_C}rdr. \qquad (25)$$

where

$$\Omega^{(1)}=\{0\le r\le l/2;\quad 0\le z\le h_F\}-\{z=h_F-0;\quad 0\le r\le l_0/2\}$$

$$\Omega^{(2)}=\{0\le r\le l/2;\quad h_F\le z\le h_F+h_C\}-\{z=h_F+0;\quad 0\le r\le l_0/2\}$$
$$-\{z=h_F+h_C-0;\quad 0\le r\le l_0/2\}$$

$$\Omega^{(3)}=\{0\le r\le l/2;\quad h_F+h_C\le z\le 2h_F+h_C\}-$$
$$-\{z=h_F+h_C+0;\quad 0\le r\le l_0/2\}. \qquad (26)$$

Note that functional (25), with obvious changes, coincides with the functional given in the previous chapter. Therefore, the operations performed on functional (25) for FEM modeling coincides with the corresponding operations described in the previous chapter. Due to this provision, the presentation of these operations is not considered here.

Thus, determining the value of mechanical displacements and electrical potential at the nodal points, we determine the value of displacements and electrical potential within each finite element. Further, after determining the displacements and potential, using the relationship (2), (3) and (4), we determine all the required values within each finite element and within the entire plate.

Having at hand the values of the stress-strain state in the plate and using the formula[5][6]

$$\gamma=\frac{\partial u}{\pi\, l_0\,\partial l_0} \qquad (27)$$

[5] Черепанов Г.П. Механика хрупкого разрушения. М.: Наука, 1974, 640с.

[6] Черепанов Г.П. Механика разрушения композиционных материалов. М.: Наука, 1983, 298с.

define the ERR (i.e. γ) at the front of the circular crack, where the U value is calculated from the following integral:

$$U=\frac{1}{2}2\pi\sum_{k=1}^{3}\iint_{\Omega^{(k)}}\left[t_{rr}^{(k)}\frac{\partial u_r^{(k)}}{\partial r}+t_{\theta\theta}^{(k)}\frac{u_r^{(k)}}{r}+t_{rz}^{(k)}\frac{\partial u_z^{(k)}}{\partial r}+t_{zr}^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+\right.$$
$$\left.+t_{zz}^{(k)}\frac{\partial u_z^{(k)}}{\partial z}\right]rdrdz+\frac{1}{2}2\pi\iint_{\Omega^{(1)}}\left[E_r^{(1)}D_r^{(1)}+E_z^{(1)}D_z^{(1)}\right]rdrdz+$$
$$+\frac{1}{2}2\pi\iint_{\Omega^{(3)}}\left[E_r^{(3)}D_r^{(3)}+E_z^{(3)}D_z^{(3)}\right]rdrdz. \qquad (28)$$

Note that l_0/b in (27) is the radius of a circular crack.

Section 3.3 provides numerical results and analyzes of these results. In these studies, the face layer materials will be selected from $PZT-4$, $PZT-5H$ and $BaTiO_3$, and the middle layer materials will be selected from Al (aluminum) and St (steel).

In this section, we will present and analyze the numerical results related to the electromechanical energy U and the energy release rate (ERR) γ at the front of a circular crack. To determine the effect of mutual influence of mechanical and electric fields on these numerical results, we will consider the following two cases (tables 1 and 2):

case 1

$$e_{ij}^{(1)}=e_{ij}^{(3)}=0,\quad \varepsilon_{ij}^{(1)}=\varepsilon_{ij}^{(3)}=0, \qquad (29)$$

case 2

$$e_{ij}^{(1)}=e_{ij}^{(3)}\neq 0,\quad \varepsilon_{ij}^{(1)}=\varepsilon_{ij}^{(3)}\neq 0. \qquad (30)$$

Note that the numerical results related to case 1 show the value of purely mechanical energy and ERR. However, the numerical results related to case 2 show the value of energy and ERR with full consideration of the mutual influence between mechanical and electric fields.

In clause 3.3.1 tests of finite element modeling and PC programs are analyzed.

In clause 3.3.2 presents the numerical results related to energies and ERR and analyzes them. First, we consider the case when there is no initial stress in the plate, i.e. consider the case where $q = 0$. In this case, consider case 2 (30) and distinguish the following energies:

- total electro-mechanical energy;
- purely mechanical energy, here it is assumed that $e_{ij}^{(1)} = e_{ij}^{(3)} = 0$, $\varepsilon_{ij}^{(1)} = \varepsilon_{ij}^{(3)} = 0$;
- energy of mutual influence;
- pure electrical energy.

Numerical results illustrating the effect of the crack radius on the above energies shown in $PZT-5H|Al|PZT-5H$ and $PZT-5H|St|PZT-5H$ plates are shown in the form of graphs. Note that when obtaining these results, it was assumed that $q|C_{44}^{PZT-5H} = 0$ (i.e., there are no initial stresses in the plate) and $h_F / l = 0.025$.

Table 1. Convergence of numerical results depending on the number of finite elements in the radial direction in the case when $l_0 / l = 0.5, h_F / l = 0.05$ and $h_C / l = 0.1$; number of finite elements in the direction axis Oz is 12 and is considered $PZT-5H|Al|PZT-5H$ plate.

Number of finite elements in the radial direction	Number of free variations	$\gamma_i\left(c_{44}^{PZT-5H} l\right)$	
		Case 1	Case 2
40	5039	5.17384	3,74138
60	7559	5.25945	3.80980
80	10079	5.31453	3.85354
100	12599	5.35750	3.88634
120	15119	5.39413	3.91292
140	17639	5.42579	3.93496

160	20159	5.45335	3.95347
200	25199	5.49826	3.98258
300	37799	5.56864	4.02635
400	50399	5.60464	4.04842
500	62999	5.62642	4.06177

Table 2. Convergence of numerical results depending on the number of finite elements in the direction of the Oz axis in the case when $l_0 / l = 0.5, \quad h_F / l = 0.05, \quad h_F / l = 0.1,$ number of endpoints elements in the direction is 100 and seen $PZT-5H|Al|PZT-5H$ plate.

Number of finite elements in the Oz direction	Number of free variations	$\gamma\left|\left(c_{44}^{PZT-5H} l\right)\right.$	
		Case 1	Case 2
12	12599	5.35750	3.88634
18	18599	5.33552	3.86970
20	20599	5.33017	3.86545
24	24599	5.32377	3.85928
28	28599	5.32045	3.85535
30	39599	5.31515	3.85174
40	40599	5.30807	3.84444

The author is sincerely grateful to his scientific advisor, Doctor of Physical and Mathematical Sciences, prof. Orujali Huseyngulu oglu Rzayev for his constant attention and valuable advice.

CONCLUSION

In this dissertation work, two types of axisymmetric problems are considered and investigated. The first of them is the problem of local buckling of a three-layer PZT+Elastic+PZT plate with interlayer circular cracks, the second is the problem of fracture (i.e., studying the ERR) of the specified plate, also with interlayer circular cracks. These problems are solved using the three-dimensional linearized theory of electro-elasticity for piezoelectric materials. The corresponding boundary value problems are solved numerically using the FEM.

When solving the first problem, the sought values are represented as a series in a small parameter, which characterize the degree of the initial imperfection of the crack surfaces. Using the appropriate mathematical calculations, the corresponding closed system of equations and boundary conditions for each approximation are obtained separately from the corresponding equations and relations of the geometrically nonlinear theory of electro-elasticity for piezoelectric materials. It is established that the equations and relations obtained for the first and subsequent approximations coincide with the corresponding equations and relations with the three-dimensional linearized theory of stability for piezoelectric materials. The determination of the critical values of the compressive force is carried out within the framework of only the zero and first approximations using the criterion of the initial imperfection.

Specific numerical results are obtained for $PZT-5H|Al|PZT-5H$, $PZT-4|Al|PZT-4$, $BaTiO_3|Al|BaTiO_3$, $PZT-5H|St|PZT-5H$, $PZT-4|St|PZT-4$ and $BaTiO_3|St|BaTiO_3$ plates. These results are simultaneously presented as dimensionless radial normal stress $\sigma_{cr}^{(1)}$ acting in the face piezoelectric layer, as $\sigma_{cr}^{(2)}$ dimensionnless radial normal stress in the middle elastic layer, and as $\overline{P}_{cr}$ dimensionless intensity of external uniformly distributed compressive normal forces acting on the lateral cylindrical surface of the plates. According to these results, the following specific conclusions can be drawn:

- the value of $\sigma_{cr}^{(1)}$, $\sigma_{cr}^{(2)}$ and $\overline{P}_{cr}$ decreases with decreasing thickness of the face layers and with increasing radius of circular cracks;
- in all considered cases, the piezoelectricity of the face layers leads to an increase in the $\sigma_{cr}^{(1)}$ values, and with an increase in the thickness of the face layers, this increase becomes more significant;
- the nature of the influence of the piezoelectricity of the materials of the face layers according to the $\sigma_{cr}^{(2)}$ and $\overline{P}_{cr}$ values depends on the electro-mechanical and geometric properties of the layers of the plates: as a rule, for a relatively thin (thick) face layer and for a relatively long (short) crack, the piezoelectricity of the face layers leads to a decrease (to increase) of $\sigma_{cr}^{(2)}$ and $\overline{P}_{cr}$ values.

However, the values of the indicated "increases" and "decreases" are significantly less than the corresponding "increases" or "decreases" obtained for $\sigma_{cr}^{(1)}$:

- the $\sigma_{cr}^{(1)}$, $\sigma_{cr}^{(2)}$ and $\overline{P}_{cr}$ values obtained for wafers with an St core material (steel) greater than their corresponding values obtained for wafers with an Al core material (aluminum);
- a more sensitive critical parameter for determining the effect of piezoelectricity on local buckling of the face layer near the interlayer crack is $\sigma_{cr}^{(1)}$.

When solving the second problem, it is assumed that a circular three-layer plate having interlayer circular cracks is first compressed by uniformly distributed normal forces in the radial direction and this force acts on the cylindrical side surface of the plate. The stress state in the plate caused by these forces is called the initial stress. After that, it is assumed that uniformly distributed normal "opening" forces are applied on the surface of the cracks, and it is required to determine the ERR and the energy caused by these forces, taking into account the influence of the initial stresses on the magnitude caused by these forces as well.

The study is carried out using the three-dimensional linearized theory of electro-elasticity for piezoelectric materials and specific numerical results are obtained using the FEM for $PZT-5H|Al|PZT-5H$, $PZT-4|Al|PZT-4$, $BaTiO_3|Al|BaTiO_3$, $BaTiO_3|Al|BaTiO_3$ plates. Based on these results, the following specific conclusions can be drawn:

- piezoelectricity of the material of the face layers leads to a decrease in the value of the total electromechanical energy and the magnitude of this decrease increases with an increase in l_0/l and with a decrease in h_F/l, where $l_0/2$ ($l/2$) is the radius of a circular crack (radius of a circular plate), h_F is the thickness of the face layer. Consequently, the l_0/l and h_F/l parameters characterize not only the dimensions of the circular crack and the thickness of the face layer, but also the dimensions of the entire circular plate;
- initial compression (tension) of the plate in the radial direction leads to an increase (decrease) in the values of the ERR; in the case of initial compression, the value of the ERR increases indefinitely as the compressive force approaches the corresponding critical forces;
- piezoelectricity of the material of the face layers leads to a decrease in the values of the ERR;
- ERR values increase (decrease) with an increase in l_0/l (with an increase in h_F/l);
- the value of ERR depends not only on the electromechanical properties of the materials of the front layers, but also on the mechanical properties of the material of the middle layer;
- the numerical results obtained for the second problem in a qualitative and quantitative sense agree with the results obtained earlier in research works.

The main results of the dissertation were published in the following works:

1. Cafarova F.I., Rzayev O.A. A stability loss of the PZT/Metal/PZT sandwich circular plate-disc under "open-circuit" condition//Transactions of NAS of Azerbaijan, issue mechanics. 2016, v.36, №4, p.50-59.
2. Cafarova F.I., Akbarov S.D., Yahnioglu N. Buckling delamination of the PZT/Metal/PZT sandwich circular plate-disc with penny-shaped interface cracks.//Smart Structures and Systems, 2017, v.19, №2, p.163-179.
3. Akbarov S.D., Cafarova F.I., Yahnioglu N. Buckling delamination of the circular sandwich plate with piezoelectric face and elastic core layers under rotationally symmetric external pressure//AIP conference Proceeding 1815, 080001(2017); doi:10/1063/1.4976433, Turkish Physical Society 32-nd International Physics Congress (TPS32)
4. Cafarova F.I., Rzayev O.A. On the influence of the "short-and open-circuit" conditions on stability loss of the PZT/Metal/PZT sandwich circular plate-disc condition//Caspian Journal of Applied Mathematics, ecology and economics. Inter. Academy, Baku,ISSN:1560-4055, 2017,v.5, №2, p.26-28.
5. Cafarova F.I. FEM analysis of the problem related to the penny-shaped interface cracks contained in the PZT/Metal/PZT sandwich circular plate// "Actual problems of modern natural and economic sciences". International Scientific Conference. May 04-05, 2018, pp. 271-277.
6. Cafarova F.I. On the problem formulation and solution method of the penny shaped interface crack problems related to the Elastic/PZT/Elastic sandwich circular plate // "Actual problems of modern natural and economic sciences". International Scientific Conference. May 04-05, 2018, pp. 309-312.
7. Cafarova F.I. Electro-mechanical energies of the PZT/Elastic/PZT sandwich circular plate with penny-shaped interface cracks under action of the normal opening forces on the cracks edges// Transactions of NAS of Azerbaijan, issue mechanics. 2018, v.38, №7, p.

8. Cafarova F.I. Energy release rate at the front of penny-shaped interface cracks contained in the PZT/Elastic/PZT sandwich circular plate under action of the normal opening forces on the cracks edges// Journal of Cont. Applied Math. v.8, 2018, p.25-46.
9. Akbarov S.D., Cafarova F.I., Yahnioglu N. The influence of initial stresses on energy release rate and total electro-mechanical patential energy for penny-shaped interface cracks in PZT/Elastic/PZT sandwich circular plate-disc// Smart Structures and Systems. v.22, №3, 2018, p.259-276.

The defense will be held on **14 December 2021** at **14^{00}** at the meeting of the Dissertation council FD 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University.

Address: Baku city, Acad. Z. Khalilov street, 23.

Dissertation is accessible at the Baku State University Library

Electronic versions of dissertation and its abstract are available on the official website of the Baku State University

Abstract was sent to the required addresses on **12 November 2021**.

Signed for print: 19.10.2021

Paper format: 60 × 84 1/16

Volume: 40000

Number of hard copies: 20